Bi - Level Multiobjective Geometric Programming

(Bl - MOGP) Problem under Fuzziness

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Abstract

Geometric programming (GP) problem is a powerful tool for solving some special type nonlinear programming problems. In this paper, we have developed a method to solve bi – level multiobjective geometric programming (BL-MOGP) problem under fuzziness. We shall describe the fuzzy optimization approach through geometric programming technique in order to solve (BL-MOGP) problem. Also the concepts of tolerance membership function and multiobjective optimization at each level are used to find a compromise solution. The solution procedure of the fuzzy approach is illustrated by a numerical example.

Keywords: Bi-Level programming; Multiobjective decision making; Fuzzy decision approach; Pareto optimal solution; Stackelberg game; geometric programming

1. Introduction

A multi – level programming (MLP) problem is identified as kind of mathematical programming that solves decentralized planning problems with decision makers in multi – level systems. Each decision maker independently maximizes (or minimizes) their own objective but is affected by the actions of other decision makers at the different levels.
Bi–level programming (BLP) problem is closely related to the economics problem addressed by Stackelberg (1952) through its development of strategic game theory. The original formulation for bi–level programming appeared in (1973), in a paper authored by Bracken and McGill (1973), although it was Candler and Norton (1977) who first used the designation “bi–Level” programming. However, it was not until the early 1980s that these problems started to receive the attention they deserved. [4, 9, 10, 14, 15]. Bi-level programming (BLP) problem is a special case of (MLP) in which, there are two independent decision makers. The higher level decision making (HLDM) makes its decision in full view of the lower level decision making (LLDM). Each decision makers (DMs) attempt to optimize its objective function and is affected by the actions of the other DM. several (BLP) and their solution methods have been presented, such as [6, 11, 12, 13, 19]. Geometric programming (GP) problem is an effective method to solve a nonlinear programming problem [2].

Since late 1960’s, (GP) problem has been known and used in various fields (like Operations Research (OR), Engineering, Sciences,…). There are many references on applications and methods of (GP) [1, 8, 16, 17, 18].

Today, most of the real–world decision making problems in economic, environmental, social, and technical areas are multidimensional and multi objectives since multi objective optimization problems different from single objective optimization problem. It is significant to realize that multiple objectives are often non–commensurable and in conflict with each other in optimization problems [3, 16].

Zadeh [7] first gave the concept of fuzzy set theory. Latter on, Bellman and Zadeh used the fuzzy set theory to the decision – making problem. Tanaka introduced the objective as fuzzy goal over the $\alpha$ - cut of a fuzzy constraint set and Zimmermann give the concept to solve multiobjective linear programming problem. Fuzzy mathematical programming has been applied to several fields.

Geometric programming is a special method used to solve a class of nonlinear programming problems. In recent years there has been an increase in research on multiobjective optimization methods. Decisions with multiobjectives are quite successful in government, military and other organizations. Researchers from a wide variety of disciplines such as mathematics, management science, economics, engineering and others have contributed to the solution methods of multiobjective optimization problems. The situation is formulated as a multiobjective optimization problem in which the goal is to minimize (or maximize) not a single objective function but several objective functions simultaneously. The purpose of multiobjective problems in the mathematical programming framework is to optimize the different objective problems.

Now, we have developed a method to solve a bi–level multiobjective geometric programming (BL-MOGP) problem based on fuzzy programming approach. Bi–level geometric programming problem can be thought as a static version of a Stackelberg Leader – follower game in which a Stackelberg strategy is used by the leader (or the higher – level decision maker HLDM), given the rational reaction of the follower (or the lower – Level decision maker LLDM).
We have proposed a two-planner bi-level multiobjective decision making model and the solution method by using the concept of condensation technique. This method uses the concepts of tolerance membership function and multiobjective optimization at each level to maximizing decision to find the optimal solution by using max–addition operator and concept of geometric programming (GP) technique.

The HLDM specifies his/her objective function and decisions with possible tolerances which are described by membership functions of fuzzy set theory. Then, the LLDM uses this preference information for HLDM and solves his/her problem subject to HLDMs restrictions.

The solution procedure of the fuzzy approach is illustrated by a numerical example.

2. Problem Formulation

A Bi-Level multiobjective geometric programming (BL – MOGP) problem can be stated as:

\[
\text{Min} F_o (x) = \text{Min} \left( f_{o1}(x), f_{o2}(x), \ldots, f_{oN_1}(x) \right)
\]

where \( x_2 \) solves

\[
\text{Min} F_1 (x) = \text{Min} \left( f_{11}(x), f_{12}(x), \ldots, f_{1N_2}(x) \right)
\]

s.t.

\[
g_j (x) \leq 1, \quad j = 1, \ldots, m
\]

\[x = (x_1, x_2) > 0, \quad x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2}, x \in \mathbb{R}^{n_1+n_2}\]

where

\[
f_{ok}(x) = \sum_{i=1}^{T_{ok}} c_{ok} \prod_{r=1}^{n} x_{r}^{a_{okir}}, \quad k = 1, \ldots, N_1, n = n_1 + n_2
\]

\[
f_{1q}(x) = \sum_{i=1}^{T_{1q}} c_{1qi} \prod_{r=1}^{n} x_{r}^{a_{1qir}}, \quad q = 1, \ldots, N_2
\]

\[
g_j (x) = \sum_{i=1}^{T_j} c_{ji} \prod_{r=1}^{n} x_{r}^{d_{jir}} \leq 1, \quad j = 1, \ldots, m
\]

c_{ok}, c_{1qi} and c_{ji} for all \( k, q, i \) and \( j \) are positive real numbers.

The decision mechanism of (BL – MOGPP) is that the HLDM and LLDM adopt the Leader – Followers Stackelberg game respectively. The definitions of the solutions of the (BL – MOGPP) can be defined as:
Definition 1.
For any \( x_i \in G_o = \{ x_i \mid (x_i, x_j) \in G \} \) given by HLDM, if the decision making variable \( x_2 \in G = \{ x_2 \mid (x_i, x_j) \in G \} \) at the lower – level is the pareto optimal solution of LLDM, then \( (x_1, x_2) \) is a feasible solution of (BL-MOGPP), where \( G = \{(x_1, x_2) \in R^n \mid g_j(x) \leq 1, j = 1, \ldots, m \} \) (8)

Definition 2.
\( (x_1^*, x_2^*) \) is a feasible solution of the (BL-MOGP) problem, no other solution \( (x_1, x_2) \in G \) exists, such that \( f_{ok}(x_1^*, x_2^*) \leq f_{ok}(x_1, x_2) \); at least one \( k \), \( k = 1, \ldots, N_l \) is strict inequality, then \( (x_1^*, x_2^*) \) is the pareto optimal solution of (BL-MOGP).

3. Bi-Level Multiobjective Geometric Programming Problem

To solve the (BL – MOGP) problem, first gets the satisfactory solution that acceptable to HLDM, and then give the HLDM decision variables and goals with some leeway to the LLDM for him/ her to seek the satisfactory solution and to arrive at the solution which is closest to the satisfactory solution of the HLDM. This due to, the LLDM should not only optimize his/her objective functions but also try to satisfy the HLDM’s goals and preference as much as possible. In this way, the solution method simplifies a (BL-MOGP) problem by transforming it into separate (MOGP) problem at higher – and lower – level, by that means the difficulty associated with non – convexity to arrive at an optimal solution is avoided.

3.1 Higher – Level Decision Maker (HLDM) Problem
First, the HLDM solves the following (MOGP)_o problem:
\[
\begin{align*}
\min_{x_1} F_o(x) &= \min_{x_1} \left( f_{o1}(x), f_{o2}(x), \ldots, f_{oN_l}(x) \right) \\
\text{s.t. } x &\in G
\end{align*}
\] (9)

To solve this problem, we use the Zimmermann’s solution procedure. This procedure consists of the following steps:

Step 1: Solve problem (9) as a single objective GP problem using only one objective at a time and ignoring the others. These solutions are known as ideal solutions.
Step 2: From the results of step1, determine the corresponding values for every objective at each solution derived. With the values of all objectives at each ideal solution, pay – off matrix can be formulated as follows:
Bi-level multiobjective geometric programming (BL-MOGP) problem

\[
\begin{align*}
& f_{o1}(x) \quad f_{o2}(x) \quad \ldots \quad f_{oN_1}(x) \\
& x \begin{bmatrix}
  f_{o1}^*(x_1^1) & f_{o2}^*(x_1^1) & f_{oN_1}^*(x_1^1) \\
  f_{o1}^*(x_2^2) & f_{o2}^*(x_2^2) & f_{oN_1}^*(x_2^2) \\
  \vdots & \vdots & \vdots \\
  f_{o1}^*(x_n^k) & f_{o2}^*(x_n^k) & f_{oN_1}^*(x_n^k)
\end{bmatrix}
\end{align*}
\]

where \( x_1^1, x_2^2, \ldots, x_n^k \) are the ideal solutions of the objectives \( f_{o1}(x), f_{o2}(x), \ldots, f_{oN_1}(x) \) respectively, \( U_{or} = \max \{ f_{or}(x_1), f_{or}(x_2), \ldots, f_{or}(x_n) \} \), and \( L_{or} = f_{or}^*(x_r) \) for \( r = 1, 2, \ldots, N_1 \), \( L_{or} \) and \( U_{or} \) be lower and upper bounds of the \( r \)th objective function \( f_{or}(x) \) for \( r = 1, \ldots, N_1 \).

**Step 3:** Using a aspiration levels of each objective of the (MOGP)\(_o\) problem (9) may be written as follows:

Find \( x \) that satisfy

\[
\begin{align*}
& f_{or}(x) \leq L_{or} , \quad (r = 1, \ldots, N_1) \\
& \text{s.t.} \\
& g_j(x) = \sum_{i=1}^{T_j} C_{ji} \prod_{r=1}^{n} x_d^{jir} \leq 1 , \quad j = 1, \ldots, m \\
& x > 0
\end{align*}
\]

where the corresponding membership function is:

\[
\mu_{or} (f_{or}(x)) = \begin{cases} 
0 & \text{or } 0 \quad \text{if } f_{or}(x) \geq U_{or} \\
\mu_{or}(x) & \text{if } L_{or} \leq f_{or}(x) \leq U_{or} \\
1 & \text{or } 1 \quad \text{if } f_{or}(x) \leq L_{or} , \quad r = 1, \ldots, N_1
\end{cases}
\]

and \( \mu_{or}(x) \) is a strictly monotonic decreasing function with respect to \( f_{or}(x) \).

**Step 4:** Formulate the fuzzy multiobjective decision making problem as:

\[
\begin{align*}
& \text{Max } \mu_{oD^+}(x) \\
& \text{s.t.} \\
& g_j(x) = \sum_{i=1}^{T_j} C_{ji} \prod_{r=1}^{n} x_d^{jir} \leq 1 , \quad j = 1, \ldots, m \\
& x > 0
\end{align*}
\]

According to max – addition operator
Max $\mu_{oD^r}(x^*) = \operatorname{Max} \left( \sum_{j=1}^{N_l} \lambda_{oj} \mu_{oj} \left(f_{oj}(x)\right) \right)$  
\hspace{1cm} \text{(13)}

s.t.

$$
\mu_{or}(f_{or}(x)) = \frac{U_{or} - f_{or}(x)}{U_{or} - L_{or}}, \hspace{1cm} r = 1, \ldots, n_1
$$

$$
g_j(x) = \sum_{i=1}^{T_i} C_{ji} \prod_{r=1}^{n} x_{r^{d_{jir}}} \leq 1, \hspace{1cm} j = 1, \ldots, m
$$

$$
x > 0
$$

$$
0 \leq \mu_{or}(f_{or}(x)) \leq 1
$$

The above problem (13) reduces to:

$$
\operatorname{Max} V(x) = \sum_{j=1}^{N_l} \lambda_{oj} \frac{U_{oj} - f_{oj}(x)}{U_{oj} - L_{oj}}
$$

or

$$
\operatorname{Max} V(x) = \sum_{j=1}^{N_l} \lambda_{oj} \frac{U_{oj} - \sum_{i=1}^{T_{ik}} C_{oki} \prod_{r=1}^{n} x_{r^{a_{okir}}}}{U_{oj} - L_{oj}}
$$
\hspace{1cm} \text{(14)}

s.t.

$$
g_j(x) = \sum_{i=1}^{T_i} C_{ji} \prod_{r=1}^{n} x_{r^{d_{jir}}} \leq 1, \hspace{1cm} j = 1, \ldots, m
$$

$$
x > 0
$$

**Step 5:** Find the optimal decision variable $x^*$ with optimal objective value

$V^*(x^*)$ can be obtained by

$$
V^*(x^*) = \sum_{j=1}^{N_l} \lambda_{oj} \frac{U_{oj}}{U_{oj} - L_{oj}} - U^*(x),
$$

where $x^*$ is optimal decision variable of the unconstrained geometric programming problem (for given $\lambda_{oj}, j = 1, \ldots, N_l$)

$$
\operatorname{Min} U(x) = \sum_{j=1}^{N_l} \lambda_{oj} \frac{U_{oj}}{U_{oj} - L_{oj}} \sum_{i=1}^{T_{ik}} C_{oki} \prod_{r=1}^{n} x_{r^{a_{okir}}}
$$

s.t.

$$
g_j(x) = \sum_{i=1}^{T_i} C_{ji} \prod_{r=1}^{n} x_{r^{d_{jir}}} \leq 1, \hspace{1cm} j = 1, \ldots, m
$$

$$
x > 0,
$$

whose solution is assumed to be $[x_1^H, x_2^H, f_{ok}^H, k = 1, \ldots, N_l]$
3.2 Lower – Level Decision Maker (LLDM) Problem

In the same way, the LLDM independently solves the following problem (MOGP):  

$$\text{Min}_{x_1} F_i(x) = \text{Min}_x \left( f_{11}(x), f_{12}(x), \ldots, f_{1N_2}(x) \right)$$  

s.t  

$$x \in G.$$  

whose solution is assumed to be  

$$[x_1^L, x_2^L, f_{1q}^L, q = 1, \ldots, N_2]$$

3.3. Bi – Level Multiobjective Geometric Programming Problem

Now the following steps are used for solving bi – level multiobjective GP problem with a tolerance and linear membership function by using GP technique to find an optimal compromise solution.

**Step 1:** Find the solutions of HLDM and LLDM which are usually different because of conflicts between two level objective functions.

The HLDM knows that using the optimal decision $x_1^H$ as a control factor for LLDM is not practical. It is more reasonable to have some tolerance that gives the LLDM an extent feasible region to search for his/her optimal solution, and also reduce searching time or interactions. In this way, the range of the decision $x_1$ should be around $x_1^H$ with its maximum tolerances $t_1$ and the following membership function can be stated as:

$$\mu_{i1}(x) = \begin{cases} 
\frac{\left[x_1^\prime - (x_i^{\prime\prime} - t_1)\right]}{t_1} & \text{if } x_i^{\prime\prime} - t_1 \leq x_1^\prime \leq x_i^{\prime\prime} \\
\frac{\left[(x_i^{\prime\prime} + t_1) - x_1\right]}{t_1} & \text{if } x_i^{\prime\prime} - x_1 \leq x_i^{\prime\prime} + t_1 \\
0 & \text{otherwise}
\end{cases}$$  

(16)

where $x_1^H$ is the most preferred solution; the $(x_1^H - t_1)$ and $(x_1^H + t_1)$ is the worst acceptable decision; and that satisfaction is linearly increasing within the interval of $[x_1^H - t_1, x_1^H]$, and other decision are not acceptable [1].
Step 2: Find the membership functions of the HLDM as:

\[
\mu_{f_{ok}}^f \left[ f_{ok} (x) \right] = \begin{cases} 
1 & \text{if } f_{ok} (x) \geq f_{ok}^H , \\
\frac{f_{ok} (x) - f_{ok}'}{f_{ok}^H - f_{ok}'} & \text{if } f_{ok}' \leq f_{ok} (x) < f_{ok}^H \\
0 & \text{if } f_{ok} (x) \leq f_{ok}', k = 1, \ldots, N_1
\end{cases}
\] (17)

The HLDM’s goals may reasonably consider the all \( f_{ok} \geq f_{ok}^H , k = 1, \ldots, N_1 \) are absolutely acceptable and all \( f_{ok} < \left[ f_{ok}' = f_{ok} \left[ x_1^L , x_2^L \right] \right], k = 1, \ldots, N_1 \) are absolutely unacceptable, and that the preference within \( f_{ok}' , f_{ok}^H , k = 1, \ldots, N_1 \) is linearly increasing.

Step 3: Form the membership functions of the LLDM as:

\[
\mu_{f_{lq}}^f \left[ f_{lq} (x) \right] = \begin{cases} 
1 & \text{if } f_{lq} (x) \geq f_{lq}^L \\
\frac{f_{lq} (x) - f_{lq}'}{f_{lq}^L - f_{lq}'} & \text{if } f_{lq}' \leq f_{lq} (x) < f_{lq}^L \\
0 & \text{if } f_{lq} (x) \leq f_{lq}', q = 1, \ldots, N_2
\end{cases}
\] (18)

where \( f_{lq}' = f_{lq} \left[ x_1^H , x_2^H \right] \).

Step 4: Solve the following Tchebycheff problem to generate the satisfactory solution as follows:

\[
\begin{align*}
\text{Max} & \quad \delta \\
\text{s.t.} & \quad x \in G, \\
& \quad \left[ \left( x_1^H + t_1 \right) - x_1 \right] / t_1 \geq \delta \cdot \mathbf{1} , \\
& \quad \left[ x_1 - \left( x_1^H - t_1 \right) \right] / t_1 \geq \delta \cdot \mathbf{1} , \\
& \quad \mu_{f_{ok}}^f \left[ f_{ok} (x) \right] = \left[ f_{ok} (x) - f_{ok}' \right] / \left[ f_{ok}^H - f_{ok}' \right] \geq \delta , \quad k = 1, \ldots, N_1 , \\
& \quad \mu_{f_{lq}}^f \left[ f_{lq} (x) \right] = \left[ f_{lq} (x) - f_{lq}' \right] / \left[ f_{lq}^L - f_{lq}' \right] \geq \delta , \\
& \quad x = (x_1, x_2) > 0, \delta \in [0,1] , \quad q = 1, \ldots, N_2
\end{align*}
\] (19)

where \( \delta \) is the overall satisfaction, and \( \mathbf{1} \) is a column vector with all elements equal to 1 and the same dimension as \( x_1 \).
**Step 5:** If the HLDM is satisfied with this solution, then a satisfactory solution is reached. Otherwise, he/she should provide new membership functions for the control variable and objectives to LLDM until a satisfactory solution is reached. In general, combined with set of control decision and objectives with tolerance, the solution of problem (19) become a Pareto optimal (satisfactory) solution for HLDM problem (1) and LLDM problem (2).

To demonstrate the solution method for (BL – MOGP). Problem Let us Consider the following example.

### 4. Numerical example

Consider the following bi-level multiobjective geometric programming (BL- MOGP) problem as:

\[ \text{Min } F_i (x_i, x_2) = \text{Min } \{ Z_i(x) = x_i^{-1} x_2^{-1} , Z_2(x) = 2x_1^2 x_2^{-1}\} \quad (I) \]

where \( x_2 \) solves

\[ \text{Min } F_i (x_i, x_2) = \text{Min } \{ Z_i = 2x_1^2 x_2^{-1} , Z_2(x) = x_i^{-1} x_2^{-1}\} \]

s.t.
- \( x + x_2 \leq 1 \)
- \( 0.2 \leq x_1 \leq 2 \)
- \( 0.3 \leq x_2 \leq 4 \)
- \( x_1, x_2 > 0 \)

In order to solve this problem, we shall first solve the (HLDM) problem by dividing it into two sub-problems as:

**sub-problem 1:**
\[ \text{Min } Z_i(x) = x_i^{-1} x_2^{-2} \]

s.t.
- \( x + x_2 \leq 1 \)
- \( 0.2 \leq x_1 \leq 2 \)
- \( 0.3 \leq x_2 \leq 4 \)
- \( x_1, x_2 > 0 \)

**sub-problem 2:**
\[ \text{Min } Z_2(x) = 2x_1^2 x_2^{-3} \]

s.t.
- \( x + x_2 \leq 1 \)
- \( 0.2 \leq x_1 \leq 2 \)
- \( 0.3 \leq x_2 \leq 4 \)
- \( x_1, x_2 > 0 \)
Solving the above sub-problems by condensation technique we have:

For (sub-problem 1) \[ x_1 = \left( \frac{1}{3}, \frac{2}{3} \right), \quad Z_1(x) = 6.75 \]

For (sub-problem 2) \[ x_2 = \left( \frac{2}{5}, \frac{3}{5} \right), \quad Z_2(x) = 57.8703 \]

Now the pay-off matrix is given by:

\[
\begin{bmatrix}
Z_1 & Z_2 \\
x_1 & \begin{bmatrix} 6.75 & 60.75 \end{bmatrix} \\
x_2 & \begin{bmatrix} 6.94 & 57.87 \end{bmatrix}
\end{bmatrix}
\]

From the pay-off matrix the lower and upper bound of \( Z_1(x) \) and \( Z_2(x) \) be:

6.75 \( \leq \) \( Z_1(x) \leq 6.94 \) and 57.87 \( \leq \) \( Z_2(x) \leq 60.75 \)

Let \( \mu_1(x) \), \( \mu_2(x) \) be the fuzzy membership function of the objective function \( Z_1(x) \) and \( Z_2(x) \) respectively and they are defined as:

\[
\mu_1(x) = \begin{cases} 
1 & \text{if } Z_1(x) \leq 6.75 \\
\frac{6.94 - Z_1(x)}{0.19} & \text{if } 6.75 \leq Z_1(x) \leq 6.94 \\
0 & \text{if } Z_1(x) \geq 6.94 
\end{cases}
\]

\[
\mu_2(x) = \begin{cases} 
1 & \text{if } Z_2(x) \leq 57.87 \\
\frac{60.75 - Z_2(x)}{2.88} & \text{if } 57.87 \leq Z_2(x) \leq 60.75 \\
0 & \text{if } Z_2(x) \geq 60.75 
\end{cases}
\]

According to max-addition operator, the 1st level reduces to:

\[
\text{Max} \left[ 57.64 - \left( \frac{Z_1(x)}{0.19} + \frac{Z_2(x)}{2088} \right) \right] \quad \text{(II)}
\]

s.t.
\[
x_1 + x_2 \leq 1, \\
0.2 \leq x_1 \leq 2, \quad 0.3 \leq x_2 \leq 4 \\
x_1, x_2 > 0.
\]

To maximize the problem (II), we minimize

\[
\frac{Z_1(x)}{0.19} + \frac{Z_2(x)}{2088} \quad \text{s.t.} \quad x_1 + x_2 \leq 1, \quad 0.2 \leq x_1 \leq 2, \quad 0.3 \leq x_2 \leq 4, \quad x_1, x_2 > 0.
\]

So, the new problem takes the form:

\[
\text{Min} \quad g(x) = 5.269 \quad x_1^{-1} x_2^{-2} + 0.699 \quad x_1^{-2} x_2^{-3}
\]

s.t.
\[
x_1 + x_2 \leq 1, \\
0.2 \leq x_1 \leq 2, \quad 0.3 \leq x_2 \leq 4 \\
x_1, x_2 > 0
\]

\[
\text{(III)}
\]
Solving problem (III) by condensation technique, we get the value of the objective function of problem (III) is $g(x^*) = 56.8339$, and the values of decisions variables are $x_1^* = 0.36577$, $x_2^* = 0.63422$

Thus, the values of the objective functions of the HLDM are $Z_1(x^*) = 6.796$ and $Z_2(x^*) = 58.599$.

Second, solves the (LLDM) problem by dividing it into two sub – problems as:

(Sub – problem 3)

$$\begin{align*}
\text{Min } Z_3(x) &= 2x_1^{-2}x_2^{-1} \\
\text{s.t.} & \quad x_1 + x_2 \leq 1, \\
& \quad 0.2 \leq x_1 \leq 2, \quad 0.3 \leq x_2 \leq 4, \\
& \quad x_1, x_2 > 0.
\end{align*}$$

and (sub – problem 4)

$$\begin{align*}
\text{Min } Z_4(x) &= 2x_1^{-3}x_2^{-2} \\
\text{s.t.} & \quad x_1 + x_2 \leq 1, \\
& \quad 0.2 \leq x_1 \leq 2, \quad 0.3 \leq x_2 \leq 4, \\
& \quad x_1, x_2 > 0.
\end{align*}$$

Solving the above sub – problems by condensation technique we have:

For (sub – problem 3):

$$x_3 = \left(\frac{2}{3}, \frac{1}{3}\right), \quad Z_3(x) = 13.5$$

For (sub – problem 4):

$$x_4 = \left(\frac{3}{5}, \frac{2}{5}\right), \quad Z_4(x) = 28.935$$

Now the pay – off matrix, for the lower and upper bound of $Z_3(x)$ and $Z_4(x)$ be:

13.5 $\leq Z_3(x) \leq$ 13.89 and 28.935 $\leq Z_4(x) \leq$ 30.375

Let $\mu_3(x)$, $\mu_4(x)$ be the fuzzy membership function of the objective function $Z_3(x)$ and $Z_4(x)$ respectively, and they are defined as:

$$\mu_3(x) = \begin{cases} 
1 & \text{if } Z_3(x) \leq 13.5 \\
13.89 - Z_3(x) & \text{if } 13.5 \leq Z_3(x) \leq 13.89 \\
0.39 & \text{if } Z_3(x) \geq 13.89
\end{cases}$$

$$\mu_4(x) = \begin{cases} 
1 & \text{if } Z_4(x) \leq 28.935 \\
30.375 - Z_4(x) & \text{if } 28.935 \leq Z_4(x) \leq 30.375 \\
1.440 & \text{if } Z_4(x) \geq 30.375
\end{cases}$$
According to max – addition operator, the 2nd level reduces to:

\[ \text{Max} \left[ 56.709 - \left( \frac{z_1(x)}{0.39} + \frac{z_2(x)}{1.44} \right) \right] \]

s.t.

\[
\begin{align*}
x_1 + x_2 & \leq 1, \\
0.2 & \leq x_1 \leq 2, \quad 0.3 & \leq x_2 \leq 4, \\
x_1, x_2 & > 0.
\end{align*}
\] (IV)

To maximize the problem (IV), we minimize

\[
\frac{z_1(x)}{0.39} + \frac{z_2(x)}{1.44}
\]

s.t. \( x_1 + x_2 \leq 1, \quad 0.2 \leq x_1 \leq 2, \quad 0.3 \leq x_2 \leq 4, \quad x_1, x_2 > 0. \)

So, the new problem is:

\[
\text{Min} \ g'(x) = 5.128 \left( x_1^{-2} x_2^{-1} \right) + 0.694 \left( x_1^{-3} x_2^{-2} \right)
\]

s.t.

\[
\begin{align*}
x_1 + x_2 & \leq 1, \\
0.2 & \leq x_1 \leq 2, \quad 0.3 & \leq x_2 \leq 4, \\
x_1, x_2 & > 0.
\end{align*}
\] (V)

Solving problem (V), by condensation technique, we get the value of the objective function of problem (5), is \( g'(x^*) = 40.144. \) and the values of decision variables are \( x_1^* = 0.634 \), \( x_2^* = 0.464 \)

Thus the values of the objective functions of LLDM are \( Z_3(x) = 10.723 \) and \( Z_4(x) = 18.226 \)

Third assume the HLDM’s control decision \( x_1^H \) is 0.36577 with the tolerance \( t = 2 \) and, consider the membership functions \( \mu_{x_1}(.) \), \( \mu_{f_{ok}}(.) \), \( \mu_{f_{iq}}(.) \) as defined in equations (16), (17) and (18).

Then, the LLDM solves the following Tchebycheff problem to generate the satisfactory solution as:

\[
\text{Max} \ \delta
\]

s.t.

\[
\begin{align*}
x_1 + x_2 & \leq 1, \\
x_1 + 2\delta & \geq 2.366, \\
2\delta - x_1 & \leq 1.634, \\
x_1^{-1}x_2^{-2} + 0.53\delta & \geq 7.326, \\
x_1^{-2}x_2^{-3} - 13.511\delta & \geq 15.783, \\
x_1^{-2}x_2^{-1} + 6.413\delta & \geq 11.775, \\
x_1^{-3}x_2^{-2} + 32.517\delta & \geq 50.743, \\
0 & \leq \delta \leq 1 \\
0.2 & \leq x_1 \leq 2, \\
0.3 & \leq x_2 \leq 4
\end{align*}
\]
The above problem can be solved by using the condensation technique, and then the solution is given by \((x_1, x_2) = (0.6332, 0.3)\) and \(\delta = 1\).

5. Conclusion

Here, we have discussed a bi-level multi-objective geometric programming problem based on fuzzy programming technique. The solution method uses the concept of tolerance membership function and multiobjective optimization at each level. Based on [5], the proposed solution method proceeds from higher level to lower level in a natural and straightforward manner. The HLDM specifies his/her objectives and decisions with possible leeway, which are described by membership function of fuzzy set theory. This information then contains the LLDM's feasible space. Finally, an illustrative numerical example has been given to clarify the proposed solution method.

References


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