\{C_k, P_k, S_k\}-Decompositions of Balanced Complete Bipartite Graphs

Jenq-Jong Lin and Min-Jen Jou

Ling Tung University, Taichung 40852, Taiwan

Abstract

Let \( L = \{H_1, H_2, \ldots, H_r\} \) be a family of subgraphs of a graph \( G \). An \( L \)-decomposition of \( G \) is an edge-disjoint decomposition of \( G \) into positive integer \( \alpha_i \) copies of \( H_i \), where \( i \in \{1, 2, \ldots, r\} \). Let \( C_k, P_k \) and \( S_k \) denote a cycle, a path and a star with \( k \) edges, respectively. In this paper, we prove that a balanced complete bipartite graph with \( 2n \) vertices has a \( \{C_k, P_k, S_k\} \)-decomposition if and only if \( k \) is even, \( 4 \leq k \leq n \) and \( n^2 \equiv 0 \pmod{k} \).

Mathematics Subject Classification: 05C51

Keywords: cycle, path, star, decomposition, balanced complete bipartite graph

1 Introduction

Let \( L = \{H_1, H_2, \ldots, H_r\} \) be a family of subgraphs of a graph \( G \). An \( L \)-decomposition of \( G \) is an edge-disjoint decomposition of \( G \) into positive integer \( \alpha_i \) copies of \( H_i \), where \( i \in \{1, 2, \ldots, r\} \). Furthermore, if each \( H_i(i \in \{1, 2, \ldots, r\}) \) is isomorphic to a graph \( H \), then we say that \( G \) has an \( H \)-decomposition.

For positive integers \( m \) and \( n \), \( K_{m,n} \) denotes the complete bipartite graph with parts of sizes \( m \) and \( n \). A complete bipartite graph is balanced if \( m = n \). A \( k \)-cycle, denoted by \( C_k \), is a cycle of length \( k \). A \( k \)-star, denoted by \( S_k \), is
the complete bipartite graph $K_{1,k}$. A $k$-path, denoted by $P_k$, is a path with $k$ edges.

Decompositions of some families of graphs into $k$-cycles has been a popular topic of research in graph theory; see [4, 7] for surveys of this topic. Articles of $P_k$-decompositions of interest include [9, 11]. Decompositions of graphs into $k$-stars have also attracted a fair share of interest; see [16, 17, 18]. The study of $\{G, H\}$-decomposition was introduced by Abueida and Daven in [1]. Abueida and Daven [2] investigated the problem of $\{K_k, S_k\}$-decomposition of the complete graph $K_n$. Abueida and O’Neil [3] settled the existence problem for $\{C_k, S_{k-1}\}$-decomposition of the complete multigraph $\lambda K_n$ for $k \in \{3, 4, 5\}$. In [10], Priyadharsini and Muthusamy gave necessary and sufficient conditions for the existence of a $\{G, H\}$-factorization of $\lambda K_n$ where $G, H \in \{C_n, P_{n-1}, S_{n-1}\}$. Furthermore, Shyu [12] investigated the problem of decomposing $K_n$ into paths and stars with $k$ edges, giving a necessary and sufficient condition for $k = 3$. In [13], Shyu considered the existence of a decomposition of $K_n$ into paths and cycles with $k$ edges, giving a necessary and sufficient condition for $k = 4$. Shyu [14] investigated the problem of decomposing $K_n$ into cycles and stars with $k$ edges, settling the case $k = 4$. Recently, Lee [5, 6] established necessary and sufficient conditions for the existence of a $\{C_k, S_k\}$-decomposition of a complete bipartite graph and $\{P_k, S_k\}$-decomposition of a balanced complete bipartite graph. In this paper, we consider the existence of a $\{C_k, P_k, S_k\}$-decomposition of the balanced complete bipartite graph, giving necessary and sufficient conditions.

2 Preliminaries

Let $G$ be a graph. The degree of a vertex $x$ of $G$, denoted by $\deg_G x$, is the number of edges incident with $x$. The vertex of degree $k$ in $S_k$ is the center of $S_k$. For $A \subseteq V(G)$ and $B \subseteq E(G)$, we use $G[A]$ and $G - B$ to denote the subgraph of $G$ induced by $A$ and the subgraph of $G$ obtained by deleting $B$, respectively. When $G_1, G_2, \ldots, G_m$ are graphs, not necessarily disjoint, we write $G_1 \cup G_2 \cup \cdots \cup G_m$ or $\bigcup_{i=1}^m G_i$ for the graph with vertex set $\bigcup_{i=1}^m V(G_i)$ and edge set $\bigcup_{i=1}^m E(G_i)$. When the edge sets are disjoint, $G = \bigcup_{i=1}^m G_i$ expresses the decomposition of $G$ into $G_1, G_2, \ldots, G_m$. $nG$ is the short notation for the union of $n$ copies of disjoint graphs isomorphic to $G$. Let $(v_0, v_1, \ldots, v_{k-1})$ denote the cycle $C_k$ with vertices $v_0, v_1, \ldots, v_{k-1}$ and edges $v_0v_1, v_1v_2, \ldots, v_{k-1}v_0$, let $v_0v_1\ldots v_k$ denote the path $P_k$ with vertices $v_0, v_1, \ldots, v_k$ and edges $v_0v_1, v_1v_2, \ldots, v_{k-1}v_k$ and let $(v_0; v_1, v_2, \ldots, v_k)$ denote the star $S_k$ with centered at vertex $v_0$ and $v_1, v_2, \ldots, v_k$ are other vertices. For any vertex $x$ of a digraph $G$, the outdegree $\deg^+_G x$ (respectively, indegree $\deg^-_G x$) of $x$ is the number of arcs incident from (respectively, to) $x$. 
Proposition 2.1. (Sotteau [15]) For positive integers \( m, n \) and \( k \), the graph \( K_{m,n} \) has a \( C_k \)-decomposition if and only if \( m, n \) and \( k \) are even, \( k \geq 4 \), \( \min\{m, n\} \geq k/2 \), and \( mn \equiv 0 \pmod{k} \).

Proposition 2.2. (Ma et al. [8]) For positive integers \( n \) and \( k \), the graph obtained by deleting a 1-factor from \( K_{n,n} \) has a \( C_k \)-decomposition if and only if \( n \) is odd, \( k \) is even, \( 4 \leq k \leq 2n \), and \( n(n-1) \) is divisible by \( k \).

Lemma 2.3. If \( k \) is an even integer with \( k \geq 4 \), then there exist \( (k/2-1) \) edge-disjoint \( k \)-cycles in \( K_{k/2,k} \).

Proof. If \( k \equiv 0 \pmod{4} \), then \( k/2 \) is even. By Proposition 2.1, there exists a \( C_k \)-decomposition \( \mathcal{H} \) of \( K_{k/2,k} \) with \( |\mathcal{H}| = k/2 \), in which \( k \)-cycles are edge-disjoint. If \( k \equiv 2 \pmod{4} \), then \( k/2 \) is odd. Proposition 2.2 implies that \( K_{k/2,k/2} \) with a 1-factor removed has a \( C_k \)-decomposition \( \mathcal{H}' \) with \( |\mathcal{H}'| = (k-2)/4 \). Hence there exist \( 2(k-2)/4 = k/2 - 1 \) edge-disjoint \( k \)-cycles in \( K_{k/2,k} \). This completes the proof.

Proposition 2.4. (Parker [9]) There exists a \( P_k \)-decomposition of \( K_{m,n} \) if and only if \( mn \equiv 0 \pmod{k} \) and one of the following cases holds.

<table>
<thead>
<tr>
<th>Case</th>
<th>( k )</th>
<th>( m )</th>
<th>( n )</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>even</td>
<td>even</td>
<td>even</td>
<td>( k \leq 2m, k \leq 2n ), not both equalities</td>
</tr>
<tr>
<td>2</td>
<td>even</td>
<td>even</td>
<td>odd</td>
<td>( k \leq 2m - 2, k \leq 2n )</td>
</tr>
<tr>
<td>3</td>
<td>even</td>
<td>odd</td>
<td>even</td>
<td>( k \leq 2m, k \leq 2n - 2 )</td>
</tr>
<tr>
<td>4</td>
<td>odd</td>
<td>even</td>
<td>even</td>
<td>( k \leq 2m - 1, k \leq 2n - 1 )</td>
</tr>
<tr>
<td>5</td>
<td>odd</td>
<td>even</td>
<td>odd</td>
<td>( k \leq 2m - 1, k \leq n )</td>
</tr>
<tr>
<td>6</td>
<td>odd</td>
<td>odd</td>
<td>even</td>
<td>( k \leq m, k \leq 2n - 1 )</td>
</tr>
<tr>
<td>7</td>
<td>odd</td>
<td>odd</td>
<td>odd</td>
<td>( k \leq m, k \leq n )</td>
</tr>
</tbody>
</table>

By Proposition 2.4, the following result can be obtained.

Lemma 2.5. If \( k \) is an even integer with \( k \geq 4 \), then there exist \( k/2 \) edge-disjoint \( k \)-paths in \( K_{k/2,k} \).

Proposition 2.6. (Yamamoto et al. [18]) For integers \( m \) and \( n \) with \( m \geq n \geq 1 \), the graph \( K_{m,n} \) has an \( S_k \)-decomposition if and only if \( m \geq k \) and

\[
\begin{cases}
  m \equiv 0 \pmod{k} & \text{if } n < k \\
  mn \equiv 0 \pmod{k} & \text{if } n \geq k.
\end{cases}
\]

3 Main results

The goal of this paper is to settle the \( \{C_k, P_k, S_k\} \)-decomposition problem for \( K_{n,n} \). We prove the following theorem.
Main Theorem. Let $k$ and $n$ be positive integers. The graph $K_{n,n}$ has a \{C_k, P_k, S_k\}-decomposition if and only if $k$ is even, $4 \leq k \leq n$ and $n^2$ is divisible by $k$.

We first give necessary conditions for a \{C_k, P_k, S_k\}-decomposition of $K_{n,n}$.

Lemma 3.1. If $K_{n,n}$ has a \{C_k, P_k, S_k\}-decomposition, then $k$ is even, $4 \leq k \leq n$ and $n^2 \equiv 0 \pmod{k}$.

Proof. Since bipartite graphs contain no odd cycle, $k$ is even. In addition, the minimum length of a cycle and the maximum size of a star in $K_{n,n}$ are 4 and $n$, respectively, we have $4 \leq k \leq n$. Finally, the size of each member in the decomposition is $k$ and $|E(K_{n,n})| = n^2$; thus $n^2 \equiv 0 \pmod{k}$. 

Throughout this paper, let $(A, B)$ denote the bipartition of $K_{n,n}$, where $A = \{a_0, a_1, \ldots, a_{n-1}\}$ and $B = \{b_0, b_1, \ldots, b_{n-1}\}$. We begin the discussion with the smallest value of $k$, namely $k = 4$.

Lemma 3.2. For an even integer $n \geq 4$, then $K_{n,n}$ has a \{C_4, P_4, S_4\}-decomposition.

Proof. First, $K_{4,4}$ can be decomposed into the following one copy of $C_4$, two copies of $P_4$ and one copy of $S_4$: $(b_0, a_0, b_1, a_1)$, $b_2a_0b_3a_1b_1$, $b_3a_1b_2a_0b_0$ and $(a_3; b_0, b_1, b_2, b_3)$. Note that $K_{n,n} = K_{4,4} \cup K_{n-4,4} \cup K_{n,n-4}$ for $n \geq 6$. In addition, by Proposition 2.1, $K_{n-4,4}$ and $K_{n,n-4}$ have $C_4$-decompositions. Hence there exists a \{C_4, P_4, S_4\}-decomposition of $K_{n,n}$ for even $n \geq 4$.

With Lemma 3.2 in mind, it is assumed that $k \geq 6$ in the sequel. We now show that the necessary conditions are also sufficient. The proof is divided into cases $n = k$, $k < n < 2k$, and $n \geq 2k$, which are treated in Lemmas 3.3, 3.4, and 3.5, respectively.

Lemma 3.3. For an even integer $k \geq 6$, then $K_{k,k}$ has a \{C_k, P_k, S_k\}-decomposition.

Proof. We distinguish two cases by the values of $k$.

Case 1. $k \equiv 0 \pmod{4}$.

Then $k/2$ is even and $K_{k,k} = 2K_{k/2,k/2+2} \cup K_{k,k/2-2}$ for $k \geq 6$. By Propositions 2.1 and 2.4, $2K_{k/2,k/2+2}$ has a $C_k$-decomposition and a $P_k$-decomposition. In addition, by Proposition 2.6, $K_{k,k/2-2}$ has an $S_k$-decomposition. Hence, $K_{k,k}$ has a \{C_k, P_k, S_k\}-decomposition.

Case 2. $k \equiv 2 \pmod{4}$.

Let $G = K_{k,k}[[a_0, a_1, \ldots, a_{k/2-1}] \cup \{b_0, b_1, \ldots, b_{k/2}\}]$, $F = K_{k,k}[[a_0, a_1, a_{k/2+1}, \ldots, a_{k-1}] \cup \{b_0, b_1, \ldots, b_{k/2}\}]$ and $H = K_{k,k}[[a_0, a_1, \ldots, a_{k-1}] \cup \{b_{k/2+1}, b_{k/2+2} \ldots,$
Finally, \( C_{k,k} = G \cup F \cup H \). We will show that \( G \cup F \) can be decomposed into two copies of \( C_k \) and \((k/2 - 1)\) copies of \( P_k \) as follows.

First, a decomposition of \( G \cup F \) into \( k \)-paths is given by the \((k/2 + 1)\) following paths:

\[
P^{(i,j)} = b_{2j}a_{ik/2}b_{2j+1}a_{ik/2+1} \ldots b_{2j+k/2-1}a_{ik/2+(k/2-1)}b_{2j+k/2}
\]

for \( i = 0, 1 \) and \( j = 0, 1, \ldots, (k-2)/4 \), where the subscripts of \( b \) are taken modulo \((k/2 + 1)\).

Next, let \( P^{(0,1)'} \) and \( P^{(1,0)'} \) be two new \( k \)-paths obtained by

\[
P^{(0,1)'} = P^{(0,1)} \cup \{a_{k/2-1}b_0, a_{k/2}b_1\} - \{a_{k/2-1}b_0, a_{k/2}b_1\},
\]

\[
P^{(1,0)'} = P^{(1,0)} \cup \{a_{k/2-1}b_0, a_{k/2}b_1\} - \{a_{k/2}b_0, a_{k/2}b_1\}.
\]

Finally, \( C^{(1)} \) and \( C^{(2)} \) are two \( k \)-cycles are obtained by

\[
C^{(1)} = P^{(0,0)} \cup \{a_{k/2-1}b_0\} - \{a_{k/2-1}b_k/2\},
\]

\[
C^{(2)} = P^{(1,0)'} \cup \{a_{k/2-1}b_k/2\} - \{a_{k/2-1}b_0\}.
\]

Thus \( G \cup F \) can be decomposed into \( C^{(1)}, C^{(2)}, P^{(0,1)'}, P^{(1,1)} \) and \( P^{(i,j)} \) for \( i = 0, 1, j = 2, 3, \ldots, (k-2)/4 \). On the other hand, by Proposition 2.6 \( H \) has an \( S_k \)-decomposition. Hence \( K_{k,k} \) has a \( \{C_k, P_k, S_k\} \)-decomposition.

\[\square\]

**Lemma 3.4.** Let \( k \) be a positive even integer and let \( n \) be a positive integer with \( 6 \leq k < n < 2k \). If \( n^2 \) is divisible by \( k \), then \( K_{n,n} \) has a \( \{C_k, P_k, S_k\} \)-decomposition.

**Proof.** Let \( n = k + r \). From the assumption \( k < n < 2k \), we have \( 0 < r < k \). Let \( t = r^2/k \). Since \( k \mid n^2 \), we have \( k \mid r^2 \), which implies that \( t \) is a positive integer. The proof is divided into two parts according to the value of \( t \).

**Case 1.** \( t = 1 \).

Then \( k = r^2 \). This implies that \( r \geq 4 \) and \( k \geq 4r \). Let

\[
G_1 = K_{k,n}[\{a_r, a_{r+1}, \ldots, a_{k-1}\}, \{b_r, b_{r+1}, \ldots, b_{k-1}\}],
\]

\[
G_2 = K_{k,n}[\{a_0, a_1, \ldots, a_{r-1}\}, \{b_0, b_1, \ldots, b_{k-1}\}],
\]

\[
G_3 = K_{k,n}[\{a_r, a_{r+1}, \ldots, a_{k+r-1}\}, \{b_0, b_1, \ldots, b_r\}],
\]

\[
G_4 = K_{k,n}[\{a_k, a_{k+1}, \ldots, a_{k+r-1}\}, \{b_r, b_{r+1}, \ldots, b_{k+r-1}\}],
\]

\[
G_5 = K_{k,n}[\{a_0, a_1, \ldots, a_{k-1}\}, \{b_k, b_{k+1}, \ldots, b_{k+r-1}\}].
\]

Note that \( G_1 = K_{k-k,k-r} \) and \( G_i = K_{k,r} \) (or \( K_{r,k} \)) for \( 2 \leq i \leq 5 \). Clearly \( K_{n,n} = G_1 + G_2 + G_3 + G_4 + G_5 \). By Propositions 2.1 and 2.6, \( G_5 \) has a \( C_k \)-decomposition and \( G_i \) has a \( S_k \)-decomposition for \( 2 \leq i \leq 5 \).

By Sotteau ([15], p.77), there are \((r-1)^2\) copies of \( k \)-cycles in the decom-
position \( \mathcal{D} \) of \( G_1 \). We take two \( k \)-cycles in \( \mathcal{D} \):

\[
C_{0,0} = (a_1, b_{k/4-1}, a_{k/2-2}, b_{k/4-2}, a_{k/2-4}, b_{k/4-3}, a_{k/2-6}, b_{k/4-4}, \ldots,
, a_2, b_0, a_0, b_{k/2-1}, a_{k/2-1}, b_{k/2-2}, a_{k/2-3}, b_{k/2-3}, a_{k/2-5}, b_{k/2-4}, \ldots, a_3, b_{k/4}),
\]

\[
C_{0,1} = (a_{r+k/2-2}, b_{k/4-1}, a_{r+1}, b_{k/4}, a_{r+3}, b_{k/4+1}, a_{r+5}, b_{k/4+2}, \ldots,
, a_{r+k/2-1}, b_{k/2-1}, a_r, b_r, a_{r+2}, b_1, a_{r+4}, b_2, a_{r+6}, b_3, \ldots, a_{r+k/2-4}, b_{k/4-2})
\]

and interchange the two edges \( a_1b_{k/4-1} \) and \( a_{r+k/2-2}b_{k/4-1} \). In doing so, we obtain two paths: \( P_{0,0} = C_{0,0} - \{a_1b_{k/4-1}\} \cup \{a_{r+k/2-2}b_{k/4-1}\} \) and \( P_{0,1} = C_{0,1} - \{a_{r+k/2-2}b_{k/4-1}\} \cup \{a_1b_{k/4-1}\} \). Thus \( K_{n,n} \) can be decomposed into \((r-1)^2 - 2\) copies of \( C_k \), two copies of \( P_k \) and \( 4r \) copies of \( S_k \). This settles the case 1.

Case 2. \( t \geq 2 \).

Let \( G'_0 = K_{n,n}[\{a_0, a_1, \ldots, a_{k/2-1}\} \cup \{b_0, b_1, \ldots, b_{k-1}\}] \) and \( G'_1 = K_{n,n}[\{a_{k/2}, a_{k/2+1}, \ldots, a_{k-1}\} \cup \{b_0, b_1, \ldots, b_{k-1}\}] \); and \( H' = K_{n,n}[\{a_0, a_1, \ldots, a_{k+r-1}\} \cup \{b_0, b_1, \ldots, b_{k+r-1}\}] \). Clearly \( G_{n,n} = G'_0 \cup G'_1 \cup F' \cup H' \). Note that \( G'_0 \) and \( G'_1 \) are isomorphic to \( K_{k/2,k} \), \( H' \) is isomorphic to \( K_{r,r} \), and \( F' \) is isomorphic to \( K_{r,k} \), which can be decomposed into \( r \) copies of \( S_k \) by Proposition 2.6.

Let \( p_0 = \lceil t/2 \rceil \) and \( p_1 = \lfloor t/2 \rfloor \). In the following, we will show that \( G'_0 \) can be decomposed into \( p_0 \) copies of \( C_k \) and \( k/2 \) copies of \( S_{k-2p_0} \), \( G'_1 \) can be decomposed into \( p_1 \) copies of \( P_k \) and \( k/2 \) copies of \( S_{k-2p_1} \), \( H' \) can be decomposed into \( k/2 \) copies of \( S_{2p_0} \), \( k/2 \) copies of \( S_{2p_1} \) and \( r \) copies of \( S_k \).

We first show the required decomposition of \( G'_0 \) and \( G'_1 \). Since \( r < k \), we have \( t < r \). Thus, \( p_0 = \lceil t/2 \rceil \leq (t+1)/2 \leq r/2 < k/2 \), which implies \( p_i \leq k/2 - 1 \) for \( i \in \{0,1\} \). This assures us that there exist \( p_0 \) edge-disjoint \( k \)-cycles in \( G'_0 \) and \( p_1 \) edge-disjoint \( k \)-paths in \( G'_1 \) by Lemmas 2.3 and 2.5, respectively. Suppose that \( Q_{0,0}, Q_{0,1}, \ldots, Q_{0,p_0-1} \) and \( Q_{1,0}, Q_{1,1}, \ldots, Q_{1,p_1-1} \) are edge-disjoint \( k \)-cycles and \( k \)-paths in \( G'_0 \) and \( G'_1 \), respectively. Let \( W'_i = G'_i - E(\bigcup_{h=0}^{p_i-1} Q_{i,h}) \) and \( X_{i,j} = W'_i[\{a_{ik/2+j}\} \cup \{b_0, b_1, \ldots, b_{k-1}\}] \) where \( i = 0,1 \) and \( j = 0,1, \ldots, k/2 - 1 \). Since \( \deg_{G'_i} a_{ik/2+j} = k \) and each \( Q_{i,h} \) uses two edges incident with \( a_{ik/2+j} \) for each \( i \) and \( j \), we have \( \deg_{W'_i} a_{ik/2+j} = k - 2p_i \). Hence \( X_{i,j} \) is a \((k-2p_i)\)-star with center \( a_{ik/2+j} \) for \( i = 0,1 \) and \( j = 0,1, \ldots, k/2 - 1 \).

Next we show the required star-decompositions of \( H' \). Equivalently we need show that there exists an orientation of \( H' \) such that, for \( i = 0,1 \), \( j = 0,1, \ldots, k/2 - 1 \), and \( w = k,k+1, \ldots, k+r-1 \),

\[
\begin{align*}
\deg_{H'}^+ a_{ik/2+j} &= 2p_i \quad (1) \\
\deg_{H'}^+ b_w &= k \quad (2)
\end{align*}
\]

We begin the orientation. For \( j = 0,1, \ldots, k/2 - 1 \) the edges \( a_j b_{k+(2p_0)j}, a_j b_{k+(2p_0)j+1}, \ldots, a_j b_{k+(2p_0)j+2p_0-1} \) and \( a_{k/2+j} b_{(p_0+1)k+(2p_1)j}, a_{k/2+j} b_{(p_0+1)k+(2p_1)j+1}, \ldots, a_{k/2+j} b_{(p_0+1)k+(2p_1)j+2p_1-1} \) are oriented from \( a_{ik/2+j} \) where the subscripts of
Lemma 3.5. Let \( k \) be a positive even integer and let \( n \) be a positive integer with \( 6 \leq k \leq n/2 \). If \( n^2 \) is divisible by \( k \), then \( K_{n,n} \) has a \( \{C_k,P_k,S_k\} \)-decomposition.

Proof. Let \( n = qk + r \) where \( q \) and \( r \) are integers with \( 0 \leq r < k \). From the assumption of \( k \leq n/2 \), we have \( q \geq 2 \). Note that

\[
K_{n,n} = K_{qk+r,qk+r} = K_{(q-1)k,(q-1)k} \cup K_{k+r,(q-1)k} \cup K_{(q-1)k,k+r} \cup K_{k+r,k+r}.
\]

Trivially, \(|E(K_{(q-1)k,(q-1)k})|, |E(K_{k+r,(q-1)k})| \) and \(|E(K_{(q-1)k,k+r})|\) are multiples of \( k \). Thus \((k + r)^2 \equiv 0 \pmod{k}\) from the assumption that \( n^2 \) is divisible by \( k \). By Proposition 2.6, \( K_{(q-1)k,(q-1)k} \), \( K_{k+r,(q-1)k} \) and \( K_{(q-1)k,k+r} \) have \( S_k \)-decomposition.
The case of \( r = 0 \), by Lemma 3.3, we obtain that \( K_{k,k} \) has a \( \{C_k, P_k, S_k\} \)-decomposition. In addition, by Lemma 3.4, \( K_{k+r,k+r} \) has a \( \{C_k, P_k, S_k\} \)-decomposition for \( 0 < r < k \). Hence there exists a \( \{C_k, P_k, S_k\} \)-decomposition of \( K_{n,n} \).

Now Lemmas 3.1, 3.2, 3.3, 3.4 and 3.5 serve to prove the Main Theorem.

References


\{C_k, P_k, S_k\}\text{-decompositions of balanced complete bipartite graphs} \hspace{0.5cm} 321


\textbf{Received: June 3, 2016; Published: July 20, 2016}