International Journal of Contemporary Mathematical Sciences
Vol. 11, 2016, no. 1, 1 - 8
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ijcms.2016.5943

Some Properties of $\Gamma_{q,k}(t)$ and Related Functions

Chrysi G. Kokologiannaki

Department of Mathematics
University of Patras
26500 Patras, Greece

Abstract

Properties, monotonicity results and inequalities for the functions $\Gamma_{q,k}(t)$, $\psi_{q,k}(t)$ and $B_{q,k}(t,s)$, for $t, s, k > 0$ and $0 < q < 1$ are derived. The obtained results are new and the monotonicity results of the functions $\Gamma_{q,k}(t)$ and $\psi_{q,k}(t)$ generalize known results for the functions $\Gamma_q(t)$, $\psi_q(t)$ and $\Gamma_k(t)$, $\psi_k(t)$.

Mathematics Subject Classification: 33D05; 33D99

Keywords: q,k-Gamma functions, q,k-psi functions, q,k-Beta functions, monotonicity, inequalities

1 Introduction

In [5] the authors introduced the q,k-generalized Pochhammer symbol as

$$[t]_{n,k} = [t][t+k]...[t+(n-1)k] = \prod_{j=0}^{n-1}[t+jk] \tag{1}$$

where

$$[t] = \frac{1-q^t}{1-q}, \quad t \in R, \quad 0 < q < 1, \tag{2}$$

and using this generalization they constructed $\Gamma_{q,k}(t)$ and $B_{q,k}(t,s)$ the q,k generalized gamma and beta functions as follows:

$$\Gamma_{q,k}(t) = \frac{(1-q^k)_{q,k}^{\frac{t}{k}-1}}{(1-q)^{\frac{t}{k}-1}}, \quad t > 0 \tag{3}$$

and

$$B_{q,k}(t,s)=\frac{\Gamma_{q,k}(t)\Gamma_{q,k}(s)}{\Gamma_{q,k}(t+s)},\quad t,s>0. \tag{4}$$

By using the relations

$$(x+y)^n_{q,k}=\prod_{j=0}^{n-1}(x+q^{jk}y) \tag{5}$$

and

$$(1+x)^t_{q,k}=\frac{(1+x)^\infty_{q,k}}{(1+q^{kt}x)^\infty_{q,k}} \tag{6}$$

in (3) the function $\Gamma_{q,k}(t)$ becomes:

$$\Gamma_{q,k}(t)=\frac{1}{(1-q)^{\frac{t}{k}-1}}\prod_{j=0}^{\infty}\frac{1-q^{(j+1)k}}{1-q^{jk+t}}. \tag{7}$$

From (7) it is obvious that for $k\to 1$ then $\Gamma_{q,k}(t)\to\Gamma_q(t)$ and for $q\to 1^-$ it becomes $\Gamma_k(t)$.

According to the definition of $\psi(t)$, $\psi_q(t)$ and $\psi_k(t)$ we define $\psi_{q,k}(t)$ as follows:

$$\psi_{q,k}(t)=\frac{\frac{d}{dt}\Gamma_{q,k}(t)}{\Gamma_{q,k}(t)}. \tag{8}$$

We mention that the functions $\Gamma_{q,k}(t)$ and $B_{q,k}(t,s)$ generalize both $\Gamma_k(t)$ and $B_k(t,s)$ defined in [6] and $\Gamma_q(t)$ and $B_q(t,s)$ defined in [4]. They are also used in [7] in order to define the k-gamma q-distribution and k-beta q-distribution.

In the next section we'll give some properties, monotonicity results and inequalities for the functions $\Gamma_{q,k}(t)$, $\psi_{q,k}(t)$ and $B_{q,k}(t,s)$. The results generalize known results [1, 2, 6, 8, 9, 10] for the functions $\Gamma_q(t)$, $\psi_q(t)$, $B_q(t,s)$ and $\Gamma_k(t)$, $\psi_k(t)$, $B_k(t,s)$. The obtained results are new and for their proof have been used simple methods of analysis. The inequalities (13), (14)and (36) have been proved in [3] using more complicated methods.

2 Main results

Theorem 2.1 *The function $\psi_{q,k}(t)$ is a completely monotonic function of t, for $t>0$, $k>0$ and $0<q<1$.*

Proof Taking into account that the relation $\frac{dq^{at}}{dt}=alnqq^{at}$ holds, the derivative of $\Gamma_{q,k}(t)$ with respect to t is:

$$\frac{d}{dt}\Gamma_{q,k}(t) = \frac{d}{dt}\{\frac{1}{(1-q)^{\frac{t}{k}-1}}\prod_{j=0}^{\infty}\frac{1-q^{(j+1)k}}{1-q^{qk+t}}\}$$

$$= -\frac{1}{k}ln(1-q)\Gamma_{q,k}(t) - \Gamma_{q,k}(t)\frac{1}{\prod_{j=0}^{\infty}(1-q^{jk+t})}\frac{d}{dt}\prod_{j=0}^{\infty}(1-q^{jk+t})$$

$$= \Gamma_{q,k}(t)[-\frac{1}{k}ln(1-q) + lnq\sum_{j=0}^{\infty}\frac{q^{jk+t}}{1-q^{jk+t}}]. \quad (9)$$

Using (8) we obtain

$$\psi_{q,k}(t) = -\frac{1}{k}ln(1-q) + lnq\sum_{j=0}^{\infty}\frac{q^{jk+t}}{1-q^{jk+t}}. \quad (10)$$

From (10) taking the first and second derivative of $\psi_{q,k}(t)$ with respect to t, we obtain correspondingly:

$$\frac{d}{dt}\psi_{q,k}(t) = ln^2 q\sum_{j=0}^{\infty}\frac{q^{jk+t}}{(1-q^{jk+t})^2} \quad (11)$$

and

$$\frac{d^2}{dt^2}\psi_{q,k}(t) = ln^3 q\sum_{j=0}^{\infty}[\frac{q^{jk+t}}{(1-q^{jk+t})^2} + \frac{2q^{2(jk+t)}}{(1-q^{jk+t})^3}]. \quad (12)$$

Since $lnq < 0$, from (11) and (12)it follows that the function $\psi_{q,k}(t)$ is an increasing and concave function of of t, for $t > 0$, $k > 0$ and $0 < q < 1$.

We notice that the derivative of each positive function of the form $\frac{q^{\lambda(jk+t)}}{(1-q^{jk+t})^m}$, for $\lambda = 1, 2, ...$ and $m = 2, 3, ...$ is a sum of two positive functions $\frac{\lambda q^{\lambda(jk+t)}}{(1-q^{jk+t})^m} + \frac{mq^{(\lambda+1)(jk+t)}}{(1-q^{jk+t})^{m+1}}$ multiplying with lnq. So the $n-$th derivative with respect to t, of $\psi_{q,k}(t)$ is equal to the product of $ln^{n+1}q$ and a series with positive terms. That means that the function $\psi'_{q,k}(t)$ is completely monotonic.

Remark 2.2 *The monotonicity results for the $\psi_{q,k}(t)$ are analogous with the results proved in [1, 2, 6, 8, 9, 10] concerning the functions $\psi(t)$, $\psi_q(t)$ and $\psi_k(t)$.*

Corollary 2.3 *The function $\Gamma_{q,k}(t)$ is a logarithmically convex function of t for $t > 0$, $k > 0$ and $0 < q < 1$ so it satisfies the inequality:*

$$\Gamma_{q,k}(\lambda x + (1-\lambda)y) \leq [\Gamma_{q,k}(x)]^{\lambda}[\Gamma_{q,k}(y)]^{1-\lambda}, \quad 0 < \lambda < 1. \quad (13)$$

The equality holds for $x = y$.

Proof Since the right part of the equation (11) is positive and taking into account (8) we easily obtain that the function $\Gamma_{q,k}(t)$ is logarithmically convex in $(0, \infty)$. Due to this property the inequality (13) follows immediately.

Corollary 2.4 *The inequality*

$$\Gamma_{q,k}(\frac{x+y}{2}) \leq \sqrt{\Gamma_{q,k}(x)\Gamma_{q,k}(y)} \tag{14}$$

is true for $x, y > 0$, $k > 0$ and $0 < q < 1$. The equality holds for $x = y$.

Proof It follows from (13) for $\lambda = \frac{1}{2}$.

Remark 2.5 *The results of corollary 2.3 and corollary 2.4 have been also proved in [3] using a more complicated method.*

Theorem 2.6 *For $t > 0$, $k > 0$, $0 < q < 1$ and $n \in N$ the inequality*

$$\Gamma_{q,k}(nt) \leq \frac{\Gamma_{q,k}(t)}{(1-q)^{\frac{(n-1)t}{k}}} \tag{15}$$

holds true. The equality holds for $n = 1$.

Proof We use the definition (7) for t and nt, $n \in N$ so we get

$$\frac{\Gamma_{q,k}(nt)}{\Gamma_{q,k}(t)} = \frac{1}{(1-q)^{\frac{(n-1)t}{k}}} \prod_{j=0}^{\infty} \frac{1-q^{jk+t}}{1-q^{jk+nt}} \tag{16}$$

Since $jk + t < jk + nt$ for $k, j, t > 0$ and $n \geq 1$, the desired inequality follows from (16).

Corollary 2.7 *The inequality*

$$\Gamma_{q,k}(x+y) \leq \frac{1}{(1-q)^{\frac{x+y}{2k}}} \sqrt{\Gamma_{q,k}(x)\Gamma_{q,k}(y)} \tag{17}$$

is true for $x, y > 0$, $k > 0$ and $0 < q < 1$. The equality holds for $x = y$.

Proof The equality (14) becomes

$$\Gamma_{q,k}(x+y) \leq \sqrt{\Gamma_{q,k}(2x)\Gamma_{q,k}(2y)}. \tag{18}$$

From (18) and using (15) for $n = 2$ we obtain the desired inequality.

Proposition 2.8 *For $k > 0$, $t > 0$ and $0 < q < 1$ the function $\psi_{q,k}(t)$ satisfies the equation:*

$$\psi_{q,k}(t+k) = \psi_{q,k}(t) - lnq\frac{q^t}{1-q^t}. \tag{19}$$

Proof Using the definition (7) of $\Gamma_{q,k}(t)$, for $t+k$ we can easily derive the equation:

$$\Gamma_{q,k}(t+k) = [t]\Gamma_{q,k}(t) \quad or \quad \Gamma_{q,k}(t+k) = \frac{1-q^t}{1-q}\Gamma_{q,k}(t) \tag{20}$$

Differentiating with respect to t both parts of (20), it follows:

$$\frac{d}{dt}\Gamma_{q,k}(t+k) = -lnq\frac{q^t}{1-q}\Gamma_{q,k}(t) + \frac{1-q^t}{1-q}\frac{d}{dt}\Gamma_{q,k}(t). \tag{21}$$

By dividing both parts of (21) by $\Gamma_{q,k}(t+k)$, taking in mind the definition of $\psi_{q,k}(t)$ and (20) we obtain the desired equation (19).

Remark 2.9 *By induction and using (20) we can easily prove the known [5] relation*

$$\Gamma_{q,k}(t+nk) = [t]_{n,k}\Gamma_{q,k}(t) \tag{22}$$

for $t>0$, $k>0$, $0<q<1$ and $n \in N$.

Theorem 2.10 *The function $\psi_{q,k}(t)$ satisfies the equality*

$$\psi_{q,k}(t+nk) = \psi_{q,k}(t) - lnq\sum_{j=0}^{n-1}\frac{q^{t+jk}}{1-q^{t+jk}}, \quad n \in N \tag{23}$$

for $t>0$, $k>0$ and $0<q<1$.

Proof The equality will be proved by induction. For $n=1$ it holds, because of (20). We suppose that (23) holds for n and we'll prove that it holds also for $n+1$. That is we have to prove the equation:

$$\psi_{q,k}(t+(n+1)k) = \psi_{q,k}(t) - lnq\sum_{j=0}^{n}\frac{q^{t+jk}}{1-q^{t+jk}}. \tag{24}$$

But

$$\begin{aligned}\psi_{q,k}(t+(n+1)k) = \psi_{q,k}((t+nk)+k) &= \psi_{q,k}(t+nk) - lnq\frac{q^{t+nk}}{1-q^{t+nk}} \\ &= \psi_{q,k}(t) - lnq\sum_{j=0}^{n-1}\frac{q^{t+jk}}{1-q^{t+jk}} - lnq\frac{q^{t+nk}}{1-q^{t+nk}} \\ &= \psi_{q,k}(t) - lnq\sum_{j=0}^{n}\frac{q^{t+jk}}{1-q^{t+jk}}\end{aligned} \tag{25}$$

which is the desired result. So the equality (23) holds for every $n \in N$.

Theorem 2.11 *The function $\psi_{q,k}(t)$ satisfies the inequalities:*

$$\frac{1}{k}ln[t] + \frac{q^t}{1-q^t}lnq < \psi_{q,k}(t) < \frac{1}{k}ln[t] \tag{26}$$

for $t > 0$, $k > 0$ and $0 < q < 1$.

Proof We apply the mean value theorem of differential calculus on the function $f(t) = ln\Gamma_{q,k}(t)$ in the interval $(t, t+k)$. So there is $t_0 \in (t, t+k)$ such that the equality

$$f(t+k) - f(t) = f'(t_0)(t+k-t) \tag{27}$$

holds, or

$$ln\Gamma_{q,k}(t+k) - ln\Gamma_{q,k}(t) = \psi_{q,k}(t_0)k \tag{28}$$

and using (20) we obtain

$$\psi_{q,k}(t_0) = \frac{1}{k}ln[t]. \tag{29}$$

Since the function $\psi_{q,k}(t)$ increases with respect to $t > 0$, the inequalities $\psi_{q,k}(t) < \psi_{q,k}(t_0) < \psi_{q,k}(t+k)$ hold and using (29) and (19) we obtain the inequalities (26).

Corollary 2.12 *For $n \in N$, $0 < q < 1$ the following inequalities hold:*

$$\frac{1}{k}ln[k] + \frac{q^k}{1-q^k}lnq < \psi_{q,k}(nk) < \frac{1}{k}ln[nk], \quad if \quad k > 1 \tag{30}$$

and

$$\frac{1}{k}ln[nk] < \psi_{q,k}(nk) < \frac{1}{k}ln[k] + \frac{q^{nk}}{1-q^{nk}}lnq, \quad if \quad 0 < k < 1. \tag{31}$$

Proof By putting k and nk instead of t in (26) and taking into account that if $k > 1$ then $nk > k$ so $\psi_{q,k}(k) < \psi_{q,k}(nk)$ and if $0 < k < 1$ then $k > nk$ so $\psi_{q,k}(k) > \psi_{q,k}(nk)$ we obtain the desired inequalities (30) and (31) correspondingly.

Theorem 2.13 *The equality*

$$B_{q,k}(t+nk, s+nk) = \frac{[t]_{n,k}[s]_{n,k}}{[t+s]_{2n,k}}B_{q,k}(t,s) \tag{32}$$

holds true, for $t, s > 0$, $k > 0$ and $n \in N$.

Proof It follows immediately using (4) and (22).

Theorem 2.14 *The function $B_{q,k}(t,s)$ satisfies the inequalities*

$$\sqrt{B_{q,k}(t,t)B_{q,k}(s,s)} \leq B_{q,k}(t,s) \leq \frac{B_{q,k}(t,t)B_{q,k}(s,s)}{\Gamma_{q,k}(t+s)(1-q)^{\frac{t+s}{k}}} \tag{33}$$

Proof According to the definition (4) we obtain

$$B_{q,k}(t,t) = \frac{(\Gamma_{q,k}(t))^2}{\Gamma_{q,k}(2t)} \quad or \quad \Gamma_{q,k}(t) = \sqrt{B_{q,k}(t,t)\Gamma_{q,k}(2t)} \tag{34}$$

Hence $B_{q,k}(t,s)$ can be written

$$B_{q,k}(t,s) = \frac{\sqrt{B_{q,k}(t,t)B_{q,k}(s,s)\Gamma_{q,k}(2t)\Gamma_{q,k}(2s)}}{\Gamma_{q,k}(t+s)}. \tag{35}$$

Because of the inequalities (18) and (15) for $n = 2$, the last equality becomes

$$B_{q,k}(t,s) \geq \sqrt{B_{q,k}(t,t)B_{q,k}(s,s)} \tag{36}$$

and

$$B_{q,k}(t,s) \leq \sqrt{\frac{B_{q,k}(t,t)B_{q,k}(s,s)}{\Gamma_{q,k}(t+s)}}\sqrt{\frac{\Gamma_{q,k}(t)\Gamma_{q,k}(s)}{(1-q)^{\frac{t+s}{k}}}} \tag{37}$$

or

$$B_{q,k}(t,s) \leq \sqrt{\frac{B_{q,k}(t,t)B_{q,k}(s,s)B_{q,k}(t,s)}{\Gamma_{q,k}(t+s)(1-q)^{\frac{t+s}{k}}}}. \tag{38}$$

The inequalities (36) and (38) prove (33).

Remark 2.15 *The inequality (36) has been proved in [3] using a more complicated method.*

References

[1] M. Abramowitz and I.A. Stegun (Eds.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Applied Mathematics Series, 55, fourth ed., with corrections, National Bereau of Standards, Washington, 1965.

[2] H. Alzer, Sharp inequalities for the digamma and polygamma functions, *Forum Mathematicum*, **16** (2004), no. 2, 181-221. http://dx.doi.org/10.1515/form.2004.009

[3] K. Brahim and Y. Sidomou, Some inequalities for the q,k-Gamma and Beta functions, *Malaya Journal of Matematik*, **2** (2014), no. 1, 61-71.

[4] A. De Sole and V. Kac, On integral representations of q-gamma and q-beta functions, *Rendiconti Lincei, Matematica e Applicazioni*, Series 9, **16** (2005), no. 1, 11-29.

[5] R. Diaz and C. Teruel, q, k-Generalized Gamma and Beta functions, *Journal of Nonlinear Mathematical Physics*, **12** (2005), no. 1, 118-134. http://dx.doi.org/10.2991/jnmp.2005.12.1.10

[6] R. Diaz and E. Pariguan, On hypergeometric functions and Pochhammer k-symbol, *Divulgaciones Matematicas*, **15** (2007), no. 2, 179-192.

[7] R. Diaz, C. Ortiz and E. Pariguan, On the k-gamma q-distribution, *Central European Journal of Mathematics*, **8** (2010), no. 3, 448-458. http://dx.doi.org/10.2478/s11533-010-0029-0

[8] H.H. Kairies and M.E. Muldoon, Some characterizations of q-factorial functions, *Aequationes Math.*, **25** (1982), 67-76. http://dx.doi.org/10.1007/bf02189598

[9] C.G. Kokologiannaki, Properties and inequalities of generalized k-Gamma, Beta and Zeta functions, *International Journal of Contemporary Mathematical Sciences*, **5** (2010), no. 14, 653-660.

[10] C.G. Kokologiannaki, Monotonicity of functions involving q-Gamma functions, *Mathematical Inequalities and Applications*, **14** (2011), no. 2, 413-419. http://dx.doi.org/10.7153/mia-14-33

Received: September 23, 2015; Published: November 21, 2015