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Quasi-Forest Graphs with the k-th Largest Number of Maximal Independent Sets

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Abstract

In this paper we complete the determination of the k-th ($3 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$) largest numbers of maximal independent sets among all quasi-forest graphs of order $n \geq 8$ and characterize the extremal graphs.

Mathematics Subject Classification: 05C51

Keywords: maximal independent set, quasi-tree graphs, quasi-forest graphs, extremal graph

1 Introduction

Let G be a graph with vertex set and edge set being $V(G)$ and $E(G)$, respectively. A subset $I \subseteq V(G)$ is *independent* if there is no edge of G between any two vertices of I. A *maximal independent set* is an independent set that is not a proper subset of any other independent set. The set of all maximal independent sets of G is denoted by $\mathrm{MI}(G)$ and its cardinality by $mi(G)$.

Around 1960, Erdös and Moser proposed the problem of determining the maximum number of $mi(G)$ in the family of graphs of order n and characterizing structure of graphs attaining the maximum value. Shortly after, Moon and Moser [10] solved the problem. It was then studied for various families of graphs, including trees, forests, (connected) graphs with at most one cycle, (connected) triangle-free graphs, (k-)connected graphs, bipartite graphs; for a survey see [3].

A connected graph (respectively, graph) G with vertex set $V(G)$ is called a *quasi-tree graph* (respectively, *quasi-forest graph*), if there exists a vertex $x \in V(G)$ such that $G-x$ is a tree (respectively, forest). The concept of quasi-tree graphs was mentioned by Liu and Lu in [9]. The problem of determining the largest and the second largest numbers of $mi(G)$ among all quasi-tree graphs and quasi-forest graphs of order n was solved by Lin [7, 8].

The purpose of this paper is to determine the k-th ($3 \le k \le \lfloor \frac{n-1}{2} \rfloor$) largest number of maximal and maximum independent sets among all quasi-forest graphs of order $n \ge 8$. Extremal graphs achieving these values are also given.

2 Preliminary

For a graph $G = (V, E)$, the cardinality of $V(G)$ is called the *order*, and it is denoted by $|G|$. The *neighborhood* $N_G(x)$ of a vertex x is the set of vertices adjacent to x in G and the *closed neighborhood* $N_G[x]$ is $\{x\} \cup N_G(x)$. The *degree* of x is the cardinality of $N_G(x)$, denoted by $\deg_G(x)$. A vertex x is called a *leaf* if $\deg_G x = 1$. For a set $A \subseteq V(G)$, the *deletion* of A from G is the graph $G - A$ obtained from G by removing all vertices in A and their incident edges. Two graphs G_1 and G_2 are *disjoint* if $V(G_1) \cap V(G_2) = \emptyset$. The *union* of two disjoint graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with vertex set $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. Let nG be the short notation for the union of n copies of disjoint graphs isomorphic to G. For a connected graph H and a graph G with components $G_1, G_2, \ldots, G_k$, $H * G$ is the set of *clasps*, which are the graphs with vertex set $V(H * G) = V(H) \cup V(G)$ and edge set $E(H * G) = E(H) \cup E(G) \cup \{xu_i : i = 1, 2, \ldots, k\}$, where x is a vertex with maximum degree in H and u_i is a vertex with maximum degree in G_i for $i = 1, 2, \ldots, k$. A *path* P_n of order n is a graph with $V(P_n) = \{x_1, x_2, \ldots, x_n\}$ and $E(P_n) = \{x_1x_2, x_2x_3, \ldots, x_{n-1}x_n\}$, where the x_i are all distinct. We refer to a path by the natural sequence of its vertices, writing, say, $P_n : x_1, x_2, \ldots, x_n$. The vertex $x_{\lceil \frac{n}{2} \rceil}$ is called the *central vertex* of P_n. For positive integers m and n, $P_m \oplus P_n$ is the graph obtained from P_m by adding a P_n and a new edge joining the leaf of P_m and the central vertex of P_n. Denote by C_n a *cycle* with n vertices.

Throughout this paper, for simplicity, let $r = \sqrt{2}$.

Lemma 2.1. ([2]) *If G is the union of two disjoint graphs G_1 and G_2, then $mi(G) = mi(G_1) \cdot mi(G_2)$.*

Lemma 2.2. ([1, 2]) *For any vertex v in a graph G, $mi(G) \le mi(G - v) + mi(G - N_G[v])$.*

The results of the largest, the second largest and the third largest numbers of maximal independent sets among all forests are described in Theorems 2.3, 2.4 and 2.5, respectively.

Theorem 2.3. ([2, 4]) *If F is a forest with $n \geq 1$ vertices, then $mi(F) \leq f_1(n)$, where*

$$f_1(n) = \begin{cases} r^n, & \text{if } n \text{ is even},\\ r^{n-1}, & \text{if } n \text{ is odd.}\end{cases}$$

Furthermore, $mi(F) = f_1(n)$ if and only if $F = F_1(n)$, where

$$F_1(n) = \begin{cases} \frac{n}{2}P_2, & \text{if } n \text{ is even},\\ (P_1 * \frac{n-1-2s}{2}P_2) \cup sP_2 \text{ for } 0 \leq s \leq \frac{n-1}{2}\ , & \text{if } n \text{ is odd.}\end{cases}$$

Theorem 2.4. ([5]) *If F is a forest with $n \geq 4$ vertices having $F \neq F_1(n)$, then $mi(F) \leq f_2(n)$, where*

$$f_2(n) = \begin{cases} 3r^{n-4}, & \text{if } n \text{ is even},\\ 3, & \text{if } n = 5,\\ 7r^{n-7}, & \text{if } n \text{ is odd.}\end{cases}$$

Furthermore, $mi(F) = f_2(n)$ if and only if $F = F_2(n)$, where

$$F_2(n) = \begin{cases} P_4 \cup \frac{n-4}{2}P_2, & \text{if } n \geq 4 \text{ is even},\\ P_2 \oplus P_3 \text{ or } P_4 \cup P_1, & \text{if } n = 5,\\ P_7 \cup \frac{n-7}{2}P_2, & \text{if } n \geq 7 \text{ is odd.}\end{cases}$$

Theorem 2.5. ([6]) *If F is a forest with $n \geq 8$ vertices having $F \neq F_i(n)$, $i = 1, 2$, then $mi(F) \leq f_3(n)$, where*

$$f_3(n) = \begin{cases} 5r^{n-6}, & \text{if } n \text{ is even},\\ 13r^{n-9}, & \text{if } n \text{ is odd.}\end{cases}$$

Furthermore, $mi(F) = f_3(n)$ if and only if $F = F_3(n)$, where

$$F_3(n) = \begin{cases} P_6 \cup \frac{n-6}{2}P_2 \text{ or } (P_1 \oplus P_5) \cup \frac{n-6}{2}P_2, & \text{if } n \text{ is even},\\ (P_4 \oplus P_5) \cup \frac{n-9}{2}P_2, & \text{if } n \text{ is odd.}\end{cases}$$

The results of the largest numbers of maximal independent sets among all quasi-tree graphs and quasi-forest graphs are described in Theorems 2.6 and 2.7, respectively.

Theorem 2.6. ([7])*If Q is a quasi-tree graph with $n \geq 5$ vertices, then $mi(Q) \leq q_1(n)$, where*

$$q_1(n) = \begin{cases} 3r^{n-4}, & \text{if } n \text{ is even},\\ r^{n-1} + 1, & \text{if } n \text{ is odd.}\end{cases}$$

Furthermore, $mi(Q) = q_1(n)$ if and only if $Q = Q_1(n)$ or $Q = C_5$, where $Q_1(n)$ is shown in Figure 1.

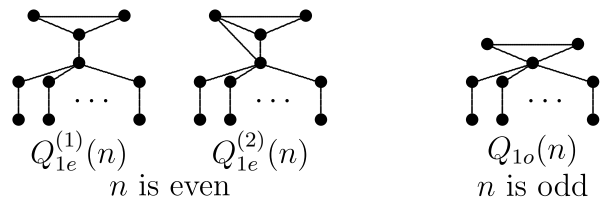

Figure 1: The graph $Q_1(n)$

Theorem 2.7. ([7])*If Q is a quasi-forest graph with $n \geq 2$ vertices, then $mi(Q) \leq \overline{q}_1(n)$, where*

$$\overline{q}_1(n) = \begin{cases} r^n, & \text{if } n \text{ is even,} \\ 3r^{n-3}, & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, $mi(Q) = \overline{q}_1(n)$ if and only if $Q = \overline{Q}_1(n)$, where

$$\overline{Q}_1(n) = \begin{cases} \frac{n}{2}P_2, & \text{if } n \text{ is even,} \\ C_3 \cup \frac{n-3}{2}P_2, & \text{if } n \text{ is odd.} \end{cases}$$

The results of the second largest numbers of maximal independent sets among all quasi-tree graphs and quasi-forest graphs are described in Theorems 2.8 and 2.9, respectively.

Theorem 2.8. ([8])*If Q is a quasi-tree graph with $n \geq 6$ vertices having $Q \neq Q_1(n)$, then $mi(Q) \leq q_2(n)$, where*

$$q_2(n) = \begin{cases} 5r^{n-6} + 1, & \text{if } n \text{ is even,} \\ r^{n-1}, & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, $mi(Q) = q_2(n)$ if and only if $Q = Q_2(n)$, where

$$Q_2(n) = \begin{cases} Q_{2e}^{(1)}(n), Q_{2e}^{(2)}(n), Q_{2e}^{(3)}(n), Q_{2e}^{(4)}(n), & \text{if } n \text{ is even,} \\ P_1 * \frac{n-1}{2}P_2, Q_{2o}^{(1)}(7), Q_{2o}^{(2)}(7), Q_{2o}^{(3)}(7), Q_{2o}^{(4)}(7), & \text{if } n \text{ is odd,} \end{cases}$$

where $Q_2(n)$ is shown in Figures 2 and 3.

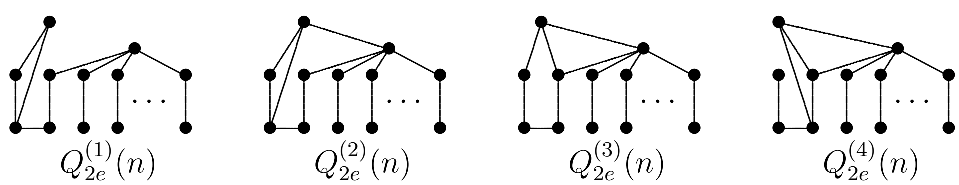

Figure 2: The graphs $Q_{2e}^{(i)}(n)$, $1 \leq i \leq 4$

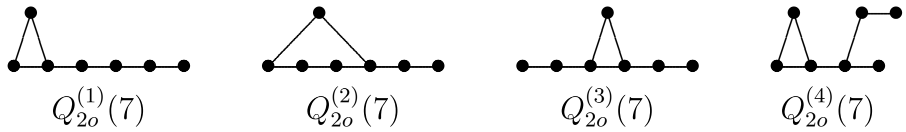

Figure 3: The graphs $Q_{2o}^{(i)}(7)$, $1 \le i \le 4$

For positive integer t, W_t is the graph of order $2t+1$ obtained by t copies of C_3 having one common vertex.

Theorem 2.9. ([8])*If Q is a quasi-forest graph with $n \ge 4$ vertices having $Q \ne \overline{Q}_1(n)$, then $mi(Q) \le \overline{q}_2(n)$, where*

$$\overline{q}_2(n) = \begin{cases} 3r^{n-4}, & \text{if } n \text{ is even,} \\ 5r^{n-5}, & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, $mi(Q) = \overline{q}_2(n)$ if and only if $Q = \overline{Q}_2(n)$, where

$$\overline{Q}_2(n) = \begin{cases} P_4 \cup \frac{n-4}{2}P_2, Q_1(n-2s) \cup sP_2, & \\ Q_2(6) \cup \frac{n-6}{2}P_2, C_3 \cup (P_1 * \frac{n-4-2s}{2}P_2) \cup sP_2, & \text{if } n \text{ is even,} \\ Q_1(5) \cup \frac{n-5}{2}P_2, W_2 \cup \frac{n-5}{2}P_2, C_5 \cup \frac{n-5}{2}P_2, & \text{if } n \text{ is odd,} \end{cases}$$

3 Main results

In this section we determine the k-th ($3 \le k \le \lfloor \frac{n-1}{2} \rfloor$) largest values of $mi(Q)$ among all quasi-forest graphs Q of order $n \ge 8$. Moreover, the extremal graphs achieving these values are also determined.

Define the graphs $\overline{Q}_i(n)$, $i = 3, 4, \ldots, \lfloor \frac{n-1}{2} \rfloor$ and $\overline{Q}'(n)$ of order $n \ge 8$ as follows.

$$\overline{Q}_i(n) = \begin{cases} Q_2(2i+2) \cup F_1(n-2i-2), & \text{if } n \ge 8 \text{ is even,} \\ (W_t * (i-t)P_2) \cup F_1(n-2i-1), & \text{if } n \ge 9 \text{ is odd,} \end{cases}$$

and

$$\overline{Q}'(n) = C_3 \cup F_2(n-3).$$

Let $\overline{q}_i(n) = mi(\overline{Q}_i(n))$. For simple calculation, we have that

$$\overline{q}_i(n) = \begin{cases} 5r^{n-6} + r^{n-2i-2}, & \text{if } n \ge 8 \text{ is even,} \\ r^{n-1} + r^{n-2i-1}, & \text{if } n \ge 9 \text{ is odd,} \end{cases}$$

and

$$mi(\overline{Q}'(n)) = \begin{cases} 21r^{n-10}, & \text{if } n \ge 8 \text{ is even,} \\ 9r^{n-7}, & \text{if } n \ge 9 \text{ is odd.} \end{cases}$$

Theorem 3.1. *For integers k and n with $n \geq 8$ is even and $3 \leq k \leq n/2-1$. If Q is a quasi-forest graph of order n having $Q \neq \overline{Q}_i(n)$, for $i = 1, 2 \ldots, k-1$, then $mi(Q) \leq \overline{q}_k(n)$. Furthermore, the equality holds if and only if $Q = \overline{Q}_k(n)$ or $\overline{Q}'(n)$ with $k = 4$.*

Proof. Let Q be a quasi-forest graph of even order $n \geq 8$ having $Q \neq \overline{Q}_i(n)$, for $i = 1, 2 \ldots, k-1$ and $3 \leq k \leq n/2-1$, such that $mi(Q)$ is as large as possible. Then $mi(Q) \geq \overline{q}_k(n)$. Since $Q \neq \overline{Q}_1(n), \overline{Q}_2(n)$, it follows that $Q \neq F_1(n), F_2(n)$. Suppose that Q is a forest, by Theorem 2.5, we have that $mi(Q) \leq 5r^{n-6}$, which is a contradiction to $mi(Q) \geq \overline{q}_k(n)$. Hence Q has at least one cycle. Let $Q = G \cup F$, where G is a quasi-tree graph of order s with at least one cycle and F is a forest of order $n-s$. We consider the following two cases.

Case 1. s is odd. Suppose that $G \neq Q_1(s)$, by Lemma 2.1, Theorems 2.3 and 2.8, we have that $mi(Q) = mi(G) \cdot mi(F) \leq r^{s-1} \cdot r^{n-s-1} = r^{n-2}$, which is a contradiction to $mi(Q) \geq \overline{q}_k(n)$.

Now we assume that $G = Q_1(s)$. For the case of $s \geq 5$, by Lemma 2.1, Theorems 2.3 and 2.6, we obtain that $mi(Q) = mi(G) \cdot mi(F) \leq (r^{s-1}+1) \cdot r^{n-s-1} = r^{n-2} + r^{n-s-1} \leq 5r^{n-6}$, which is a contradiction to $mi(Q) \geq \overline{q}_k$. For the other case of $s = 3$, then $F \neq F_1(n-3)$ since $Q \neq \overline{Q}_2(n)$. By Lemma 2.1, Theorems 2.4 and 2.5, we have that

$$\begin{aligned}
\overline{q}_k(n) &= 5r^{n-6} + r^{n-2k-2} \\
&\leq mi(Q) \\
&= mi(G) \cdot mi(F) \\
&\leq \begin{cases} 3 \cdot 7r^{n-10} = 21r^{n-10} = \overline{q}_4(n), & \text{if } F = F_2(n-3), \\ 3 \cdot 13r^{n-12} = 39r^{n-12} < \overline{q}_k(n), & \text{if } F \neq F_2(n-3). \end{cases}
\end{aligned}$$

Furthermore, the equalities holding imply that $k = 4$, $G = C_3$ and $F = F_2(n-3)$, that is, $Q = \overline{Q}'(n) = C_3 \cup F_2(n-3)$.

Case 2. s is even. Suppose that $F \neq F_1(n-s)$, by Lemma 2.1, Theorems 2.4 and 2.6, we have that $mi(Q) = mi(G) \cdot mi(F) \leq 3r^{s-4} \cdot 3r^{n-s-4} = 9r^{n-8}$, which is a contradiction to $mi(Q) \geq \overline{q}_k(n)$.

Now we assume that $F = F_1(n-s)$. Note that $Q \neq \overline{Q}_1(n), \overline{Q}_2(n)$, it follows that $G \neq \frac{s}{2}P_2, Q_1(s)$. On the other hand, since $Q \neq \overline{Q}_i(n)$, for $i = 1, 2 \ldots, k-1$, by Lemma 2.1, Theorems 2.3 and 2.6, we have that

$$\begin{aligned}
\overline{q}_k(n) &= 5r^{n-6} + r^{n-2k-2} \\
&\leq mi(Q) \\
&= mi(G) \cdot mi(F) \\
&\leq \begin{cases} (q_2(s)-1) \cdot f_1(n-s), & \text{if } s \leq 2k, \\ q_2(s) \cdot f_1(n-s), & \text{if } s \geq 2k+2, \end{cases} \\
&= \begin{cases} 5r^{s-6} \cdot r^{n-s}, & \text{if } s \leq 2k, \\ (5r^{s-6}+1) \cdot r^{n-s}, & \text{if } s \geq 2k+2, \end{cases} \\
&= \begin{cases} 5r^{n-6}, & \text{if } s \leq 2k, \\ 5r^{n-6} + r^{n-s}, & \text{if } s \geq 2k+2, \end{cases} \\
&\leq 5r^{n-6} + r^{n-2k-2} \\
&= \overline{q}_k(n).
\end{aligned}$$

Furthermore, the equalities holding imply that $s = 2k+2$, $G = Q_2(2k+2)$ and $F = F_1(n-2k-2)$. In conclusion, $Q = \overline{Q}_k(n) = Q_2(2k+2) \cup F_1(n-2k-2)$. $\square$

Theorem 3.2. *For integers k and n with $n \geq 9$ is odd and $3 \leq k \leq (n-1)/2$. If Q is a quasi-forest graph of order n having $Q \neq \overline{Q}_i(n)$, for $i = 1, 2 \ldots, k-1$, then $mi(Q) \leq \overline{q}_k(n)$. Furthermore, the equality holds if and only if $Q = \overline{Q}_k(n)$ or $\overline{Q}'(n)$ with $k = 3$.*

Proof. Let Q be a quasi-forest graph of odd order $n \geq 9$ having $Q \neq \overline{Q}_i(n)$, for $i = 1, 2 \ldots, k-1$ and $3 \leq k \leq (n-1)/2$, such that $mi(Q)$ is as large as possible. Then $mi(Q) \geq \overline{q}_k(n)$. Suppose that Q is a forest, by Theorem 2.3, we have that $mi(Q) \leq r^{n-1}$, which is a contradiction to $mi(Q) \geq \overline{q}_k(n)$. Hence Q has at least one cycle. Let $Q = G \cup F$, where G is a quasi-tree graph of order s with at least one cycle and F is a forest of order $n-s$. Let x be a vertex such that $Q - x$ is a forest. Then x is on some cycle of Q, it follows that $\deg_Q(x) \geq 2$. We consider the following two cases.

Case 1. $Q - x \neq F_1(n-1)$. Suppose that $\deg_Q x \geq 3$, by Lemma 2.2, Theorems 2.3 and 2.4, we have that $mi(Q) \leq mi(Q-x) + mi(Q - N_Q[x]) \leq 3r^{(n-1)-4} + r^{(n-4)-1} = 4r^{n-5}$, which is a contradiction to $mi(Q) \geq \overline{q}_k(n)$. So we assume that $\deg_Q x = 2$.

Subcase 1.1. $Q - N_Q[x] \neq F_1(n-3)$. By Lemma 2.2, Theorems 2.3 and 2.4

again, we have that

$$\begin{aligned}\overline{q}_k(n) &= r^{n-1} + r^{n-2k-1} \\ &\le mi(Q) \\ &\le mi(Q-x) + mi(Q - N_Q[x]) \\ &\le \begin{cases} 3r^{(n-1)-4} + 3r^{(n-3)-4} = 9r^{n-7} = \overline{q}_3(n), & \text{if } Q - x = F_2(n-3), \\ 5r^{(n-1)-6} + 3r^{(n-3)-4} = 8r^{n-7} < \overline{q}_k(n), & \text{if } Q - x \ne F_2(n-3). \end{cases}\end{aligned}$$

Furthermore, the equalities holding imply that $k = 3$, $G = C_3$ and $F = F_2(n-3)$, that is, $Q = \overline{Q}'(n) = C_3 \cup F_2(n-3)$.

Subcase 1.2. $Q - N_Q[x] = F_1(n-3)$. There are two possibilities for graph G. See Figure 4. By simple calculation, we have that $mi(G_1^*) = r^{s-1} + 1$ and

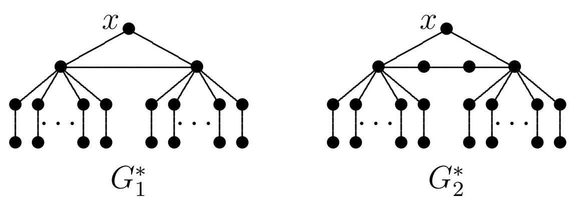

Figure 4: The graphs G_i^*, $1 \le i \le 2$

$mi(G_2^*) = 3r^{s-5} + 2$, hence, $mi(G_1^* \cup F) = mi(G_1^*) \cdot mi(F) = (r^{s-1}+1) \cdot r^{n-s} = r^{n-1} + r^{n-s}$ and $mi(G_2^* \cup F) = mi(G_2^*) \cdot mi(F) = (3r^{s-5}+2) \cdot r^{n-s} = 3r^{n-5} + 2r^{n-s}$. Note that $mi(G_2^* \cup F) = 3r^{n-5} + 2r^{n-s}$, which is a contradiction to $mi(Q) \ge \overline{q}_k(n)$.

Since $Q \ne \overline{Q}_i(n)$, $i = 1, 2, \ldots, k-1$, it follows that $G_1^* \ne Q_1(s)$, $s \le 2k-1$. Consider the graph $G_1^* \cup F$, by Lemma 2.1 and Theorem 2.6, we have that

$$\begin{aligned}\overline{q}_k(n) &= r^{n-1} + r^{n-2k-1} \\ &\le mi(Q) \\ &= mi(G_1^*) \cdot mi(F) \\ &\le \begin{cases} r^{s-1} \cdot r^{n-s}, & \text{if } s \le 2k-1, \\ (r^{s-1}+1) \cdot r^{n-s}, & \text{if } s \ge 2k+1, \end{cases} \\ &= \begin{cases} r^{n-1}, & \text{if } s \le 2k-1, \\ r^{n-1} + r^{n-s}, & \text{if } s \ge 2k+1, \end{cases} \\ &\le r^{n-1} + r^{n-2k-1} \\ &= \overline{q}_k(n).\end{aligned}$$

Furthermore, the equalities holding imply that $s = 2k+1$, $G_1^* = Q_1(2k+1)$ and $F = F_1(n-2k-1)$. Note that $Q_1(2k+1) = (W_1 * (k-1)P_2)$. In conclusion, $Q = \overline{Q}_k(n) = (W_1 * (k-1)P_2) \cup F_1(n-2k-1)$.

Case 2. $Q - x = F_1(n-1)$. Then there are one possibility for graph $Q = (W_t * (\frac{s-1}{2} - t)P_2) \cup F_1(n-s)$. Since Q is not a forest and $Q \neq \overline{Q}_i(n)$, $i = 1, 2 \ldots, k-1$, it follows that $s \geq 2k+1$. Hence we have that $\overline{q}_k(n) = r^{n-1} + r^{n-2k-1} \leq mi(Q) = (r^{s-1}+1) \cdot r^{n-s} \leq r^{n-1} + r^{n-2k-1} = \overline{q}_k(n)$ for $s \geq 2k+1$. Furthermore, the equalities holding imply that $s = 2k+1$. In conclusion, $Q = \overline{Q}_k(n) = (W_t * (k-t)P_2) \cup F_1(n-2k-1)$. □

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