An Asymmetric Putnam-Fuglede Theorem for \((p,k)\)-Quasiposinormal Operators

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Abstract
An asymmetric Putnam-Fuglede Theorem for \((p,k)\)-quasiposinormal operators is proved. As a consequence of this result, we obtain that the generalized derivation induced by these classes of operators is orthogonal to its kernel.

Mathematics Subject Classification: 47B20, 47A30, 47B47

Keywords: Hilbert Schmidt class, Putnam-Fuglede theorem, hyponormal operator, posinormal operator, orthogonality

1 Introduction
Let \(\mathcal{H}\) be a separable infinite dimensional complex Hilbert space, and let \(\mathcal{B}(\mathcal{H}), C_2\) and \(C_1\), denote the algebra of all bounded linear operators on \(\mathcal{H}\), the Hilbert Schmidt class and the trace class in \(\mathcal{B}(\mathcal{H})\) respectively. It’s well known that \(C_2\) and \(C_1\) each form a two-sided \(*\)-ideal in \(\mathcal{B}(\mathcal{H})\) and \(C_2\) is itself a Hilbert space with the inner product
\[
\langle X, Y \rangle = \sum \langle X e_i, Y e_i \rangle = tr(Y^*X) = tr(XY^*),
\]
where \( \{e_i\} \) is any orthonormal basis of \( \mathcal{H} \) and \( tr(.) \) is the natural trace on \( C_1 \).

The Hilbert-Schmidt norm of \( X \in C_2 \) is given by \( \|X\|_2 = \langle X, X \rangle^{1/2} \). For any operator \( A \in \mathcal{B}(\mathcal{H}) \), set, as usual, \( |A| = (A^*A)^{1/2} \) and \( [A^*, A] = A^*A - AA^* = |A|^2 - |A^*|^2 \) (the self-commutator of \( A \)), and consider the following definitions:

\( A \) is normal if \( A^*A = AA^* \), hyponormal if \( A^*A - AA^* \geq 0 \), \( p \)-hyponormal if \( |A|^{2p} \geq |A^*|^{2p} \) for \( 0 < p < 1 \). An 1-hyponormal is hyponormal which has been studied by many authors and it’s known that hyponormal operators have many interesting properties similar to those of normal operators [12]. An operator \( A \in \mathcal{B}(\mathcal{H}) \) is positive, \( A \geq 0 \) if \( \langle Ax, x \rangle \geq 0 \) for all \( x \in \mathcal{H} \). Posinormal operators where introduced in [13] as the class of operators for which \( AA^* = A^*PA \) for some nonnegative operator \( P \). Here \( P \) is called interrupter of \( A \). It was noticed then that this was a very large class that includes hyponormal operators as well as all invertible operators. Equivalently, an operator \( A \) is posinormal if \( AA^* \leq c^2A^*A \), where \( c > 0 \). Also note that, using the absolute value notation, we get an operator \( A \) is posinormal if and only if \( |A|^2 \leq c^2|A^*|^2 \) for some \( c > 0 \).

An operator \( A \) is said to be \( p \)-posinormal (\( 0 < p < 1 \)) if \( (A^*A)^p \leq c^2(A^*A)^p \) for some \( c > 0 \). According to [7] an operator is called \((p, k)\)-quasiposinormal if

\[
A^k(c^2(A^*A)^p - (AA^*)^p)A^k \geq 0
\]

for some \( c > 0 \), \( 0 < p < 1 \), and a positive integer \( k \). These classes are related by proper inclusion

\[
p - \text{hyponormal} \subset p - \text{posinormal} \subset (p, k) - \text{quasiposinormal}
\]

\[
p - \text{hyponormal} \subset p - \text{quasihyponormal} \subset (p, k) - \text{quasihyponormal} \subset (p, k) - \text{quasiposinormal}
\]

and

\[
\text{hyponormal} \subset k - \text{quasihyponormal} \subset (p, k) - \text{quasihyponormal} \subset (p, k) - \text{quasiposinormal},
\]

for a positive integer \( k \) and \( 0 < p \leq 1 \).

A \((p, k)\)-quasiposinormal operator is an extension of posinormal, \( p \)-posinormal, \( p \)-quasiposinormal and \( k \)-quasiposinormal. For an example of an operator in each these classes that not belong to the smaller classes (see [7, 8, 9, 13, 18]). Here we present some examples.
1. If $A$ is hyponormal and invertible and $0 < p < 1$ and $A^p$ exists, then $A^p$ is $p$-hyponormal and need not be hyponormal.

2. If $\mathcal{M}$ is the closure of $A^k$, then $A$ is $(p, k)$-quasihyponormal if and only if $A|_{\mathcal{M}}$ is $p$-hyponormal.

3. It follows from (2) above that a unilateral weighted shift with weight $\{w_n\}_{n=k}^\infty$ is $(p, k)$-quasihyponormal if and only if the sequence $\{w_n\}_{n=k}^\infty$ is increasing, (so the first $k$ terms can be arbitrary). However, a bilateral weighted shift that is $(p, k)$-quasihyponormal must be actually hyponormal (by item (2) above).

The famous Putnam-Fuglede theorem is as follows (see [6, 15]).

**Theorem 1.1** If $A, B \in \mathcal{B}(\mathcal{H})$ are normal operators and $AX = XB$ for some $X \in \mathcal{B}(\mathcal{H})$, then $A^*X = XB^*$.

Several authors have extended this theorem to nonnormal operators (see [5, 13, 14, 16]). S.K. Berberian [4] relaxes the hypothesis on $A$ and $B$ in Theorem 1 by assuming $A$ and $B^*$ hyponormal operators and $X$ to be Hilbert-Schmidt class. Patel [14] proved that if $A$ and $B^*$ are $p$-hyponormal operators such that $AX = XB$ for $X \in C_2$, then $A^*X = XB^*$. However, M.Y. Lee [7] proved that if $A$ is $p$-hyponormal operator and $B^*$ is an invertible $p$-hyponormal operator such that $AX = XB$ and for some $X \in C_2$ and $|||A|||^{1-p}.|||B^{-1}|||^{1-p} \leq 1$, then $A^*X = XB^*$.

In this paper, we will prove that this result remains true without the condition $|||A|||^{1-p}.|||B^{-1}|||^{1-p} \leq 1$. We also prove that the above result remains true for $(p, k)$-quasihyponormal operators without the additional condition $|||A|||^{1-p}.|||B^{-1}|||^{1-p} \leq 1$ showing that we don’t need this additional condition as in ([8], Theorem 4).

Let $A, B \in \mathcal{B}(\mathcal{H})$, we define the generalized derivation $\delta_{A,B}$ induced by $A$ and $B$ as follows

$$\delta_{A,B}(X) = AX - XB \forall X \in \mathcal{B}(\mathcal{H}).$$

It is clear that $\delta_{A,B}(C_2) \subseteq C_2$. However it can also happen that $\delta_{A,B}(X) \in C_2$ for some $X \in \mathcal{B}(\mathcal{H})|_{C_2}$, hence $\text{ran} \ (\delta_{A,B}|_{C_2}) \subseteq \text{ran} \ (\delta_{A,B}) \cap C_2$ and then we also have $\text{ran} \ (\delta_{A,B}|_{C_2}) \subseteq \text{ran} \ (\delta_{A,B}) \cap C_2$, where $(\cdot)_{C_2}$ denotes the closure of the $C_2$ norm. A. Turnsek [17] asked the following question: When the reverse inclusion is possible? In this paper we consider the question when

$$\text{ran} \ (\delta_{A,B}|_{C_2})_{C_2} = \text{ran} \ (\delta_{A,B}) \cap C_2.$$  \quad (1)

We show that (1) holds in the case when $A$ is $(p, k)$-quasiposinormal and $B^*$ is an invertible $(p, k)$-quasihyponormal.
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J. Anderson and C. Foias [2] proved that if \( A \) and \( B \) are normal, \( S \) is an operator such that \( AS = SB \), then

\[
\|\delta_{A,B}(X) - S\| \geq \|S\|, \forall X \in \mathcal{B}(\mathcal{H}),
\]

where \( \|\cdot\| \) is the usual norm. Hence the range of \( \delta_{A,B} \) is orthogonal to the null space of \( \delta_{A,B} \). The orthogonality here is understood in the sense of Definition 1.2 in [1].

The purpose of this paper is to prove that the Fuglede-Putnam theorem remains true if \( A \) is is \((p,k)\)-quasiposinormal and \( B^* \) is an invertible \((p,k)\)-quasihyponormal. As a consequence of this result, we give a similar orthogonality result by proving that the range of the generalized derivation induced by the above classes of operators is orthogonal to its kernel, also we give of positive answer of the question raised by A. Turnsek [17].

2 Main Results

the basic operator \( \Gamma \) induced by \( A \) and \( B \) is defined on \( C_2 \) by \( \Gamma X = AXB \), and the adjoint of \( \Gamma \) is given by the formula \( \Gamma^* X = A^*XB^* \) [4].

Proposition 2.1 Let \( A, B \in \mathcal{B}(\mathcal{H}) \). If \( A \geq 0 \) and \( B \geq 0 \), then \( \Gamma \geq 0 \).

Proof. Let \( X \in C_2 \),

\[
\langle \Gamma X, X \rangle = tr(AXBX^*)
= tr(A^{1/2}XBA^{1/2})
= tr((A^{1/2}XB^{1/2})(A^{1/2}X^*B^{1/2})) \geq 0.
\]

Proposition 2.2 If \( A \) is \((p,k)\)-quasiposinormal and \( B^* \) is \((p,k)\)-quasihyponormal, then \( \Gamma \) is \((p,k)\)-quasiposinormal.

Proof. It is well known that \( \Gamma^*\Gamma X = A^*AXBB^* \) and \( \Gamma\Gamma^* X = A^*AXBB^*B \).

So,

\[
|\Gamma^k|(X) = |A|X|B^*|, \quad |\Gamma^k|(X) = |A^*|X|B|.
\]

Thus for all \( X \in C_2 \),

\[
\Gamma^k(c^2|\Gamma|^{2p} - |\Gamma^*|^{2p})\Gamma^k X
= \Gamma^k(c^2|\Gamma|^{2p} - |\Gamma^*|^{2p})A^kX^kB^k
= c^2A^k|A|^{2p}A^kB^k|B^*|^{2p}B^{*k} - A^{*k}|A^*|^{2p}A^kXB^k|B^*|^{2p}B^{*k}
= A^{*k}(c^2|A|^{2p} - |A^*|^{2p})A^kXB^k|B^*|^{2p}B^{*k}
+ A^{*k}|A^*|^{2p}A^kXB^k(|B^*|^{2p} - |B|^{2p})B^{*k}.
\]

Now, since \( A \) is \((p,k)\)-quasiposinormal and \( B^* \) is \((p,k)\)-quasihyponormal,

\[
\Gamma^k(c^2|\Gamma|^{2p} - |\Gamma^*|^{2p})\Gamma^k \geq 0
\]

and so, \( \Gamma \) is \((p,k)\)-quasiposinormal.
Lemma 2.3 [11] Let \( T \in \mathcal{B}(\mathcal{H}) \) be \((p,k)\)-quasiposinormal for \( 0 < p \leq 1 \) and a positive integer \( k \). If \( \lambda \neq 0 \) and \((T - \lambda)x = 0\) for some nonzero \( x \in \mathcal{H} \), then \((T - \lambda)^*x = 0\).

Theorem 2.4 If \( A \in \mathcal{B}(\mathcal{H}) \) is \((p,k)\)-quasihyponormal and \( B^* \in \mathcal{B}(\mathcal{H}) \) is an invertible \((p,k)\)-quasihyponormal operator such that \( AX = XB \) for some \( X \in C_2 \), then \( AX^* = XB^* \).

Proof. Let \( \mathcal{K} \) defined on \( C_2 \) by \( \mathcal{K}Y = AYB^{-1} \) for all \( Y \in C_2 \). Since an invertible \((p,k)\)-quasihyponormal operator is \((p,k)\)-quasihyponormal (see [8]). Then it follows from Proposition 2.2 that \( \mathcal{K} \) is \((p,k)\)-quasiposinormal, furthermore, \( \mathcal{KX} = AXB^{-1} = X \). Hence \( X \) is an eigenvector of \( \mathcal{K} \). By applying Lemma 2.3 we get \( \mathcal{K}^*X = A^*X(B^{-1})^* = X \), that is \( AX^* = XB^* \). This completes the proof.

Remark 2.5 It is proved in [7] that if \( A \) is \( p \)-quasihyponormal operator an \( B^* \) is an invertible \( p \)-quasihyponormal operator such that \( AX = XB \) for \( X \in C_2 \) and \( ||A||^{1-p} \cdot ||B^{-1}||^{1-p} \leq 1 \), then \( A^*X = XB^* \), then \( AX^* = XB^* \). We showed in Theorem 2.4 that we don’t need the additional condition \( ||A||^{1-p} \cdot ||B^{-1}||^{1-p} \leq 1 \). Also in [8] it is proved that if \( A \) is \((p,k)\)-quasihyponormal and \( B^* \) is an invertible \((p,k)\)-quasihyponormal operator such that \( AX = XB \) for \( X \in C_2 \) and \( ||A||^{1-p} \cdot ||B^{-1}||^{1-p} \leq 1 \), then \( A^*X = XB^* \), then \( AX^* = XB^* \). Here also we don’t need the additional condition \( ||A||^{1-p} \cdot ||B^{-1}||^{1-p} \leq 1 \).

As a consequence of Theorem 2.4, we obtain

Corollary 2.6 Let \( A \in \mathcal{B}(\mathcal{H}) \) be \( p \)-quasihyponormal and \( B^* \in \mathcal{B}(\mathcal{H}) \) be an invertible \( p \)-quasihyponormal \( (0 < p \leq 1) \) such that \( AX = XB \) for some \( X \in C_2 \), then \( AX^* = XB^* \).

Corollary 2.7 Let \( A \in \mathcal{B}(\mathcal{H}) \) be \((p,k)\)-quasihyponormal and \( B^* \in \mathcal{B}(\mathcal{H}) \) be an invertible \((p,k)\)-quasihyponormal \( (0 < p \leq 1) \) such that \( AX = XB \) for some \( X \in C_2 \), then \( AX^* = XB^* \).

Corollary 2.8 Let \( A \in \mathcal{B}(\mathcal{H}) \) be \( p \)-hyponormal and \( B^* \in \mathcal{B}(\mathcal{H}) \) be an invertible \( p \)-hyponormal \( (0 < p \leq 1) \) such that \( AX = XB \) for some \( X \in C_2 \), then \( AX^* = XB^* \).

Corollary 2.9 [1] Let \( A \in \mathcal{B}(\mathcal{H}) \) be posinormal and \( B^* \in \mathcal{B}(\mathcal{H}) \) be an invertible hyponormal operator such that \( AX = XB \) for some \( X \in C_2 \), then \( AX^* = XB^* \).

Now, we ready to extend the orthogonality result to the \((p,k)\)-quasiposinormal class operators.
Theorem 2.10 Let $A, B$ be in $\mathcal{B}(H)$ and $S \in C_2$. Then
\[ \| \delta_{A,B}(X) + S \|_2^2 = \| \delta_{A,B}(X) \|_2^2 + \| S \|_2^2 \] (2)
and
\[ \| \delta^*_{A,B}(X) + S \|_2^2 = \| \delta^*_{A,B}(X) \|_2^2 + \| S \|_2^2 \] (3)
if and only if $\delta_{A,B}(S) = 0 = \delta_{A^*,B^*}(S)$ for all $S \in C_2$.

Proof. We know that the Hilbert-Schmidt class $C_2$ is a Hilbert space. Note that
\[ \| \delta_{A,B}(X) + S \|_2^2 = \| \delta_{A,B}(X) \|_2^2 + \| S \|_2^2 + \text{Re}\langle \delta_{A,B}(X), S \rangle \]
and
\[ \| \delta^*_{A,B}(X) + S \|_2^2 = \| \delta^*_{A,B}(X) \|_2^2 + \| S \|_2^2 + \text{Re}\langle X, \delta_{A,B}(S) \rangle \].
Hence by the equality $\delta_{A,B}(S) = 0 = \delta_{A^*,B^*}(S)$ we obtain (2) and (3). The claim is verified and the proof is complete.

Corollary 2.11 Let $A, B$ be operators in $\mathcal{B}(H)$ and $S \in C_2$. Then
\[ \| \delta_{A,B}(X) + S \|_2^2 = \| \delta_{A,B}(X) \|_2^2 + \| S \|_2^2 \] (4)
and
\[ \| \delta^*_{A,B}(X) + S \|_2^2 = \| \delta^*_{A,B}(X) \|_2^2 + \| S \|_2^2 \] (5)
if and only if either of the following hold.
(i) $A$ and $B^*$ are hyponormal operators.
(ii) $A$ is $p$-hyponormal and $B^*$ is an invertible $p$-hyponormal.
(iii) $A$ is $k$-quasihyponormal and $B^*$ is an invertible $k$-quasihyponormal.
(iv) $A$ is $p$-quasihyponormal and $B^*$ is an invertible $p$-quasihyponormal.
(v) $A$ is posinormal and $B^*$ is an invertible $(p,k)$-quasihyponormal.

Now we answer the question when
\[ \overline{\text{ran}\left( \delta_{A,B|C_2} \right)^{C_2}} = \overline{\text{ran}(\delta_{A,B}) \cap C_2^{C_2}}. \]

Theorem 2.12 Let $A, B \in \mathcal{B}(H)$ such that $\ker \delta_{A,B} \subseteq \ker \delta_{A^*,B^*}$, then
\[ \overline{\text{ran}\left( \delta_{A,B|C_2} \right)^{C_2}} = \overline{\text{ran}(\delta_{A,B}) \cap C_2^{C_2}}. \]
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Proof. Since \( \text{ran}^\perp (\delta_{A,B}|_{C_2}) = \ker (\delta_{A,B}|_{C_2}) \). Hence if \( S \in \text{ran}^\perp (\delta_{A,B}|_{C_2}) \), then \( S \in \ker (\delta_{A,B}|_{C_2}) \). Hence Theorem 2.10 would imply that \( S \in (\text{ran} (\delta_{A,B}) \cap C_2)^\perp \), that is,
\[
\text{ran}^\perp (\delta_{A,B}|_{C_2}) \subseteq (\text{ran} (\delta_{A,B}) \cap C_2)^\perp.
\]
But
\[
(\text{ran} (\delta_{A,B}) \cap C_2)^\perp \subseteq \text{ran}^\perp (\delta_{A,B}|_{C_2}).
\]
Thus
\[
\text{ran}^\perp (\delta_{A,B}|_{C_2}) = (\text{ran} (\delta_{A,B}) \cap C_2)^\perp.
\]
Consequently
\[
\frac{\text{ran} (\delta_{A,B}|_{C_2}) C_2}{C_2} = \frac{\text{ran} (\delta_{A,B}) C_2}{C_2}.
\]

**Corollary 2.13** Let \( A, B \in \mathcal{B}(\mathcal{H}) \). Then
\[
\frac{\text{ran} (\delta_{A,B}|_{C_2}) C_2}{C_2} = \frac{\text{ran} (\delta_{A,B}) C_2}{C_2}.
\]
under any of the statements (i)-(v) in Corollary 2.11.

**References**


Received: November 11, 2015; Published: February 2, 2016