On the Laplacian Energy of Windmill Graph

and Graph $D_{m,C_n}$

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Abstract

Let $G$ be a graph with $n$ vertices and $m$ edges, and also let $\mu_i (i = 1, 2, \cdots, n)$ be the eigenvalues of the Laplacian matrix of the graph $G$. The Laplacian energy of the graph $G$ is defined as

$$LE = LE(G) = \sum_{i=1}^{n} |\mu_i - \frac{2m}{n}|$$

In this paper, we present some upper bounds of the Laplacian energy for special graphs, which are the windmill graph and the graph $D_{m,C_n}$. The bounds of the Laplacian energy are given by the methods of partitioning of matrices.

Keywords: Laplacian energy, windmill graph, graph $D_{m,C_n}$

1 Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V(G) = \{\nu_1, \nu_2 \cdots \nu_n\}$ and edge set $E(G)$. Let $A(G)$ be the $(0, 1)$-adjacency matrix of $G$ and $D(G)$ be the diagonal matrix of vertex
degrees. The Laplacian matrix of $G$ is $L(G) = D(G) - A(G)$. The Laplacian matrix has nonnegative eigenvalues, which are denoted by $\mu_1, \mu_2, \cdots \mu_n$, arranged in a non-increasing order, $n \geq \mu_1 \geq \mu_2 \geq \mu_3 \cdots \geq \mu_n = 0$ [3, 4, 5]. When we consider more than one graph, then we write $\mu_i(G)$ instead of $\mu_i$. Let $d_i$ be the degree of the vertex $v_i$ for $i = 1, 2, \ldots, n$. The maximum vertex degree is denoted by $\Delta$. The intention for Laplacian energy stems from graph energy $E(G)$ [2]. The energy $E(G)$ of a graph $G$ is equal to the sum of the absolute values of the eigenvalues of the adjacency matrix of $G$. We have recently proposed an energy-like quantity $LE(G)$, based on the eigenvalues of the Laplacian matrix of $G$. The Laplacian energy of the graph $G$ is defined as [4, 5]

$$LE = LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|$$

Laplacian energy is a broad measure of graph complexity. Song et al [11] have introduced component-wise Laplacian graph energy, as a complexity measure useful to filter image description hierarchies. This paper is organized as followed. In section 2, we start with showing some preliminary lemmas and theorems relying on the pigeonhole principle. Then, In section 3-4, we distinguish calculate the Laplacian energy and the upper bound of the Laplacian energy of above special graphs, which are given by the methods of partitioning of matrices [12].

## 2 Preliminaries

In this section, we shall list some previously known results that will be needed in the next two sections. The eigenvalues of a real matrix $X$ are the square roots of the eigenvalues of the matrix $XX^t$, where $X^t$ represent the transpose of the matrix $X$. Then for the graph $G$ with $n$ vertices, we have $LE(G) = \sum_{i=1}^{n} \mu_i(L(G) - d(G)E_n)$.

**Lemma 2.1** ([7,9]) Let $X$ and $Y$ be two $n \times n$ real matrices. Then

$$\sum_{i=1}^{n} \mu_i(X + Y) \leq \sum_{i=1}^{n} \mu_i(X) + \mu_i(Y). \quad (1)$$

**Lemma 2.2** ([8]) Let $G$ be a graph on $n$ vertices which has at least one edge. Then the Laplacian eigenvalues are labeled so that $\mu_1 \geq \mu_2 \geq \cdots \mu_n$, then

$$\mu_1 \geq \Delta + 1. \quad (2)$$

**Lemma 2.3** ([1]) Let $G$ be a graph on $n$ vertices with Laplacian spectrum $\mu_1, \mu_2, \cdots, \mu_{n−1}, \mu_n = 0$. Then

$$LE(G) \leq \sum_{i=1}^{n} \max\{\mu_i, d(G)\}. \quad (3)$$
The windmill graph $K_n^{(m)}$ are defined as only one centre vertex, which included $m$ of the complete graph $K_n$. The number of vertices are $N = m(n - 1) + 1$, and the number of edges are $M = \frac{mn(n-1)}{2}$, the average of degree is $d = \frac{mn(n-1)}{m(n-1)+1}$.

The principle of labeling vertices of the windmill graph $K_n^{(m)}$ is the first of the centre vertex is labeled, the following each complete graph is labeled in proper sequence. See Figure 1.

By massive calculation, we discover the following rules. The eigenvalues of Laplacian have below regulation.

(a) $\mu_1 = m(n - 1) + 1, \mu_{m(n-1)+1} = 0$.

(b) The number of eigenvalues of 1 is equal to $m - 1$, the remaining eigenvalues are equal to $n$.

**Theorem 3.1** The Laplacian energy of windmill graph $K_n^{(m)}$ is given by

$$LE(K_n^{(m)}) = 2 - mn - 2n + n^2m + d(3m - nm - 1).$$
Proof First we have to prove that
\[
LE(K_n^{(m)}) = \sum_{i=1}^{m(n-1)+1} |\mu_i - d|
\]
\[
= \sum_{i=1}^{m(n-1)+1-m} \mu_i - \sum_{i=m(n-1)+2-m}^{m(n-1)+1-m} d + \sum_{i=m(n-1)+2-m}^{m(n-1)+1} \mu_i - \sum_{i=m(n-1)+2-m}^{m(n-1)} d
\]
\[
= \mu_1 + \sum_{i=2}^{m(n-1)+1} \mu_i - \sum_{i=1}^{m(n-1)+2-m} d + \sum_{i=m(n-1)+2-m}^{m(n-1)+1} d - \sum_{i=m(n-1)+2-m}^{m(n-1)} \mu_i
\]
By the equality (a)(b), we have
\[
= m(n-1) + 1 + (mn - 2m)n - (mn - 2m + 1)d + md - (m - 1)
\]
\[
= m(n - 2) + 2 + n(mn - 2m) + d(3m - mn - 1)
\]
\[
= 2 - mn - 2n + n^2m + d(3m - nm - 1).
\]
This completes the proof.

**Theorem 3.2** The upper bound of Laplacian energy of the windmill graph \(K_n^{(m)}\) is given by
\[
LE(K_n^{(m)}) \leq mn^2 + m(d + n - 1) + 1.
\]

**Proof** From the above equality (a)(b) and the equality (3), we have
\[
LE(K_n^{(m)}) \leq \sum_{i=1}^{m(n-1)+1} \max\{\mu_i, d(K_n^{(m)})\}
\]
\[
\leq \sum_{i=1}^{mn-2m+1} \mu_i + \sum_{i=mn-2m+2}^{m(n-1)+m(n-1)+1} d
\]
\[
\leq \mu_1 + \sum_{i=2}^{m(n-1)+1} \mu_i + \sum_{i=m(n-1)+2-m}^{m(n-1)+1} d
\]
\[
\leq mn^2 + m(d + n - 1) + 1.
\]
This completes the proof.

**Theorem 3.3** Let \(K_n^{(m)}\) is a windmill graph, then the upper bound of Laplacian energy is given by
\[
LE(K_n^{(m)}) \leq 2\sqrt{n(n-1)} + (n - 1)(n - 1 - m) + nd + (m - 1)(n - 1)(d + n - 1).
\]

**Proof** Note that
\[
L(K_n^{(m)}) - d(K_n^{(m)})E_{m(n-1)+1} = \begin{pmatrix}
D_n(K_n^{(m)}) - d(K_n^{(m)})E_n - A(K_n) & B \\
B^t & R(K_n^{(m)})
\end{pmatrix}
\]
since

\[
BB^t = \begin{pmatrix}
n-1 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0
\end{pmatrix}_{n \times n}
\]

The Essay [11] had introduced the inequation, so we get

\[
\sum_{i=1}^{n} \mu_i \begin{pmatrix}
n-1 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0
\end{pmatrix} \leq \sum_{i=1}^{n} \mu_i \begin{pmatrix}
n-1 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0
\end{pmatrix} = \sqrt{n(n-1)}.
\]

As we are knowed, the part of the Laplacian matrix is

\[
R(K_n^{(m)}) = \begin{pmatrix}
R(K_n^{(2)}) & 0 \\
0 & R(K_n^{(m-1)})
\end{pmatrix}.
\]

And

\[
R(K_n^{(2)}) = \begin{pmatrix}
n-1-d(K_n^{(2)}) & \cdots & -1 \\
-1 & \ddots & \ddots \\
-1 & \cdots & n-1-d(K_n^{(2)})
\end{pmatrix}
\]

This is ,

\[
LE(K_n^{(m)}) \leq \sum_{i=1}^{n} |\mu_i \left( \begin{pmatrix} 0 & B \\ B^t & 0 \end{pmatrix} \right) | + \sum_{i=1}^{n} |\mu_i (D_n(K_n^{(m)}) - d(K_n^{(m)})E_n - A(K_n))| + \sum_{i=1}^{n} |\mu_i R(K_n^{(m)})| \leq 2\sqrt{n-1} + (n-1)(n-1+m) + nd + (m-1)(n-1)(d-n+1).
\]

This completes the proof.

4 The Graph $D_{m,C_n}$

The graph $D_{m,C_n}$ [10] consists of $m$ cycles with one common vertex, which denoted by $v_1$. And each cycle has $n$ vertices besides the center point $v_1$. So the number of vertices are
Figure 2: The graph $D_{m,C_6}$

$(n - 1)m + 1$ and $mn$ edges. Then the average of degree is $d = \frac{2mn}{(n-1)m+1}$.

The principle of labelling vertices of the $D_{m,C_n}$ graph is the first of the common vertex is labeled, the following each cycle is labelled in proper sequence. See Figure 2.

**Theorem 4.1** The upper bound of the Laplacian energy of the $D_{m,C_n}$ graph is given by

$$LE(D_{m,C_n}) \leq 2(m + n - 1) + nd + 2\sqrt{n(m - 1)} + (d - 2)(m - 1), \quad m \geq 2.$$  
$$LE(D_{m,C_n}) \leq 2(m + n - 1) + 2\sqrt{2n(m - 1)} + (n - 1)(m - 1)(2 + d) + nd, \quad m \geq 3, \quad n \geq 3.$$

**Proof.** According to the summary, we found the Laplacian matrix of graph of $D_{2,C_2}$, we may see

$$L(D_{2,C_2}) - d(D_{2,C_2})E_{(n-1)m+1} = \left( D_2(D_{2,C_2}) + \frac{-d(D_{2,C_2})E_2 - M(D_{2,C_2})}{P^t} P \right)$$  \hspace{1cm} (4)

Which
On the Laplacian energy of Windmill graph

\[ M = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \]

\[ PP^t = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \]

We conclude the Laplacian matrix of the graph of \( D_{m,C_n} \), using the above result, we get

\[ L(D_{m,C_n}) - d(D_{m,C_n})E_{(n-1)m+1} = \begin{pmatrix} D_n(D_{m,C_n}) + \{-d(D_{m,C_n})E_n - M(D_{m,C_n}) \} & P' \\ P'^t & W(D_{m,C_n}) \end{pmatrix} \]

Which

\[ W(D_{m,C_n}) = \begin{pmatrix} W(D_{m-1,C_n}) & 0 \\ 0 & 2 \end{pmatrix} \]

\[ P' P'^t = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} m - 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \]

It is known \( W(D_{2,C_n}) = 2 \). We get

\[ LE(D_{m,C_n}) \leq 2(m + n - 1) + nd + 2\sqrt{n(m-1)} + (d-2)(m-1), m \geq 2. \]

First part of the proof is done.

Now we suppose that the equality holds in (4). Then we continue found the relationship with the Laplacian matrix of the \( D_{2,C_n}, (n \geq 3) \), we found the following the expressions.

\[ L(D_{2,C_n}) - d(D_{2,C_n})E_{(n-1)m+1} = \begin{pmatrix} D_n(D_{2,C_n}) + \{-d(D_{2,C_n})E_n - M(D_{2,C_n}) \} & P'' \\ P''t & W_{(n-1)(n-1)}(D_{2,C_n}) \end{pmatrix} \]

Which,

\[ W(D_{2,C_n}) = \begin{pmatrix} W(D_{2,C_{n-1}}) & X_{(n-2)\times1} \\ X_{1\times(n-2)}^t & 2 \end{pmatrix} \]

\[ P'' P''t = \begin{pmatrix} 2(m - 1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} X_{(n-2)\times1}X_{1\times(n-2)}^t = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & 0 \\ 0 & \cdots & 1 \end{pmatrix}_{n \times n} \]

Thus we explore that the matrix of \( W(D_{2,C_n}) \) is a symmetric matrix. Merely be divided by us to research more better, which by partitioning of matrix. So in fact

\[ \sum_{i=1}^{(n-1)(m-1)} |\mu_i(W(D_{2,C_n}))| = \sum_{i=1}^{(n-1)(m-1)} |\mu_i(D_{(n-1)(m-1)}(D_{2,C_n}) - d(D_{2,C_n})E_{(n-1)(m-1)})| \]
Using the results, we get the Laplacian matrix of the graph $D_{m,C_n}(m \geq 3, n \geq 3)$ is given by

$$L(D_{m,C_n}) - d(D_{m,C_n})E_{(n-1)m+1} = \begin{pmatrix} D_n(D_{m,C_n}) + \{-d(D_{m,C_n})E_n - M(D_{m,C_n}) & P'' \nonumber \\
\end{pmatrix}$$

Which

$$W(D_{m,C_n}) = \begin{pmatrix} W(D_{m-1,C_n}) & 0 \\
0 & w(D_{m-1,C_n}) \end{pmatrix} P'' P''t = \begin{pmatrix} 2(m-1) & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0 \end{pmatrix}$$

Although the matrix of $M(D_{m,C_n})$ are different, but all of the $M(D_{m,C_n})$ are symmetric matrix. And the element of the $M(D_{m,C_n})$ are 0 and -1. So by contradiction, we get

$$\sum_{i=1}^{n} |\mu_i[D_n(D_{m,C_n}) + \{-d(D_{m,C_n})E_n - M(D_{m,C_n})\}]| \leq \sum_{i=1}^{n} |\mu_i[D_n(D_{m,C_n})]| + \sum_{i=1}^{n} |\mu_i[-d(D_{m,C_n})E_n]|$$

Then we have

$$LE(D_{m,C_n}) \leq \sum_{i=1}^{n} \mu_i\left( \begin{pmatrix} 0 & P'' \\
P''t & 0 \end{pmatrix} \right) + \sum_{i=1}^{n} |\mu_i[D_n(D_{m,C_n}) + \{-d(D_{m,C_n})E_n - M(D_{m,C_n})\}]|$$

$$\text{m(n-1)+1-n}$$

$$\sum_{i=1}^{n} |\mu_i(W(D_{m,C_n})|.$$
References


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