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Deployment with Property Monodrome Group Topology

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Abstract

The purpose of this paper is to explain, how we extend a topological local group X to a group topology G.

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1 Introduction

Elie Cartan [1] showed that every local lie group contains a neighborhood of identity which is homeomorphic to a neighborhood of the identity of a lie group [1], [5, Theorem 84] and Pontryagin spotted that a local lie group is basis for a lie group [5].Olver showed that if a local lie group has the associative law then it embeds into a lie group [4].

Topological local groups (or local topological groups) are local lie groups without manifold property, which means that the group multiplication and inversion operations only being defined for elements sufficiently near the identity.

The question then arose as to whether every topological local group is contained in a group topology. In this paper we show that a topological local group can be extended to a group topology.

2 Preliminary Notes

Definition 2.1. *If X is a set, $D^{(n)} \subset X^n$ is subset of the cartesian product X^n of n copies of X and $f^{(n)} : D^{(n)} \to X$, $f^{(n)}(x_1, ..., x_n) = x_1...x_n$, then $f^{(n)}$ will be called an n-array local operation on X. Denote $f^{(2)}(x, y)$ by xy.*

Definition 2.2. *[6] Let X be a local group [8, 9, 10]. If there exist:*

a) a distinguished element $e \in X$, identity element,

b) a continuous product map $f^{(2)} : D^{(2)} \to X$ defined on an open subset

$$(e \times X) \cup (X \times e) \subset D^{(2)} \subset X \times X.$$

c) a continuous inversion map $\nu : X \to X$
satisfying the following properties:

(i) Identity: $f^{(2)}(e, x) = x = f^{(2)}(x, e)$ for every $x \in X$

(ii) Inverse: $f^{(2)}(\nu(x), x) = e = f^{(2)}(x, \nu(x))$ for every $x \in X$

(iii) Associativity: If (x, y), (y, z), $(f^{(2)}(x, y), z)$ and $(x, f^{(2)}(y, z))$ all belong to $D^{(2)}$, then

$$f^{(2)}(f^{(2)}(x, y), z) = f^{(2)}(x, f^{(2)}(y, z))$$

then X is called a *topological local group.*

Definition 2.3. *Let X be a topological local group, we call X an $n-$assocative if*

1. *local operation $f^{(n)}$ is defined for every $k < n$.*
2. *there exists $f^{(k)}(x_1, ..., x_k)f^{(l)}(x_{k+1}, ..., x_n)$ for every $k, l < n$ such that $k + l = n$ and*

$$f^{(k)}(x_1, ..., x_k)f^{(l)}(x_{k+1}, ..., x_n) = f^{(n)}(x_1, ..., x_n).$$

Definition 2.4. *We call X the global associative if:*

1. *conditions in Definition 2.3 for every n;*
2. *for all local operation $f_1^{(n)}$, $f_2^{(n)}$ and for every n-tuple $(x_1, x_2, ..., x_n) \in D^{(n)} \subset X^n$,*

$$f_1^{(n)}(x_1, ..., x_n) = f_2^{(n)}(x_1, ..., x_n).$$

Definition 2.5. *A $n-$array local opration in a local group X is called a word if it is an $n-$assocative.*

A continuous map of topological local groups $\phi : X \to X'$, will be called a *homomorphism of topological local group*, if $x, y \in X$ and $xy \in X$ then $\phi(x)\phi(y)$ exists in X' and $\phi(xy) = \phi(x)\phi(y)$.

With these morphisms the topological local groups form a category which contains the subcategory of topological groups.
A homomorphism $\phi : X \to X'$ will be called *strong* if, for every $x, y \in X$, the existence of $\phi(x)\phi(y)$ implies the existence of $xy \in X$. A morphism will be called a monomorphism (epimorphism) if, it is injective (surjective).
A subset H of a local group will be called *sublocal group* (*symmetric* subset) if it contains the identity and also if $x \in H$ then $x^{-1} \in H$.

Definition 2.6. *Let $(X, f^{(2)}, D^{(2)})$ be a local group and*

$$H = \{x \in X | f^{(2)}(x, y), f^{(2)}(y, x) \in D^{(2)}, \forall y \in X\}$$

$(\star)$ $N \leq H$, *and* $f^{(2)}(x, N) = f^{(2)}(N, x)$ *for every* $x \in X$.

Let $\frac{X}{N} = \{xN | x \in X\} = \{[x] | x \in X\}$. *(We show* $f^{(2)}(x, N)$ *by* xN*.)*
The cosets $\{xN : x \in X\}$ *form a local group called the quotient local group.*

Suppose $[x], [y] \in \frac{X}{N}$ and $[x] \cap [y] \neq \emptyset$, then $[x] = [y]$. We define $[x]^{-1} = [x^{-1}] = x^{-1}N$. ($x^{-1}$ which means x inverse by continuous inversion map $\nu : X \to X$.)

The action on the quotient local group topology is given by $f^{(2)}([x], [y]) = f^{(2)}(x, f^{(2)}(y, N))$ where $f^{(2)}(x, y) \in D^{(2)}$. This action is well define. Because, if $xN = x'N$ and $yN = y'N$ for $x, x', y, y' \in X$, we will have

$$(xy)^{-1}x'y' = y^{-1}x^{-1}x'y' \in y^{-1}Ny' = y^{-1}y'N \in N.$$

So $xyN = x'y'N$. It is easy to show that the quotient local group topology is a local group topology.

3 Deploying T.L.G to Group Topology

Enlargement of a local group X and monodrome were introduced algebraicly[7].

Suppose U_1 and U_2 are open neighborhoods of e in topological local group X and $f_1 : U_1 \to X'$ and $f_2 : U_2 \to X'$ are both morphisms. We say that f_1 and f_2 are *equivalent* if there exists an open neighborhood U_3 of e in X such that $U_3 \subseteq U_1 \cap U_2$ and $f_1|_{U_3} = f_2|_{U_3}$.

Definition 3.1. *We say that a topological local group X is enlargeable if there exists a group Topology G and a morphism $\phi : X \to G$ such that $\phi : X \to \phi(X)$ is a homeomorphism related to the equivalent class.*

If a topological local group is not globally associative then it may not be extended to a group topology or topological group.

Example 3.2. *Let* $X = \{1, a, b, c, d, ab, bc, cd, de, (ab)c, a(bc), (cd)e, c(de), b(cd)e, h = ((a(bc))d)e, k = a(b(cd)e) | (ab)c = a(bc), (cd)e = c(de)\}$, *with discrete topology. Now* X *is a topological local group, with the action*

$$1 * x = x * 1 = x,\ x * x^{-1} = 1 \text{ and } x = x^{-1} \text{ for every } x \in X,$$

but X *can not be a topological local subgroup of a group topology, since* $h \neq k$.

If a group topology G *is an enlargement of* X, *then* X *is homeomorphic to* $\phi(X)$ *(*ϕ *as in Definition 3.1). Since* $h, k \in X$, $h \neq k$ *then* $\phi(h) \neq \phi(k)$ *in* G, *that is* G *is not a group and this is a contradiction.[6]*

So a topological local group X may not be extended to a group topology, but there exists a topological sublocal group of X in which it is enlargeable to a group topology.

Example 3.3. *Let* $X' = \{1, a, b, ab\}$ *with the action as in the Example 3.2, such that* $a^{-1} = a, b^{-1} = b$, $((ab)^{-1} = b^{-1}a^{-1})$. *Then* X' *is a topological sublocal group of* X, *which embeds in* S_3 *(permutation group of order 3) with discrete topology:*

$$1 \mapsto I,\ a \mapsto (231),\ b \mapsto (321).$$

We see that X' *is a non-abelian topological local group and is enlargeable to a group topology. So, there exists a topological sublocal group* X' *of a topological local group* X *which is enlargeable but* X *is not enlargeable to a group topology. [6]*

Definition 3.4. *Let* X *be a topological local group and* G *a group topology and* $X \subset G$. *Then* G *is called an* X*-monodrome if*

(1) X *generates* G *topologically (i.e:* X *is the smallest closed sublocal group in* G *which generate* G*)*

(2) For a group topology H *and every continuous homomorphism* $\psi : X \to H$ *there exists a continuous homomorphism* $\nu : G \to H$ *such that the following diagram commutes.*

$$\begin{tikzcd} X \arrow[d] \arrow[dr, "\psi"] & \\ G \arrow[r, "\nu"'] & H \end{tikzcd} \tag{1}$$

Remark 3.5. *Let G be an enlargement of X. Note that $\phi : X \to \phi(X)$ should be a strong homeomorphism of topological local groups.*

We consider $X = \{-1, 0, 1\}$ with the following actions:

$$1 * -1 = -1 * 1 = 0 \quad \textit{and} \quad x * 0 = 0 * x = x \quad \forall x \in X$$

Then X is a local group. It can not be extended to $\mathbb{Z}_3$. For if, $\phi : X \to Z_3$ is the identity map, we have $1 * 1 = 2$ in $\mathbb{Z}_3$. On the other hand, we know that $1 * 1 \notin X$ and $2 = 1 * 1 = \phi(1) * \phi(1)$ which is a contradictions. But, X can be extend to $\mathbb{Z}_5$ and $\mathbb{Z}$ such that $\mathbb{Z}$ is an X-monoderome.

Lemma 3.6. *(Uniqueness) Let G be a group topology which is an X-monodrome with embedding ϕ. Suppose H is a group topology, $\psi : X \to H$ a continuous homomorphism, H is generated by $\psi(X)$ and $\nu' : H \to G$ is a continuous homomorphism. If the following diagram commutes*

$$\begin{tikzcd} & X \arrow[dl, "\psi"'] \arrow[d] \\ H \arrow[r, "\nu'"] & G \end{tikzcd} \tag{2}$$

then ν' is a homeomorphism and H is an X-monodrome with embedding ψ.

Proof: Combining the diagrams (2.1) and (2.2), we obtain

$$\begin{tikzcd} & X \arrow[dl, "\psi"'] \arrow[d, "\phi"] \arrow[dr, "\psi"] & \\ H \arrow[r, "\nu'"] & G \arrow[r, "\nu"] & H \end{tikzcd}$$

Since $\nu(\nu'(\psi(x))) = \psi(x)$ for every $x \in X$ and H is topologically generated by $\psi(X)$, then $\nu \circ \nu' = Id_H$

$$H \xrightarrow{\nu'} G \xrightarrow{\nu} H$$

In fact, $\nu' : H \to G$ is surjective, because $\nu' \circ \psi(X) = \phi(X)$ generates G. Hence, ν' is a homeomorphism with the inverse ν.

In next theorem show that a topological local group with global associative law extend to a group topology.

Definition 3.7. *Let F be a free group on a local group X. Then, $u \in F$ is called an e-element if $u = x_1^{\epsilon_1} x_2^{\epsilon_2} ... x_n^{\epsilon_n}$, where $\epsilon_i \in \{1, -1\}$ and $x_1, x_2, ..., x_n \in X$ such that $f^{(n)}(x_1^{\epsilon_1}, x_2^{\epsilon_2}, ..., x_n^{\epsilon_n})$ is $n-$associative and is be equaled to the identity e in X.*

A topological local group X can be embedded in the factor group topology F over a normal subgroup N which contains the e-elements.

Theorem 3.8. *(main theorem) Let X be a topological local group with the global associative property, and F the free topological group on X.*

Let $N \subset F$ be the set of all e-elements of F. Then N is a normal subgroup of F and if $\phi : F \to \frac{F}{N}$ denotes the natural homomorphism, then the restriction of ϕ to X is injective and $\frac{F}{N}$ is a monodrome for X with embedding $\phi : X \to \frac{F}{N}$ group topology.

Proof: We can easily obtain that N is normal and that $\phi : X \to \frac{F}{N}$ is a homomorphism.
Let $\psi : X \to H$ be a homomorphism into a group topology H that is generated with X . We will prove that there exists a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\psi"] \arrow[d, "\phi"'] & H \\ \frac{F}{N} \arrow[ur, "\nu"'] & \end{tikzcd} \tag{3}$$

Indeed, since F is freely generated by X, the mapping $\psi : X \to H$ can be extended to a homomorphism $\gamma : F \to H$. This gives a commutative diagram $\gamma\iota = \psi$.

$$\begin{tikzcd} X \arrow[d, "\iota"'] \arrow[dr, "\psi"] & \\ F \arrow[r, "\gamma"'] & H \end{tikzcd} \tag{4}$$

where ι is the inclusion map. Since $\psi : X \to H$ is a homomorphism, we have $\gamma(u) = e$ for every e-element $u \in F$. Thus $\gamma(N) = e$. It follows that $\gamma : F \to H$ factors through $\phi : F \to \frac{F}{N}$

$$\begin{tikzcd} F \arrow[r, "\gamma"] \arrow[d, "\phi"'] & H \\ \frac{F}{N} \arrow[ur, "\nu"'] & \end{tikzcd} \tag{5}$$

Hence $\frac{F}{N}$ is a monodrome. We have $\psi : X \to H$ is the embedding map then ϕ is an embedding.

Conclusion:

We define Topological local groups without locally Euclidean property [6]. In our topological local groups, if the products $x.y$; $y.z$; $x.(y.z)$; and $(x.y)z$ are all defined, then $x.(y.z) = (x.y).z$. This condition is called "local associativity" by Olver in [4]. A much stronger condition for a local group to satisfy is

"global associativity" in which, given any finite sequence of elements from the topological local group and two different ways of introducing parentheses in the sequence, if both products thus formed exist, then these two products are in fact equal.

We extended a topological local group to a group topology by methods from free topological groups.

This question can be raised whether a topological local group can embed to a topological group?

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