Properties of GV-Semigroups

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Abstract

A semigroup $S$ is called a GV-semigroup if for every element $a$ of $S$ there exists a positive integer $n$ such that $a^n$ is regular and every regular element of $S$ is completely regular element. Some new properties of GV-semigroups are shown in this paper. Some characters of Greens relations in GV-semigroups are first given. Moreover, idempotents and ideals in GV-semigroups are investigated.

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1 Introduction and preliminaries

A semigroup $S$ is called an eventually regular semigroup if for every $a \in S$ there exists a positive integer $n$ such that $a^n$ is regular. Let us denote by $r(a)$ the least positive integer $n$ such that $a^n$ is regular element of $S$ and call it the regular index of $a$. If every regular element of an eventually regular semigroup $S$ is completely regular, then $S$ is called a GV-semigroup. From the definition of GV-semigroups, we obtain GV-semigroups are one special case of eventually regular semigroups. Furthermore, we can get GV-semigroups are extentions of completely regular semigroups in eventually regular semigroups. So the strategy to study GV-semigroups is to generalize known results for completely regular semigroups to GV-semigroups. Completely regular semigroups have been explored extensively. Especially, Petrich, M. [7] and Howie,
J.M. [6] generalized many properties of completely regular semigroups which were investigated by some algebraists such as Lajos, S. and Jones, P.R. etc. Bogdanovic, S. [1] generalized the part of the results for completely regular semigroups to GV-semigroups and obtained some results which were even new for completely regular semigroups.

The aim of this paper is to investigate some basic properties of GV-semigroups that have not been shown by anyone else. Firstly, Greens relations in GV-semigroups are studied, we can get some useful results for GV-semigroups. Secondly, we provide some characteristics of ideals and idempotents of GV-semigroups.

We shall use the standard terminology and notation of semigroup theory as in [1, 6, 7].

Let $S$ be an eventually regular semigroup, we will denote by $E_S$ the set of all idempotents of $S$, by $\langle E_S \rangle$ the subsemigroup of $S$ generated by $E_S$, by $\text{Reg} S$ the set of all regular elements of $S$, and by $\text{Gr} S$ the set of all completely regular elements of $S$. A semigroup $S$ is called a $\pi$-group if $S$ is an eventually regular semigroup and $S$ only has an idempotent. An eventually regular semigroup $S$ is called $r$-semigroup if $\langle ab \rangle^r = a^r(a).b^r(b)$ for all $a, b \in S$. Let $S$ be a semigroup and $I$ be an ideal of $S$, if $I \neq S$ then $I$ is called a proper ideal of $S$. We will denote by $I(S)$ the union of all the proper ideals, by $L(S)$ the union of all the left proper ideals, by $R(S)$ the union of all the right proper ideals, and by $I(a)$ the principle ideals of $S$ generated by $a$. A subsemigroup $B$ of semigroup $S$ is called a bi-ideal of $S$ if $BSB \subseteq B$. A subset $Q$ of a semigroup $S$ is called a quasi-ideal of $S$ if $QS \cap SQ \subseteq Q$.

Lemma 1.1 ([1]) $S$ is a GV-semigroup if and only if $S$ is eventually regular and every $H^*$ class contains an idempotent, i.e. $S$ is the union of $\pi$-group.

Lemma 1.2 ([2, 3]) Let $S$ be an eventually regular semigroup and $\rho$ be a congruence on $S$, $ap$ is an idempotent of $S/\rho$, then an idempotent $e$ can be found in $S$ such that $ape$.

Remark. $S$ is an eventually regular semigroup and $\rho$ is a group congruence on $S$, then $x\rho$ is an idempotent of $S/\rho$ for all $x \in \langle E_S \rangle$.

Lemma 1.3 ([7]) Let $S$ be a semigroup, then every ideal (left ideal, right ideal) of $S$ is a quasi-ideal of $S$ and every quasi-ideal of $S$ is a bi-ideal of $S$.

2 Greens relations

Let $S$ be an eventually regular semigroup, we define the equivalent relations $L^*, R^*, H^*, J^*, D^*$ on $S$ which generated by Greens relations on regular semigroup.
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\[ aL^*b \Rightarrow a^{r(a)}L^r(b), \quad aR^*b \Leftrightarrow a^{r(a)}R^r(b), \]
\[ aH^*b \Leftrightarrow a^{r(a)}H^r(b), \quad aJ^*b \Leftrightarrow a^{r(a)}J^r(b), \quad D^* = L^* \vee R^*. \]

Suppose \( \kappa = \{ L, R, D, H, J \} \), \( \kappa^* = \{ L^*, R^*, D^*, H^*, J^* \} \) be equivalent relations on \( S \). Suppose \( U \subseteq S \) then the subset \( \kappa^*|_U = \{(x, y) \mid x, y \in U, x \kappa^* y \} \) is called the restriction of \( \kappa^* \) to \( U \). \( \kappa^*_U = \{(x, y) \mid x, y \in U, \exists a, v \in U \text{ such that } xa \kappa^* y \} \).

We list some lemmas which will be used in the sequel.

**Lemma 2.1 [8]** Let \( S \) be an eventually regular semigroup and \( T \) be an eventually regular subsemigroup of \( S \), if \( \kappa \in \{ L, R, H \} \) then \( \kappa^*_T = \kappa^*|_T \) and \( D^* = L^* \cdot R^* = R^* \cdot L^* \).

**Lemma 2.2** Let \( S \) be an eventually regular semigroup and \( \kappa \in \{ L, R, H, D, J \} \), then \( \kappa|_{RegS} = \kappa^*|_{RegS} \).

**Proof.** Suppose \( a, b \in RegS \), i.e. \( r(a) = r(b) = 1 \) and \( aDb \). we have \( aD^*b \) immediately, hence \( D \subseteq D^* \). Conversely, let \( aD^*b \), \( a, b \in RegS \), that is, \( a = a^{r(a)}D^r(b) = b \), i.e. \( aDb \), hence \( D^* \subseteq D \) and so \( D|_{RegS} = D^*|_{RegS} \).

The remainder of the claim follows by the same way.

**Remark.** \( S \) is an eventually regular semigroup and \( \kappa = \{ L, R, H, D, J \} \), then \( \kappa \subseteq \kappa^* \) is not true in any condition.

**Lemma 2.3 [6, 7]** Let \( S \) be a completely simple semigroup, then \( ab \in R_a \cap L_b \) for all \( a, b \in S \) and \( \kappa = \{ L, R, H, D \} \) is a congruence on \( S \).

**Lemma 2.4** Let \( S \) be a \( \pi \)-group, then \( RegS = G_e \). (\( e \) is the only idempotent of \( S \), \( G_e \) is the maximum group generated by \( e \).)

**Lemma 2.5** Let \( S \) be a GV-semigroup, then there exists \( a' \in V(a) \) for any \( a \in Reg(S) \) such that \( aa' = a'a, aa'a = a \).

**Proposition 2.6** Let \( S \) be a GV-semigroup and \( T \) be an eventually regular subsemigroup of \( S \), then \( T \) is a GV-semigroup.

**Proof.** Suppose \( a \in T \) and \( T \) is an eventually regular subsemigroup of \( S \), then there exists \( m = r(a) \) such that \( a^m \in RegT \). Hence \( a^{2m} \in RegT \) and so there exists \( y \in T \) such that \( a^{2m}ya^{2m} = a^{2m} \). We put \( x = ya^{2m}y \in T \), whence \( x \) is a inverse of \( a^{2m} \) and \( a^mxa^m \cdot a^mxa^m = a^mxa^{2m}xa^m = a^mxa^m \), that is, \( a^mxa^m \in E_S \). Notice that \( a^mxa^m \in T \), we have \( a^mxa^m \in E_T \). By the completely regularity of \( a^m \) and lemma2.5, we put \( (a^m)^{-1} \in V(a^m) \) and \( a^m(a^m)^{-1} = (a^m)^{-1}a^m \), hence \( (a^mxa^m) \cdot a^m = a^m \cdot (a^m)^{-1}a^m \cdot xa^m = (a^m)^{-1}a^m \cdot a^m \cdot xa^m = a^m = (a^m)^{-1}a^mxa^m \cdot a^m = (a^m)^{-1}a^mxa^m = (a^m)^{-1}a^mxa^m = a^m \) and so \( a^m(xa^m) = a^mxa^m \), that is, \( a^mRa^mxa^m \). A similar argument will show that \( a^mL^*a^m \).
Consequently, \(a^mH_{a^m}xa^m\). By the lemma2.2, we have \(aH^*a^mH^*a^mxa^m, a \in T, a^mxa^m \in T\). And by lemma2.1, we know \(H^*_T = HS_S|_T = H^*_S \cap (T \times T)\), hence \(aH^*_Ta^mxa^m\), that is, every \(H^*\) class contains an idempotent and so \(T\) is a GV-semigroup by the lemma1.1.

Clearly, \((E_S)\) is an eventually regular subsemigroup of \(S\). By proposition2.6, we get the result immediately.

**Corollary 2.7** Let \(S\) be a GV-semigroup, then \((E_S)\) is a GV-semigroup.

**Proposition 2.8** Let \(S\) be a GV-semigroup, then \(D^* = J^*, D = J\).

**Proof.** Firstly, we show \(D^* = J^*\). Suppose \(aJ^*b\), then we have \(a, b\) lie in the same completely archimedean semigroup \(S_\alpha(\alpha \in LL\) is a semilattice). As \(S_\alpha\) is a completely archimedean semigroup, we obtain \(\text{Reg}_S(\alpha)\) is a completely simple semigroup \(S^*\). Hence \(a^mD^*b, r(a) = m, r(b) = n\). By lemma2.2, we know \(a^mD^*b^m\), that is, \(aD^*a^mD^*bD^*b\), i.e. \(aD^*b\). Thus, \(J^* \subseteq D^*\). Conversely, let \(aD^*b\), that is, \(a^mD^*b^m, r(a) = m, r(b) = n\). Hence \(a^mJ^*b\), i.e. \(aJ^*b\) and so \(D^* \subseteq J^*\). Therefore \(D^* = J^*\), as required.

Finally, to show \(D = J\). we first show \(J \subseteq D\). Suppose \(aJb, a, b \in S\), then there exist \(x, y, u, v \in S\) such that \(a = xyb, b = uav\). Notice that \(a = xby = x(uav)y = \ldots = (ux)^t(a(vy)^t\). Dually, \(b = (ux)^tb(vy)^t\). We put \(t = \max\{r(xu), r(ux), r(vy), r(vy)\}\) and since \(S\) is the union of \(\pi\)-group. we get \((ux)^t \in G_a, (ux)^t \in G_g, (vy)^t \in G_h, (vy)^t \in G_h, e, f, g, h \in E_S\). Whence \(ea = a, gb = b = bh\). For \(exb(y) = exby = ea = a; a.\{(vy)^t-1, v[(vy)^t-1]\} = exby.(vy)^t-1v[(vy)^t-1] = exby[(vy)^t-1] = exbh = exb[(vy)^t-1] \subseteq G_b\), thus \(aLExb\). Next to prove \(bLExb\). As \((ex)b = exb, \{(ux)^t+1-1, u(xu)^t\}.exb = [(ux)^t+1-1, u(xu)^txb = [(ux)]^t+1, (ux)^t+1, b = gb = b, [(ux)]^t+1 \subseteq G_g\), thus \(bLExb\). Hence \(aLExbLb\), that is, \(aL.Rb\), i.e. \(aDb\) and so \(J \subseteq D\). Clearly, \(D \subseteq J\) therefore, \(D = J\).

**Corollary 2.9** Let \(S\) be a completely archimedean semigroup, then \(D^* = J^*, D = J\); Let \(S\) be an eventually regular semigroup, then \(D^* \subseteq J^*\); Let \(S\) be a simple GV-semigroup, then \(S\) is completely regular semigroup.

**Proposition 2.10** Let \(S\) be a GV-semigroup and \(T\) be an eventually regular subsemigroup of \(S\), then \(D^*_T = D^*_S|_T\).

**Proof.** Firstly, \(D^*_T \subseteq D^*_S|_T\), clearly. On the other hand, Suppose \(aD^*_S|_Tb\), that is, \(aD^*b\) for \(a, b \in T\). Since \(S\) is a GV-semigroup and by proposition2.8, we have \(D^* = J^*, i.e. aJ^*b\). Hence \(a, b\) lie in the same completely archimedean semigroup \(S_\alpha(\alpha \in LL\) is a semilattice). Put \(r(a) = m, r(b) = n\), then \(a^m, b^n\) lie in the completely simple semigroup \(S^\ast\) in \(S_\alpha\). By lemma2.3, we know \(a^m b^n \in R_{a^m} \cap L_{b^n}\), so \(a^m R a^m b^n\) and \(S^\ast\) is a regular semigroup. By lemma2.2,
we get \(a^mR^*a^m b^n\), i.e. \(aR^*a^m b^n\). Dually, we obtain \(bL^*a^m b^n\), \(a, b, a^m b^n \in T\). By lemma 2.1, we have \(aR^*a^m b^n L^* b\), that is, \(aD_T b\). Hence \(D^*_T \supseteq D_S | T\) and so \(D^*_T = D_S | r\).

**Proposition 2.11** Let \(S\) be a completely archimedean r-semigroup, then \(L^*, R^*, H^*, D^*\) are congruences on \(S\).

**Proof.** Firstly, we show \(H^*\) is a congruence on \(S\). Let \(aH^*b, \forall c \in S\). As \(S\) is the nil-extension of completely simple semigroup \(S^*\), we have \(a^m, b^n, c^t \in \text{Reg}\(S = S^*\), \(r(a) = m, r(b) = n, r(c) = t\). By lemma 2.2 and \(aH^*b\), i.e. \(a^mHb^n\), hence \(a^mH^*b^n\). For \(c^t, a^m, b^n \in \text{Reg}\(S = S^*\), we have \(a^m c^t Hb^n c^t\) by lemma 2.3. And since \(S\) is r-semigroup, whence \((ac)^{r(ac)} = a^{c(a) c^t(c)} = a^m c^t Hb^n c^t = b^n(b^t c^t(c)) = (bc)^{r(bc)}\), that is, \(acH^*bc\). Thus \(H^*\) is right compatible. Dually, \(H^*\) is left compatible. Clearly, \(H^*\) is an equivalence on \(S\). Consequently, \(H^*\) is a congruence on \(S\).

Finally, the remainder of the results can be proved by the same technique.

## 3 Ideals and idempotents

**Theorem 3.1** Let \(S\) be an eventually regular semigroup, then the following conditions are equivalent:

1. \(S\) is a GV-semigroup;
2. every principle ideal of \(S\) is a GV-semigroup;
3. every ideal of \(S\) is a GV-semigroup;
4. every left(right) ideal of \(S\) is a GV-semigroup;
5. every bi-ideal of \(S\) is a GV-semigroup;
6. every quasi-ideal of \(S\) is a GV-semigroup.

**Proof.** (1) \(\iff\) (2) Firstly, we show (1) \(\Rightarrow\) (2). Let \(S\) be a GV-semigroup and \(I\) be any ideal of \(S\). We can obtain \(I\) is a subsemigroup of \(S\), immediately. For \(S\) is an eventually regular semigroup, we have there exists \(m \in N\) such that \(a^m \in \text{Reg}\(S\) for any \(a \in I\). Notice that \(a^m = a^{m-2}aa \subseteq SIS \subseteq I, (m > 2), m \leq 2, a^m \in I\) clearly, hence \(I\) is an eventually regular subsemigroup. Suppose \(\forall a \in \text{Reg}\(I \subseteq \text{Reg}\(S), then exists \(x \in S\) such that \(axa = a, ax = xa\). we put \(t = xax \in SIS \subseteq I\), then we know \(t \in V(a), ata = a, at = ta\), that is, \(a \in \text{Gr}\(I\). Therefore \(I\) is a GV-semigroup. Clearly, every principle ideal of \(S\) is also a GV-semigroup. Next, we show (2) \(\Rightarrow\) (1). For \(\forall a \in \text{Reg}\(S\), we have \(a \in \text{I}(a)\) and let \(I(a)\) be a GV-semigroup. So there exists \(x \in I(a) \subseteq S\) such that \(axa = a, ax = xa\), i.e. \(a \in \text{Gr}\(S\). Thus \(S\) is a GV-semigroup.

(2) \(\Rightarrow\) (3) By (1) \(\iff\) (2), we have (2) \(\Rightarrow\) (3), that is (1) \(\Rightarrow\) (3). From the proof of (1) \(\Rightarrow\) (2), we can get (1) \(\Rightarrow\) (3) immediately, hence (2) \(\Rightarrow\) (3).
(3) $\Rightarrow$ (4) As $S$ is an ideal of $S$ itself. By (3), we get $S$ is a GV-semigroup. Suppose $L$ is any left ideal of $S$, we have $L$ is a subsemigroup of $S$, immediately. For $S$ is an eventually regular semigroup, we have there exists $m \in N$ such that $a^m \in \text{Reg}S$ for any $a \in I$. Notice that $a^m = a^{m-1}.a \subseteq SL \subseteq L(m > 1)$, $m = 1, a^m \in I$, clearly. Suppose $\forall a \in \text{Reg}L \subseteq \text{Reg}S = \text{Gr}S$. By lemma 2.5, we get there exists $x \in V(a)$ such that $axa = a, ax = xa$. Hence $x = xax = xxa \in SL \subseteq L$ and so $L$ is a GV-semigroup. By the same way, any right ideal of $S$ is also a GV-semigroup.

(4) $\Rightarrow$ (5) Firstly, since $S$ is a left ideal of $S$ itself, by (4) we know $S$ is a GV-semigroup. Let $B$ be any bi-ideal of $S$, we get $B$ is a subsemigroup of $S$ from its definition. For $S$ is a GV-semigroup, we have there exists $m \in N$ such that $a^m \in \text{Reg}S$ for any $a \in B, a^m \in B$, and there exist $x \in V(a^m)$ such that $a^mxa^m = a^m, xa^m = a^m x, xam x = x$ by lemma 2.5. Notice that $x = xam x = xam x = a^n xxxa^m \in BSB \subseteq B$ and so $B$ is a GV-semigroup.

(5) $\Rightarrow$ (6) By lemma 1.3, we can prove it immediately.

(6) $\Rightarrow$ (1) Notice that $S.S \cap S.S \subseteq S$, so $S$ is a quasi-ideal of $S$ itself. By (6), we have $S$ is a GV-semigroup.

**Remark** This theorem means that the completely regularity of a GV-semigroup $S$ is a hereditary property concerning the all kinds of ideals of $S$.

**Proposition 3.2** Let any proper ideal (proper left ideal, proper right ideal) of $S$ be a GV-semigroup if and only if $I(S) (L(S), R(S))$ is a GV-semigroup.

**Proof.** To prove necessity, suppose any proper ideal $I$ of $S$ is a GV-semigroup. Let $\forall a, b \in I(S), a \in I$, hence $ab \in IS \subseteq I \subseteq I(S)$ and so $I(S)$ is a subsemigroup of $S$. For any $a \in I(S), a \in I$, then there exists $m \in N$ and $x \in I \subseteq I(S)$ such that $a^mxa^m = a^m$. Hence $I(S)$ is an eventually regular semigroup. Let $\forall a \in \text{Reg}(I(S), a \in I)$, then there exists $a' \in V(a)$ such that $aa'a = a, aa' = a'a$ by lemma 2.5. Hence $a' = a'aa' \in SIS \subseteq I \subseteq I(S)$ and so $I(S)$ is a GV-semigroup.

We now prove sufficiency. Put $I$ as any proper ideal of $S$. Then we know $I$ is a subsemigroup of $S$ immediately. For $\forall a \in I \subseteq I(S)$, there exists $m \in N$ such that $a^m \in \text{Reg}(I(S))$. By lemma 2.5, notice that $a^m \in I, x = xam x \in SIS \subseteq I$, we get $I$ is a GV-semigroup.

The condition of proper left(right) ideal can be proved as the same way.

**Proposition 3.3** Let $S$ be a GV-semigroup and $r(ab) = m, r(ba) = n$, if $(ab)^m \in E_S$ and $m \geq n$ then $(ba)^m \in E_S$.

**Proof.** Since $S$ is a GV-semigroup and $m \geq n$, then we can get $(ba)^m \in G_{(ba)^m}((ba)^m)^{-1}$ and put $(ba)^m .[(ba)^m]^{-1} = (ba)^n .[(ba)^n]^{-1} = e, [(ba)^m]^{-1} \in V[(ba)^m] \subseteq G_e, [(ba)^n]^{-1} \in V[(ba)^n] \subseteq G_e$. Notice $(ba)^m = (ba)^m .(ba)^m .[(ba)^m]^{-1} = b.(ab)^m .a.(ab)^m .[(ba)^m]^{-1} = b(ab)^m a(ba)^m .[(ba)^m]^{-1} = b(ab)^2m a(ba)^m .[(ba)^m]^{-1} = (ba)^2m .e = (ba)^{2m}$, that is, $(ba)^m \in E_S$. 
Theorem 3.4 Let $S$ be a GV-semigroup and $\rho$ be a congruence on $S$, $\forall e, f \in E_S$. Then $eSf, eS, Sf, e\rho$ are all GV-semigroups.

Proof. We get $eSf$ is a subsemigroup of $S$, immediately. $\forall a \in eSf$, then there exists $m \in N$ such that $a^m \in RegS = GrS, a^m \in eSf$. By lemma 2.5, we know there exists $x \in V(a^m)$ such that $a^m xa^m = a^m, a^m x = xa^m, xa^mx = x$ and there exists $b \in S$ such that $a^m = ebf$. Notice that $x = xa^mx = a^m xx = ebfxxa^m = xebfxxa^m = ebfxxebf$, i.e. $x \in eSf$. That is, $eSf$ is a GV-semigroup.

By the same technique, we obtain $eS, Sf$ are both GV-semigroups.

Finally, we show $e\rho(\forall e \in E_S)$ is a GV-semigroup. Suppose $\forall a, b \in e\rho$, then $abpeee = e$, that is, $e\rho$ is a subsemigroup of $S$. There exists $m \in N$ such that $a^m \in RegS, a^m \in e\rho$ for any $a \in e\rho$. By lemma 2.5, we have there exists $x \in V(a^m)$ such that $a^m xa^m = a^m, a^m x = xa^m, xa^mx = x$ and $a^m \rho e \rho(a^m)^3$. Notice that $x = xa^mx = (x^2a^m)^2(a^m)^3 = x, xa^m a^m = xa^m xa^m a^m = xa^m xa^m = a^m, i.e. xpa^m \rho e$. Consequently, $e\rho$ is a GV-semigroup.

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