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A Note on Large Numbers of Maximal Independent Sets in Forests

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Abstract

In this paper we complete the determination of the k-th ($3 \leq k \leq \lfloor n/2 \rfloor - 1$) largest numbers of maximal independent sets among all forests of order $n \geq 8$ and characterize the extremal graphs.

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Keywords: maximal independent set, forest, extremal graph

1 Introduction

Let $G = (V, E)$ be a simple undirected graph. A subset $I \subseteq V$ is *independent* if there is no edge of G between any two vertices of I. A *maximal independent set* is an independent set that is not a proper subset of any other independent set. The set of all maximal independent sets of G is denoted by $\text{MI}(G)$ and its cardinality by $mi(G)$.

The problem of determining the largest value of $mi(G)$ in a general graph of order n and those graphs achieving the largest number was proposed by Erdös and Moser, and solved by Moon and Moser [6]. It was then studied for various families of graphs, including trees, forests, (connected) graphs with at most one cycle, (connected) triangle-free graphs, (k-)connected graphs, bipartite graphs; for a survey see [4]. Later, Jin and Li [1] determined the second largest number of maximal independent sets among all graphs of order n. As for trees and forests, it was solved by Jou and Lin [5].

The purpose of this paper is to determine the k-th ($3 \leq k \leq \lfloor n/2 \rfloor - 1$) largest number of maximal and maximum independent sets among all forests of order $n \geq 8$. Extremal graphs achieving these values are also given.

2 Preliminary

For our discussions, some terminology and notation are needed. For a graph $G = (V, E)$, the cardinality of $V(G)$ is called the *order*, and it is denoted by $|G|$. For a set $A \subseteq V(G)$, the *deletion* of A from G is the graph $G - A$ obtained from G by removing all vertices in A and their incident edges. Two graphs G_1 and G_2 are *disjoint* if $V(G_1) \cap V(G_2) = \emptyset$. The *union* of two disjoint graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with vertex set $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. Let nG be the short notation for the union of n copies of disjoint graphs isomorphic to G. A component of odd (respectively, even) order is called an *odd* (respectively, *even*) *component.* Denote by P_n a *path* with n vertices. Throughout this paper, for simplicity, let $r = \sqrt{2}$.

The following results are essential for our discussions.

Lemma 2.1. ([2]) *If G is the union of two disjoint graphs G_1 and G_2, then $mi(G) = mi(G_1) \cdot mi(G_2)$.*

The results of the largest numbers of maximal independent sets among all trees and forests are described in Theorems 2.2 and 2.3, respectively.

Theorem 2.2. ([2, 3]) *If T is a tree with $n \geq 1$ vertices, then $mi(T) \leq t_1(n)$, where*

$$t_1(n) = \begin{cases} r^{n-2} + 1, & \text{if } n \text{ is even}, \\ r^{n-1}, & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, $mi(T) = t_1(n)$ if and only if $T = T_1(n)$, where

$$T_1(n) = \begin{cases} B(2, \frac{n-2}{2}) \text{ or } B(4, \frac{n-4}{2}), & \text{if } n \text{ is even}, \\ B(1, \frac{n-1}{2}), & \text{if } n \text{ is odd.} \end{cases}$$

where $B(i, j)$ is the set of batons, which are the graphs obtained from the basic path P of $i \geq 1$ vertices by attaching $j \geq 0$ paths of length two to the endpoints of P in all possible ways (*see Figure* 1).

Theorem 2.3. ([2, 3]) *If F is a forest with $n \geq 1$ vertices, then $mi(F) \leq f_1(n)$, where*

$$f_1(n) = \begin{cases} r^{n}, & \text{if } n \text{ is even}, \\ r^{n-1}, & \text{if } n \text{ is odd.} \end{cases}$$

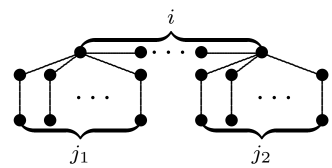


Figure 1: The baton $B(i, j)$ with $j = j_1 + j_2$

Furthermore, $mi(F) = f_1(n)$ *if and only if* $F = F_1(n)$*, where*

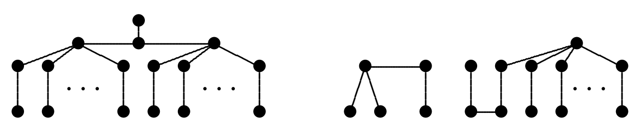
$$F_1(n) = \begin{cases} \frac{n}{2}P_2, & \textit{if } n \textit{ is even}, \\ B(1, \frac{n-1-2s}{2}) \cup sP_2 & \\ \quad \textit{for some } s \textit{ with } 0 \le s \le \frac{n-1}{2}, & \textit{if } n \textit{ is odd.} \end{cases}$$

The results of the second largest numbers of maximal independent sets among all trees and forests are described in Theorems 2.4 and 2.5, respectively.

Theorem 2.4. ([5]) *If* T *is a tree with* $n \ge 4$ *vertices having* $T \ne T_1(n)$*, then* $mi(T) \le t_2(n)$*, where*

$$t_2(n) = \begin{cases} r^{n-2}, & \textit{if } n \textit{ is even}, \\ 3, & \textit{if } n = 5, \\ 3r^{n-5} + 1, & \textit{if } n \textit{ is odd.} \end{cases}$$

Furthermore, $mi(T) = t_2(n)$ *if and only if* $T = T_2'(8), T_2''(8), P_{10}$*, or* $T_2(n)$*, where* $T_2(n)$ *and* $T_2'(8)$*,* $T_2''(8)$ *are shown in Figures 2 and 3, respectively.*



$T_{2e}(n)$, $n \ge 4$ is even $\quad$ $T_2(5)$ $\quad$ $T_{2o}(n)$, $n \ge 7$ is odd

Figure 2: The trees $T_2(n)$

$T_2'(8)$ $\quad$ $T_2''(8)$

Figure 3: The trees $T_2'(8)$ and $T_2''(8)$

Theorem 2.5. ([5]) *If* F *is a forest with* $n \ge 4$ *vertices having* $F \ne F_1(n)$*, then* $mi(F) \le f_2(n)$*, where*

$$f_2(n) = \begin{cases} 3r^{n-4}, & \textit{if } n \textit{ is even}, \\ 3, & \textit{if } n = 5, \\ 7r^{n-7}, & \textit{if } n \textit{ is odd.} \end{cases}$$

Furthermore, $mi(F) = f_2(n)$ *if and only if* $F = F_2(n)$*, where*

$$F_2(n) = \begin{cases} P_4 \cup \frac{n-4}{2} P_2, & \text{if } n \geq 4 \text{ is even}, \\ T_2(5) \text{ or } \ P_4 \cup P_1, & \text{if } n = 5, \\ P_7 \cup \frac{n-7}{2} P_2, & \text{if } n \geq 7 \text{ is odd.} \end{cases}$$

3 Main results

In this section we determine the k-th ($3 \leq k \leq \lfloor n/2 \rfloor - 1$) largest values of $mi(G)$ among all forests of order $n \geq 8$. Moreover, the extremal graphs achieving these values are also determined.

Define the graphs $F_i(n)$, $i = 3, 4, \ldots, \lfloor n/2 \rfloor - 1$ and $F_4'(n)$ of order $n \geq 8$ as follows.

$$F_i(n) = \begin{cases} T_1(2i) \cup F_1(n-2i), & \text{if } n \geq 8 \text{ is even}, \\ T_2(2i+3) \cup F_1(n-2i-3), & \text{if } n \geq 9 \text{ is odd}, \end{cases}$$

and

$$F_4'(n) = 2T_1(4) \cup F_1(n-8), \text{ for } n \text{ is even.}$$

Let $f_i(n) = mi(F_i(n))$. For simple calculation, we have that

$$f_i(n) = \begin{cases} r^{n-2} + r^{n-2i}, & \text{if } n \geq 8 \text{ is even}, \\ 3r^{n-5} + r^{n-2i-3}, & \text{if } n \geq 9 \text{ is odd}, \end{cases}$$

and

$$mi(F_4'(n)) = 9r^{n-8}, \text{ for } n \text{ is even.}$$

In this paper we will prove the following result.

Theorem 3.1. *For integers* k *and* n *with* $n \geq 8$ *and* $3 \leq k \leq \lfloor n/2 \rfloor - 1$*. If* F *is a forest of order* n *having* $F \neq F_i(n)$*, for* $i = 1, 2 \ldots, k-1$*, then* $mi(F) \leq f_k(n)$*. Furthermore, the equality holds if and only if* $F = F_k(n)$ *or* $F_4'(n)$ *with* n *is even,* $k = 4$*.*

Proof. Let F be a forest of order $n \geq 8$ having $F \neq F_i(n)$, for $i = 1, 2 \ldots, k-1$ and $3 \leq k \leq \lfloor n/2 \rfloor - 1$, such that $mi(F)$ is as large as possible. Then $mi(F) \geq f_k$. We consider the following two cases.

Case 1. n is even. Suppose that there exist two odd components H_1 and H_2 of F, where $|H_i| = m_i$ for $i = 1, 2$. By Lemma 2.1, Theorems 2.2 and 2.3, we have that

$$\begin{aligned} f_k(n) &= r^{n-2} + r^{n-2k} \\ &\leq mi(F) \\ &= mi(H_1) \cdot mi(H_2) \cdot mi(F - (V(H_1) \cup V(H_2))) \\ &\leq r^{m_1-1} \cdot r^{m_2-1} \cdot r^{n-m_1-m_2} \\ &= r^{n-2} \\ &< f_k(n), \end{aligned}$$

which is a contradiction. Hence F has no odd component. Since $F \neq F_1(n)$, there exists a component H of even order $m \geq 4$.

Suppose that $F - V(H) \neq F_1(n-m)$, By Lemma 2.1, Theorems 2.2 and 2.5, we have that

$$\begin{aligned} f_k(n) &= r^{n-2} + r^{n-2k} \\ &\leq mi(F) \\ &= mi(H) \cdot mi(F - (V(H))) \\ &\leq t_1(m) \cdot f_2(n-m) \\ &= (r^{m-2} + 1) \cdot 3r^{n-m-4} \\ &= 3r^{n-6} + 3r^{n-m-4} \\ &\leq 9r^{n-8} \\ &= f_4(n). \end{aligned}$$

Furthermore, the equalities holding imply that $m = k = 4$, $H = T_1(4)$ and $F - V(H) = F_2(n-4) = T_1(4) \cup F_1(n-8)$, that is, $F = F_4'(n) = 2T_1(4) \cup F_1(n-8)$.

Now we assume that $F - V(H) = F_1(n-m)$. Since $F \neq F_i(n)$ for $i = 1, 2, \ldots, k-1$, by Lemma 2.1, Theorems 2.2 and 2.3, we have that

$$\begin{aligned} f_k(n) &= r^{n-2} + r^{n-2k} \\ &\leq mi(F) \\ &= mi(H) \cdot mi(F - (V(H))) \\ &\leq \begin{cases} (t_1(m) - 1) \cdot f_1(n-m), & \text{if } m \leq 2k-2, \\ t_1(m) \cdot f_1(n-m), & \text{if } m \geq 2k, \end{cases} \\ &= \begin{cases} r^{m-2} \cdot r^{n-m}, & \text{if } m \leq 2k-2, \\ (r^{m-2} + 1) \cdot r^{n-m}, & \text{if } m \geq 2k, \end{cases} \\ &= \begin{cases} r^{n-2}, & \text{if } m \leq 2k-2, \\ r^{n-2} + r^{n-m}, & \text{if } m \geq 2k, \end{cases} \\ &\leq r^{n-2} + r^{n-2k} \\ &= f_k(n). \end{aligned}$$

Furthermore, the equalities holding imply that $m = 2k$, $H = T_1(2k)$ and $F - V(H) = F_1(n-2k)$. In conclusion, $F = F_k(n) = T_1(2k) \cup F_1(n-2k)$.

Case 2. n is odd. Suppose that there exist three odd components H_1, H_2 and H_3 of F, where $|H_i| = m_i$ for $i = 1, 2, 3$. By Lemma 2.1, Theorems 2.2 and 2.3, we have that

$$\begin{aligned} f_k(n) &= 3r^{n-5} + r^{n-2k-3} \\ &\le mi(F) \\ &= mi(H_1) \cdot mi(H_2) \cdot mi(H_3) \cdot mi(F - (V(H_1) \cup V(H_2) \cup V(H_3))) \\ &\le r^{m_1-1} \cdot r^{m_2-1} \cdot r^{m_3-1} \cdot r^{n-m_1-m_2-m_3} \\ &= r^{n-3} \\ &< f_k(n), \end{aligned}$$

which is a contradiction. Hence F has exactly one component H of odd order $m \ge 1$.

For the case that $F - V(H) \neq F_1(n - m)$, By Lemma 2.1, Theorems 2.2 and 2.5, we have that

$$\begin{aligned} f_k(n) &= 3r^{n-5} + r^{n-2k-3} \\ &\le mi(F) \\ &= mi(H) \cdot mi(F - (V(H))) \\ &\le r^{m-1} \cdot 3r^{n-m-4} \\ &\le 3r^{n-5} \\ &< f_k(n), \end{aligned}$$

which is a contradiction.

For the other case that $F - V(H) = F_1(n-m)$. Since $F \neq F_1(n)$, it follows that $H \neq T_1(m)$. By Lemma 2.1, Theorems 2.3 and 2.4 , we have that

$$\begin{aligned} f_k(n) &= 3r^{n-5} + r^{n-2k-3} \\ &\le mi(F) \\ &= mi(H) \cdot mi(F - (V(H))) \\ &\le \begin{cases} (t_2(m) - 1) \cdot f_1(n - m), & \text{if } m \le 2k + 1, \\ t_2(m) \cdot f_1(n - m), & \text{if } m \ge 2k + 3, \end{cases} \\ &= \begin{cases} 3r^{m-5} \cdot r^{n-m}, & \text{if } m \le 2k + 1, \\ (3r^{m-5} + 1) \cdot r^{n-m}, & \text{if } m \ge 2k + 3, \end{cases} \\ &= \begin{cases} 3r^{n-5}, & \text{if } m \le 2k + 1, \\ 3r^{n-5} + r^{n-m}, & \text{if } m \ge 2k + 3, \end{cases} \\ &\le 3r^{n-5} + r^{n-2k-3} \\ &= f_k(n). \end{aligned}$$

Furthermore, the equalities holding imply that $m = 2k + 3$, $H = T_2(2k + 3)$ and $F - V(H) = F_1(n - 2k - 3)$. In conclusion, $F = F_k(n) = T_2(2k + 3) \cup F_1(n - 2k - 3)$. □

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