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On Almost Sure Limiting Behavior of AQSI Sequence[1]

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Abstract

In this paper we establish the Kolmogorov type inequality and an generalized three series theorem for AQSI sequence of random variables. We obtain the strong convergence and a Chung's type strong law of large numbers for sequence of AQSI.

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Keywords: AQSI random sequences; Kolmogorov type inequality; generalized three series theorem

1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $(X_n)_{n\in N}$ be a sequence of random variables(r.v.'s) defined on $(\Omega, \mathcal{F}, P)$

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Chandra and *Ghosal* [3] introduced the notion of asymptotically quadrant sub-independent (AQSI).

Definition 1. A sequence $(X_n)_{n\in N}$ of r.v.'s is said to be asymptotically quadrant sub-independent (AQSI) if there exists a nonnegative sequence $(q_n)_{n\in N}$ such that $q(n) \to 0$, as $n \to \infty$ and for $\forall\ i \neq j$

$$P(X_i > s, X_j > t) - P(X_i > s)P(X_j > t) \leq q(|i-j|)\alpha_{ij}(s,t),\ \ s,t > 0 \quad (1.1)$$

$$P(X_i < s, X_j < t) - P(X_i < s)P(X_j < t) \leq q(|i-j|)\beta_{ij}(s,t),\ \ s,t < 0 \quad (1.2)$$

where $\alpha_{ij}(s,t) \geq 0$ and $\beta_{ij}(s,t) \geq 0$.

The concept AQSI includes a lot of r.v.'s, such as pairwise independent, negatively associated, negative quadrant dependent, asymptotically quadrant independent. Some mixing r.v.'s also satisfy the above conditions. Therefore, the study of AQSI sequence is more fundamental and difficult. So far, the research for AQSI is not promising. *Chandra* and *Ghosal* studied the law of large numbers and Marcinkiewicz-Zygmund type strong law for AQSI. Kim, Ko and Ryu [5] established the Hájeck-Rènyi type inequality for AQSI and obtained the strong law of large numbers. In this paper, we first establish the *Kolmogorov* type inequality and the three serises of theorem for AQSI sequence. Based on the research above, we study the almost sure convergence of AQSI and obtain the Chung type law of large numbers.

For a sequence $(X_n)_{n\in N}$ of r.v.'s defined on the fixed probability space, we set $S_n = \sum_{i=1}^n X_i, W_{m,n} = \max_{1\leq k\leq n} |\sum_{i=m+1}^{m+k} X_i|$, and $X_n^c = -cI_{(X_n<-c)} + X_nI_{(|X_n|\leq c)} + cI_{(X_n>c)}$, for any constant $c > 0$, Let $(a_n, b_n)_{n\in N}$ be two sequences of positive real numbers and denote $a_n = \mathcal{O}(b)_n($ resp. $a_n = o(b_n)$, if there exists a constant $C > 0$ satisfying that $a_n \leq Cb_n, n \to \infty$,(resp. $a_n/b_n \to 0$). Finally, the symbol C denotes a generic constant($0 < C < \infty$) which is not necessarily the same in different places.

Several lemmas are needed to establish the main results.

Lemma 1. [3] *Let $(X_n)_{n\in N}$ be a sequence of AQSI, if $(f_n)_{n\in N}$ is a sequence of nondecreasing (nonincreasing) functions, then $(f_n(X_n))_{n\in N}$ is also a sequence of AQSI r.v.'s.*

Lemma 2. [6] *Let $X_1, X_2, \cdots, X_n$ be a sequence of integrable r.v.'s and let $a_1^2, a_2^2, \cdots, a_n^2$ be real numbers such that*

$$E(X_{m+1} + \cdots + X_{m+p})^2 \leq a_{m+1}^2 + a_{m+p}^2 \quad (1.3)$$

for all $m, p \geq 1, m+p \leq n$. Then we have

$$EW_{0,n}^2 \leq ((\log(n)/\log 3) + 2)^2 \sum_{k=1}^n a_i^2 \quad (1.4)$$

Throughout the entire paper we will consider, unless otherwise mentioned, r.v.'s $(X_n)_{n\in N}$ as a AQSI sequence with zero means and $EX_n^2 < \infty, n = 1, 2, \cdots, \sum_{n=1}^{\infty} q(n) < \infty$ and satisfies the following conditions

$$\int_0^{\infty}\int_0^{\infty} \alpha_{ij}(s,t)dsdt \le C(EX_i^2 + EX_j^2), \tag{1.5}$$

$$\int_{-\infty}^{0}\int_{-\infty}^{0} \beta_{ij}(s,t)dsdt \le C(EX_i^2 + EX_j^2). \tag{1.6}$$

The plan of this paper is as follows. The main results will be presented in Section 2 which including a *Kolmogorov* type inequality and a generalized three series theorem for AQSI sequence, and two illustrative examples.

2. **Main Results**

With the preliminaries accounted for, the main result may be established.

Theorem 1. *(Kolmogorov type inequality) Let $(X_n)_{n\in N}$ be defined as in section 1, then*

$$E(\sum_{i=l+1}^{l+k} X_i)^2 \le C\sum_{i=l+1}^{l+k} EX_i^2, \tag{2.1}$$

$$E(\max_{1\le k\le n}(\sum_{i=l+1}^{l+k} X_i)^2) \le C\log^2 n\sum_{i=l+1}^{l+n} EX_i^2. \tag{2.2}$$

Proof. Suppose $X^+ = \max\{X, 0\}$ and $X^- = \max\{-X, 0\}$. It is easy to see $\{X_n^+\}$ and $\{X_n^-\}$ form AQSI sequence by Lemma 1. Since

$$\begin{aligned} cov(X_i^+, X_j^+) &= \int_0^{\infty}\int_0^{\infty}[P(X_i^+ > s, X_j^+ > t) - P(X_i^+ > s)P(X_j^+ > t)]dsdt \\ &\le q(|i-j|)\int_0^{\infty}\int_0^{\infty} \alpha_{ij}(s,t)dsdt \le Cq(|i-j|)(EX_i^2 + EX_j^2). \end{aligned}$$

We have by the condition $\sum_{n=1}^{\infty} q(n) < \infty$,

$$Var(\sum_{i=l+1}^{l+k} X_i^+) \le C\sum_{i=l+1}^{l+k} EX_i^2, \tag{2.3}$$

Similarly, we have

$$Var(\sum_{i=l+1}^{l+k} X_i^-) \le C\sum_{i=l+1}^{l+k} EX_i^2. \tag{2.4}$$

Noticing $EX_n = 0$, we have

$$\begin{aligned} E(\sum_{i=l+1}^{l+k} X_i)^2 &= Var(\sum_{i=l+1}^{l+k} X_i) \\ &\leq 2\left[Var(\sum_{i=l+1}^{l+k} X_i^+) + Var(\sum_{i=l+1}^{l+k} X_i^-)\right] \leq C\sum_{i=l+1}^{l+k} EX_i^2 \end{aligned}$$

By lemma 2, we have

$$E(\max_{1\leq k\leq n}(\sum_{i=l+1}^{l+k} X_i)^2) \leq ((\log n/\log 3)+2)^2 C\sum_{i=l+1}^{l+n} EX_i^2 \leq C\log^2 n\sum_{i=l+1}^{l+n} EX_i^2.$$

□

Theorem 2. *Suppose that*

$$\sum_{n=1}^{\infty} \log^2 nVarX_n < \infty. \tag{2.5}$$

Then $\sum_{n=1}^{\infty} X_n$ *a.s. convergence.*

Proof. From (2.1) and (2.5), if integer $m > n \to \infty$, we have

$$E(S_m - S_n)^2 \leq C\sum_{k=n+1}^{m} EX_k^2 \to 0.$$

Hence $(S_n)_{n\in N}$ is $Cauchy$ in $\mathcal{L}_2$. Since $\mathcal{L}^2$ is complete, there exists a unique $r.v.S_\infty$ (up to a.s. equivalence) in $\mathcal{L}^2$, such that $S_n \to S_\infty$ in $\mathcal{L}^2$, this together with (2.1) and (2.5) imply

$$\begin{aligned} &P(|S_{2^k} - S_\infty| > \varepsilon) \leq \varepsilon^{-2}E(S_{2^k} - S_\infty)^2 \\ =&\mathcal{O}[\limsup_{n\to\infty} E(S_n - S_{2^k})^2] \\ \leq& C\sum_{i=2^k+1}^{\infty} EX_i^2 = C\sum_{i=2^k+1}^{\infty} EX_i^2 \log^2 i\cdot\frac{1}{\log^2 i} \\ \leq&\frac{C}{(\log 2^k)^2}\sum_{i=2^k+1}^{\infty}\log^2 iEX_i^2 = \mathcal{O}(k^{-2}), \end{aligned}$$

Therefore

$$\sum_{k=1}^{\infty} P(|S_{2^k} - S_\infty| > \varepsilon) < \infty. \tag{2.6}$$

Theorem 1 and (2.5) imply

$$\begin{aligned}
&\sum_{k=1}^{\infty} P(\max_{2^{k-1}<j\le 2^k} |S_j - S_{2^{k-1}}| \ge \varepsilon) \\
=&\mathcal{O}(\sum_{k=1}^{\infty} (\log 2^k)^2 \sum_{j=2^{k-1}+1}^{2^k} EX_j^2) \\
=&\mathcal{O}(\sum_{k=1}^{\infty} \sum_{j=2^{k-1}+1}^{2^k} (\log j)^2 EX_j^2) = \sum_{j=1}^{\infty} (\log j)^2 EX_j^2 < \infty. \qquad (2.7)
\end{aligned}$$

By (2.6), (2.7) and Borel-Cantelli lemma, we have, as $k \to \infty$

$$S_{2^k} \to S_\infty \ a.s. \text{ and } \max_{2^{k-1}<j\le 2^k} |S_j - S_{2^{k-1}}| \to 0 \ a.s.$$

According to the method of subsequence, we have

$$S_n \to S_\infty \ a.s., \ n \to \infty.$$

□

Theorem 3. *Assume*

(1)

$$\sum_{n=1}^{\infty} P(|X_n| > c) < \infty, \qquad (2.8)$$

(2)

$$\sum_{n=1}^{\infty} EX_n^c < \infty, \qquad (2.9)$$

(3)

$$\sum_{n=1}^{\infty} \log^2 n Var X_n^c < \infty. \qquad (2.10)$$

Then

$$\sum_{n=1}^{\infty} X_n \ a.s. \ convergence. \qquad (2.11)$$

Proof. It is easy to see that $(X_n^c)_{n\in N}$ also form a sequence of AQSI and we have for every $i \ne j$

$$P(X_i^c > s, X_j^c > t) - P(X_i^c > s)P(X_j^c > t) \le q(|i-j|)\alpha_{ij}^*(s,t), \ s,t > 0$$
$$P(X_i^c < s, X_j^c < t) - P(X_i^c < s)P(X_j^c < t) \le q(|i-j|)\beta_{ij}^*(s,t), \ s,t < 0$$

where

$$\alpha_{ij}^*(s,t) = \begin{cases} \alpha_{ij}(s,t), & 0 < s,t \le c \\ 0, & otherwise \end{cases}$$

and

$$\beta^*_{ij}(s,t) = \begin{cases} \beta_{ij}(s,t), & -c \le s,t < 0 \\ 0, & otherwise \end{cases}$$

Hence, by (2.10) and Theorem 2, we have

$$\sum_{n=1}^{\infty}(X_n^c - EX_n^c) < \infty \ a.s. \tag{2.12}$$

(2.9) and (2.12) imply

$$\sum_{n=1}^{\infty} X_n^c < \infty \ a.s. \tag{2.13}$$

From (2.8), we have

$$\sum_{n=1}^{\infty} P(X_n \neq X_n^c) = \sum_{n=1}^{\infty} P(|X_n| > c) < \infty,$$

Thus the Borel-Catelli lemma implies that

$$P(X_n \neq X_n^c, i.o.) = 0, \tag{2.14}$$

(2.13) and (2.14) imply $\sum_{n=1}^{\infty} X_n$ *a.s.* convergence.

Theorem 4. *Let functions: $\varphi_n(x) : R \to R^+$ be nonnegative, even, continuous and nondecreasing on $(0,\infty)$ and suppose that one of the following two conditions prevails:*
(a) $x/\varphi_n(x)$ is nondecreasing in $x > 0$ for each $n \ge 1$;
(b) $x/\varphi_n(x)$ and $\varphi_n(x)/x^2$ are nonincreasing in $x > 0$ for each $n \ge 1$;
Furthermore, let $(a_n)_{n\in N}$ be a sequence of positive constants with $a_n \uparrow \infty$, if the series

$$\sum_{n=1}^{\infty} \frac{\log^2 nE\varphi_n(X_n)}{\varphi_n(a_n)} < \infty, \tag{2.15}$$

then the series $\sum_{n=1}^{\infty} \frac{X_n}{a_n}$ converges a.s. and

$$\lim_n a_n^{-1} S_n \to 0 \ a.s. \tag{2.16}$$

Proof. By Kronecker lemma, to prove (2.16), it is sufficent to prove the convergence of $\sum_{n=1}^{\infty} \frac{X_n}{a_n}$ *a.s.*. Since $\varphi_n(x)$ is nondecreasing on $x > 0$, we have

$$P(|X_n| \ge a_n) \le \int_{|X_n| \ge a_n} \frac{\varphi_n(X_n)}{\varphi_n(a_n)} dP \le \frac{E\varphi_n(X_n)}{\varphi_n(a_n)},$$

this and (2.15) imply

$$\sum_{n=1}^{\infty} P(|X_n| \ge a_n) < \infty. \tag{2.17}$$

In case (a), for $|x| \le a_n$, we have

$$\frac{|x|}{\varphi_n(x)} \le \frac{a_n}{\varphi_n(a_n)},$$

and

$$\frac{x^2}{a_n^2} \le \frac{\varphi_n^2(x)}{\varphi_n^2(a_n)} \le \frac{\varphi_n(x)}{\varphi_n(a_n)};$$

In case (b), by $\frac{\varphi_n(x)}{x^2} \searrow$, for $|x| \le a_n$, we have

$$\frac{x^2}{\varphi_n(x)} \le \frac{a_n^2}{\varphi_n(a_n)}.$$

Hence, in case (a) or (b), for $|x| \le a_n$, we have

$$\frac{x^2}{a_n^2} \le \frac{\varphi_n(x)}{\varphi_n(a_n)}. \tag{2.18}$$

From C_r inequality, we have for any n

$$E(X_n^{a_n})^2 \le 3E(a_n^2 I_{(X_n<-a_n)} + X_n^2 I_{(|X_n|\le a_n)} + a_n^2 I_{(X_n>a_n)})$$

$$= \mathcal{O}(Ea_n^2 I_{(|X_n|>a_n)} + EX_n^2 I_{(|X_n|\le a_n)}),$$

Noticing that $\varphi_n(x)$ are even functions and nondecreasing in $x>0$, we have

$$Ea_n^2 I_{(|X_n|>a_n)} \le Ea_n^2 \frac{\varphi_n(X_n)}{\varphi_n(a_n)} I_{(|X_n|>a_n)} \le \frac{a_n^2}{\varphi_n(a_n)} E\varphi_n(X_n).$$

By (2.18), we have

$$EX_n^2 I_{(|X_n|\le a_n)} = \int_{|X_n|\le a_n} X_n^2 dP \le \frac{a_n^2}{\varphi_n(a_n)} \int_{|X_n|\le a_n} \varphi_n(X_n) dP \le \frac{a_n^2}{\varphi_n(a_n)} E\varphi_n(X_n),$$

Hence

$$E(X_n^{a_n})^2 = \mathcal{O}(\frac{a_n^2}{\varphi_n(a_n)} E\varphi_n(X_n)), \tag{2.19}$$

This and (2.15) imply

$$\sum_{n=1}^{\infty} \frac{\log^2 n E(X_n^{a_n})^2}{a_n^2} = \mathcal{O}\left[\sum_{n=1}^{\infty} \frac{\log^2 n E\varphi_n(X_n)}{\varphi_n(a_n)}\right] < \infty. \tag{2.20}$$

In case (a), we have

$$\begin{aligned}
|EX_n^{a_n}| &= |E(-a_n I_{(X_n<-a_n)} + X_n I_{(|X_n|\le a_n)} + a_n I_{(X_n>a_n)})| \\
&\le Ea_n I_{(|X_n|>a_n)} + |EX_n I_{(|X_n|\le a_n)}| \\
&\le Ea_n \frac{\varphi_n(X_n)}{\varphi_n(a_n)} I_{(|X_n|>a_n)} + |\int_{|X_n|\le a_n} X_n dP| \\
&\le \frac{a_n}{\varphi_n(a_n)} E\varphi_n(X_n) + \frac{a_n}{\varphi_n(a_n)} \int_{|X_n|\le a_n} \varphi_n(X_n) dP \\
&\le \frac{2a_n}{\varphi_n(a_n)} E\varphi_n(X_n)
\end{aligned}$$

On the other hand, in case (b), note that $EX_n = 0$ and $x/\varphi_n(x) \searrow$, we have

$$\begin{aligned}
|EX_n^{a_n}| &\le Ea_n I_{(|X_n|>a_n)} + |EX_n I_{(|X_n|\le a_n)}| \\
&= a_n EI_{(|X_n|>a_n)} + |EX_n I_{(|X_n|>a_n)}| \\
&\le \frac{a_n}{\varphi_n(a_n)} E\varphi_n(X_n) + \frac{a_n}{\varphi_n(a_n)} \int_{|X_n|>a_n} \varphi_n(X_n) dP \\
&\le \frac{2a_n}{\varphi_n(a_n)} E\varphi_n(X_n)
\end{aligned}$$

Hence

$$\sum_{n=1}^{\infty} E\left[\frac{X_n^{a_n}}{a_n}\right] \le 2\sum_{n=1}^{\infty} \frac{E\varphi_n(X_n)}{\varphi_n(a_n)} < \infty. \tag{2.21}$$

Combine (2.17), (2.20) and (2.21) and Theorem 3, we have $\sum_{n=1}^{\infty} \frac{X_n}{a_n}$ $a.s.$convergence. Thus the proof is complete. □

Let $\varphi_n(x) = |x|^p, 0 < p \le 2$ in Theorem 4, we get the following lemmas.

Corollary 1. *Let $(a_n)_{n\in N}$ be a sequence of positive constants with $a_n \uparrow \infty, 0 < p \le 2$, if $\sum_{n=1}^{\infty} \frac{\log^2 nE|X_n|^p}{a_n^p} < \infty$, then $a_n^{-1} S_n = o(1)$ $a.s.$ as $n \to \infty$.*

Take $a_n = n^{1/p}$ or $a_n = n^{1/p}(\log n)^{(3+\delta)/p}, \delta > 0$ respectively in Corollary 1, we immediately obtain the following corollary 2 and 3.

Corollary 2. *Let $E|X_n|^p \le C \log^{-3-\delta} n, \delta > 0, 0 < p \le 2$, then $n^{-1/p} S_n = o(1)$ $a.s.$ as $n \to \infty$.*

Corollary 3. *Let $E|X_n|^p \le C, 0 < p \le 2$, then $n^{-1/p}(\log n)^{-(3+\delta)/p} S_n = o(1)$ $a.s.$ as $n \to \infty$.*

Corollary 4. *Let $(b_n)_{n\in N}$ be a positive sequence of nondecreasing real numbers. If $\sum_{n=1}^{\infty} \frac{EX_n^2}{b_n} < \infty$, then $(b_n \log n)^{-1} S_n \to 0$ as $n \to \infty$.*

Proof. By taking $a_n = b_n \log n$ in Corollary 1 which complete the proof. □

Corollary 5. *If* $\sum_{n=1}^{\infty} EX_n^2 < \infty$, *then for* $0 < p < 2$, *we have* $n^{-1/p}(\log n)^{-1}S_n \to 0$ *as* $n \to \infty$.

Corollary 6. *If* $\sup_n EX_n^2 < \infty$, *then for* $0 < p < 2$, *we have* $n^{-1/p}(\log n)^{-1}S_n \to 0$ *as* $n \to \infty$.

In recent years, two-dimension dependent r.v.'s was usually described by Copula, with the purpose of a wider usage of two-dimension dependent r.v.'s by putting forward some special r.v.'s. Let us recall the definition of copula.

Definition 2. Let X and Y be r.v.'s with distribution functions $F_X(x)$ and $F_Y(y)$, the function $C_{X,Y}(u,v)$ defined for $u, v \in [0,1]$ such that

$$P(X \le x, Y \le y) = C_{XY}(F_X(x), F_Y(y)) = C_{XY}(\mu, \nu)$$

is called the copula of X and Y.

The theory of Copula can be dated back to 1959, when Sklar related multiple distribution function to Copula in the form of Theorem. We can establish multiple distribution by Copula function and the marginal distributions.

As pointed in [4], in order to satisfy the existence of AQSI sequence in (1.5) and (1.6), we need to consider if r.v.'s can meet the following Copula:

$$C_{X_iX_j}(u,v) - uv \le \rho_{ij}uv(1-u)(1-v), \tag{2.22}$$

Where $(u,v) \in [0,1] \times [0,1]$,$\rho_{ij} \ge 0$ and if $|i-j| \to \infty$, $\rho_{i,j} \to 0$.

Denote $q(|i-j|) = \rho_{ij}$, for the sequence which satisfies (2.19), we have

$$\begin{aligned} &P(X_i \le s, X_j \le t) - P(X_i \le s)P(X_j \le t) \\ \le &q(|i-j|)P(X_i \le s)P(X_j \le t)P(X_i > s)P(X_j > t) \end{aligned}$$

It is not difficult to verify

$$\begin{aligned} &P(X_i > s, X_j > t) - P(X_i > s)P(X_j > t) \\ = &P(X_i \le s, X_j \le t) - P(X_i \le s)P(X_j \le t) \end{aligned}$$

Therefore, s is replaced by $s - \frac{1}{n}$, t is replaced by $t - \frac{1}{n}$, let $n \to \infty$, we conclude that the sequence above is AQSI, where

$$\alpha_{ij}(s,t) = P(X_i \le s)P(X_j \le t)P(X_i > s)P(X_j > t)$$

$$\beta_{ij}(s,t) = P(X_i < s)P(X_j < t)P(X_i \ge s)P(X_j \ge t)$$

Note the definitions of α_{ij} and β_{ij} and $(E|X|)^2 \le EX^2$, we have

$$\begin{aligned} &\int_0^\infty \int_0^\infty \alpha_{ij}(s,t)dsdt \le \int_0^\infty \int_0^\infty P(X_i > s)P(X_j > t)dsdt \\ = &EX_i^+ EX_j^+ \le E|X_i|E|X_j| \\ \le &\frac{(E|X_i|)^2 + (E|X_j|)^2}{2} \le \frac{EX_i^2 + EX_j^2}{2} \end{aligned}$$

$$\int_{-\infty}^{0}\int_{-\infty}^{0}\beta_{ij}(s,t)dsdt \le \int_{-\infty}^{0}\int_{-\infty}^{0}P(X_i<s)P(X_j<t)dsdt$$
$$=EX_i^- EX_j^- \le E|X_i|E|X_j|$$
$$\le\frac{(E|X_i|)^2+(E|X_j|)^2}{2}\le\frac{EX_i^2+EX_j^2}{2}$$

Example 5. *Let* $(X_n)_{n\in N}$ *be a sequence of r.v.'s such that* $EX_n^2<\infty, n=1,2,\cdots$*, and with Copula:*

$$C_{X_i,X_j}(u,v)=uv[1-\frac{1}{(i-j)^2}(1-u)(1-v)].$$

Example 6. *Let* $(X_n)_{n\in N}$ *be a sequence of r.v.'s such that* $EX_n^2<\infty, n=1,2,\cdots$*, and with Copula:*

$$C_{X_i,X_j}(u,v)=\frac{uv}{1-\frac{1}{(i-j)^2}(1-u)(1-v)}.$$

References

[1] T. Birkel, Laws of large numbers under dependence assumptions, *Statist. and Probab. Lett.*, **14** (1992), 355-362. https://doi.org/10.1016/0167-7152(92)90096-n

[2] T.K. Chandra, S. Ghosal, *Some Elementary Strong Laws of Large Numbers: A Review*, Technical Report 22-93, Indian Statistical Institute, 1993.

[3] T.K. Chandra, S. Ghosal, Extensions of the strong law of large numbers of Marcinkiewicz and Zygmund for dependent variables, *Acta Math. Hungar.*, **71** (1996), no. 4, 327-336. https://doi.org/10.1007/bf00114421

[4] P. Matula, On some families of AQSI random variables and related strong law of large numbers, *Applied Mathematics E-Notes*, **5** (2005), 31-35.

[5] T.K. Chandra, S. Ghosal, The strong law of large numbers for weighted averages under dependence assumptions, *J. Theoret. Probab.*, **9** (1996), no. 3, 797-809. https://doi.org/10.1007/bf02214087

[6] T.S. Kim, M.H. Ko and D.H. Ryu, A strong law of large numbers for asymptotically quadrant sub-independent sequences, *Appl. Math. and Computing*, **16** (2004), 419-427.

[7] M. Loève, *Probability Theory I*, Springer, New York, 1977. https://doi.org/10.1007/978-1-4684-9464-8

[8] V.V. Petrov, *Limit Theorems of Probability Theory*, Oxford, New York, 1995.

[9] R.J. Serfling, *Approximation Theorems of Mathematical Statistics*, John Wiley, New York, 1980. https://doi.org/10.1002/9780470316481

[10] R.B. Nelsen, *An introduction to Copulas*, Springer-Verlag, New York, 1999. https://doi.org/10.1007/978-1-4757-3076-0

[11] A. Bozorgnia, R.F. Patterson and R.L. Taylor, Limit theorems for dependent random-variables, *Proceedings of WCNA'92*, Tampa, Florida, (1996), 1639-1650. https://doi.org/10.1515/9783110883237.1639

[12] W. Qunying, Convergence properties of pairwise NQD random sequence, *Acta Mathematica Sinica*, **45** (2002), no. 3, 617-624. (in Chinese)

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