The Commuting Graph of
the Symmetric Group $S_n$

Timothy Woodcock

Department of Mathematics, Stonehill College
Easton, Massachusetts, USA 02357

Abstract

In this paper we analyze the commuting graph of the full symmetric group on $n$ elements, the graph being defined to have the nontrivial group elements as its vertex set, and an edge joining each commuting pair of vertices. We prove that if neither $n$ nor $n - 1$ is a prime, then the graph has diameter 5; that is, the maximum-length shortest path, over all pairs of vertices, is 5. In the cases where $n$ or $n - 1$ is a prime, we show that the graph is disconnected. Moreover, the components are completely identified, along with their diameters. This paper reproduces a number of results from the paper of Iranmanesh and Jafarzadeh, [4], but with some generalization and a new approach.

Keywords: Symmetric group, commuting graph, diameter, connected components

1 Groundwork and main result

For a positive integer $n$, let $S_n$ denote the symmetric group of all bijective functions on $\{1,2,\ldots,n\}$, under composition. Let $1_n$ denote the identity, or trivial element of $S_n$. For each subset $A$ of $S_n$, we define a graph $\Delta_A$, having $A \setminus \{1_n\}$ as its vertices, and with an edge joining each pair of elements of $A \setminus \{1_n\}$ that commute under the product of $S_n$. We refer to $\Delta_A$ as the commuting graph of $A$. 
For a general graph $G = (V,E)$ with a finite number of vertices, and for $v, w \in V$, the distance from $v$ to $w$ in $G$ is defined to be the minimum number of edges in a path joining these vertices. We denote this distance by $d_G(v,w)$. If no path exists, we let $d_G(v,w) = \infty$. The diameter of $G$ is the maximum value of $d_G(v,w)$ over all $v$ and $w$ in $V$, where $\infty$ is understood to be greater than any natural number. For each nonempty subset $W$ of $V$, we define the induced subgraph $G_W$ of $G$ to have vertex set $W$, and to include all edges of $E$ that join pairs of vertices in $W$. If $d_{G_W}(v,w)$ is finite for all $v,w \in W$, then $G_W$ is said to be connected. If $W$ is a maximal subset of $V$ such that $G_W$ is connected, then $G_W$ is called a component of $G$. If $G$ has multiple components, it is said to be disconnected.

In a commuting graph $\Delta_A$, a path will be called a commuting path. But it shall be convenient for us to allow consecutive vertices within a such a path to be alike. Of course, this does not affect the distance between any pair of vertices, and thus the diameter of $\Delta_A$ is unchanged.

The statement of our main result makes use of the following additional notation. For $n \geq 2$ and $2 \leq m \leq n$, let $C_n^m$ be the collection of all cycles of length $m$ in $S_n$. Let $R_n$ be the set of all nontrivial elements of $S_n$, but excluding $C_n^{n-1} \cup C_n^n$. For $\gamma \in S_n$, let $\langle \gamma \rangle$ denote the cyclic subgroup of $S_n$ generated by $\gamma$.

**Theorem 1.1.** Suppose $n \geq 3$.

a. If neither $n$ nor $n-1$ is a prime, then $\Delta_{S_n}$ is connected, of diameter 5.

b. If $n-1$ is a prime, then $\Delta_{S_n}$ is disconnected, with components $\Delta_{R_n \cup C_n^2}$, and $\Delta_{\langle \gamma \rangle}$ for $\gamma \in C_n^{n-1}$. The diameter of $\Delta_{R_n \cup C_n^n}$ is 1, 3, or 4 according as $n$ is 3, 4, or $> 4$; the diameter of each $\Delta_{\langle \gamma \rangle}$ is 0 or 1, according as $n$ is 3 or $> 3$.

c. If $n$ is a prime but $n-1$ is not, $\Delta_{S_n}$ is disconnected, with components $\Delta_{R_n \cup C_n^{n-1}}$, and $\Delta_{\langle \gamma \rangle}$ for $\gamma \in C_n^n$. The diameter of $\Delta_{R_n \cup C_n^{n-1}}$ is 5; the diameter of each $\Delta_{\langle \gamma \rangle}$ is 1.

Though this result was proved in the paper [4], we take a route to its realization that is substantially different. In particular, we shall prove that 5 is a lower bound for the diameter of $\Delta_{S_n}$ for all integers $n$ such that $n$ and $n-1$ are composite. In [4, p.132], the minimal case of $n = 9$ is discussed.

## 2 Commuting elements of $S_n$

The results developed in the present section are well known, but organized for completeness.
Definition 2.1. Let $\pi, \rho \in S_n$. We say that $\pi$ induces a cycle map on $\rho$ provided that, for each cycle $(a_0 \cdots a_{i-1})$ of $\rho$, possibly trivial, $(\pi(a_0) \cdots \pi(a_{i-1}))$ remains a cycle of $\rho$.

Proposition 2.2. Let $\pi$ and $\rho$ be elements of $S_n$. Then $\pi$ and $\rho$ commute if and only if $\pi$ induces a cycle map on $\rho$.

Proof. Let $(a_0 a_1 \cdots a_{i-1})$ be an arbitrary cycle of $\rho$, and let $j$ be an element of $\{0, 1, \ldots, i-1\}$. Assuming that $\pi$ and $\rho$ commute, we have

$$\rho(\pi(a_j)) = \pi(\rho(a_j)) = \pi(a_{(j+1) \mod i}).$$

Hence $(\pi(a_0) \pi(a_1) \cdots \pi(a_{i-1}))$ is a cycle of $\rho$. Conversely if $(\pi(a_0) \pi(a_1) \cdots \pi(a_{i-1}))$ is a cycle of $\rho$, then

$$\rho(\pi(a_j)) = \pi(a_{(j+1) \mod i}) = \pi(\rho(a_j)).$$

It follows that $\pi$ and $\rho$ commute, because $a_j$ may be viewed as a general element of $\{1, 2, \ldots, n\}$. $\Box$

We remark that the statement of 2.2 is symmetric with respect to $\pi$ and $\rho$. Thus, $\pi$ induces a cycle map on $\rho$ if and only if $\rho$ induces a cycle map on $\pi$.

Proposition 2.3. Let $\pi \in S_n$, and let $\gamma$ be a cycle of $\pi$. Then $\pi$ and $\gamma$ commute.

Proof. Suppose that $\gamma = (a_0 a_1 \cdots a_{i-1})$. We observe that

$$(\gamma(a_0) \gamma(a_1) \cdots \gamma(a_{i-1})) = (a_1 a_2 \cdots a_0) = (a_0 a_1 \cdots a_{i-1}).$$

Also, if $(b_0 b_1 \cdots b_{k-1})$ is a cycle of $\pi$ other than $\gamma$, then $\gamma$ fixes each $b_j$, because $\gamma$ fixes each element of $\{1, 2, \ldots, n\} \setminus \{a_0, a_1, \ldots, a_{i-1}\}$. Hence

$$(\gamma(b_0) \gamma(b_1) \cdots \gamma(b_{k-1})) = (b_0 b_1 \cdots b_{k-1}).$$

We conclude that $\gamma$ induces a cycle map on $\pi$, and therefore $\gamma$ commutes with $\pi$, by 2.2. $\Box$

Proposition 2.4. Let $\pi, \rho \in S_n$ be commuting elements, and suppose that $\gamma = (a_0 \cdots a_{i-1})$ is a cycle of $\pi$, of unique length over all cycles of $\pi$. Then $\rho$ acts on $\{a_0, \ldots, a_{i-1}\}$ as a power of $\gamma$.

Proof. Since $\pi$ and $\rho$ are commuting elements, $(\rho(a_0) \rho(a_1) \cdots \rho(a_{i-1}))$ is a cycle of $\pi$, by 2.2. Therefore, $(\rho(a_0) \rho(a_1) \cdots \rho(a_{i-1})) = \gamma$, because of the uniqueness of the length of $\gamma$. Thus we have $\rho(a_0) = a_k$ for some $k$ in $\{0, 1, \ldots, i-1\}$, and furthermore $\rho(a_j) = a_{(k+j) \mod i}$ for $0 \leq j \leq i-1$. In other words, $\rho$ acts on $\{a_0, a_1, \ldots, a_{i-1}\}$ in the same manner as $\gamma^k$. $\Box$
Definition 2.5. For $\pi \in S_n$, we define the fixed set of $\pi$ to be the set of all $a \in \{1, 2, \ldots, n\}$ such that $\pi(a) = a$. We denote this set by $Fix(\pi)$. We define the moved set of $\pi$, denoted $Move(\pi)$, to be $\{1, 2, \ldots, n\} \setminus Fix(\pi)$.

Proposition 2.6. Suppose that $\pi, \rho \in S_n$ are commuting elements. Also assume that $Fix(\pi)$ contains a unique element, $a$. Then $a \in Fix(\rho)$ as well.

Proof. Since $a \in Fix(\pi)$, $a$ makes up a trivial cycle of $\pi$. Thus by 2.2, $\rho(a)$ constitutes a trivial cycle as well. But $\pi$ has a unique trivial cycle, because $|Fix(\pi)| = 1$. Therefore, $\rho(a) = a$.

Definition 2.7. For $\pi, \rho \in S_n$, we say that $\pi$ and $\rho$ are disjoint provided that $Move(\pi) \cap Move(\rho) = \emptyset$.

Proposition 2.8. If $\pi, \rho \in S_n$ are disjoint, they commute.

Proof. Since $\pi$ and $\rho$ are disjoint, $\pi$ acts as the identity on $Move(\rho)$; thus $\pi$ permutes $Fix(\rho)$. Likewise, $\rho$ permutes $Fix(\pi)$.

If $a \in Fix(\pi)$, $(\rho \pi)(a) = \rho(a)$, of course. And $(\pi \rho)(a) = \rho(a)$, because $\rho$ permutes $Fix(\pi)$. Hence, $(\rho \pi)(a) = (\pi \rho)(a)$. Similarly, this holds for each $a \in Fix(\rho)$. But, since $\pi$ and $\rho$ are disjoint, $Fix(\pi) \cup Fix(\rho) = \{1, 2, \ldots, n\}$. Therefore $\pi$ and $\rho$ commute.

Definition 2.9. Let $H$ be a subgroup of $S_n$. For $a \in \{1, 2, \ldots, n\}$, we define its orbit under $H$ to be $\{\pi(a) \mid \pi \in H\}$. We denote this set by $[a]_H$.

Proposition 2.10. For a subgroup $H$ of $S_n$, the family of all $[a]_H$, $1 \leq a \leq n$, is a partition of $\{1, 2, \ldots, n\}$. Furthermore, $a \in [a]_H$ for each $a$.

Proof. Since $H$ is a subgroup of $S_n$, $H$ includes the identity element $1_n$ of $S_n$. Therefore, for each $a \in \{1, 2, \ldots, n\}$, we have $a \in [a]_H$, because $1_n(a) = a$.

Now suppose that for given elements $a, b \in \{1, 2, \ldots, n\}$, $[a]_H \cap [b]_H$ is nonempty. Then there exist $\pi, \rho \in H$ such that $\pi(a) = \rho(b)$. Let $c \in [a]_H$, and assume that $\chi(a) = c$, where $\chi \in H$. Then $(\chi \pi^{-1} \rho)(b) = c$. And, since $H$ is a group, $\chi \pi^{-1} \rho \in H$. Thus $c \in [b]_H$, so $[a]_H \subseteq [b]_H$. The reverse containment holds by a parallel argument.

Proposition 2.11. Let $H$ be a subgroup of $S_n$. If $\pi$ is in the center of $H$, and $a \in Fix(\pi)$, then $[a]_H \subseteq Fix(\pi)$.

Proof. Let $b \in [a]_H$, and assume that $\rho(a) = b$, where $\rho \in H$. Then we have

$$\pi(b) = \pi(\rho(a)) = \rho(\pi(a)) = \rho(a) = b.$$ 

Therefore, $b \in Fix(\pi)$. 

\[\square\]
3 The commuting graph $\Delta_{R_n}$

Theorem 3.1. Suppose $n \geq 4$. Then $\Delta_{R_n}$ is connected, of diameter 3 or 4, according as $n = 4$ or $n > 4$.

The proof of the theorem is realized through a sequence of results. We remark that $R_n$ is empty for $n \in \{1, 2, 3\}$.

Proposition 3.2. The diameter of $\Delta_{R_4}$ is 3.

Proof. Let

$$T = \{(1 \ 2), (1 \ 3), (1 \ 4), (2 \ 3), (2 \ 4), (3 \ 4)\},$$

$$D = \{(1 \ 2)(3 \ 4), (1 \ 3)(2 \ 4), (1 \ 4)(2 \ 3)\}.$$  

We observe that any element of $S_4 \setminus \{1_4\}$ that is neither a cycle of length 3 nor a cycle of length 4 is a member of $T \cup D$; in other words, $R_4 = T \cup D$.

Given an arbitrary $\pi \in R_4$, we claim that $\pi$ commutes with some $\varphi \in D$. If $\pi \in D$, we may take $\varphi = \pi$, since a group element commutes with itself. If $\pi \in T$, let us write $\pi = (ab)$. Then by 2.3, $\pi$ commutes with $\varphi = (a \ b)(c \ d) \in D$, where $c$ and $d$ are the elements of $\{1, 2, 3, 4\} \setminus \{a, b\}$, in arbitrary order.

Now let $\rho \in R_4$ as well, and assume that $\psi \in D$ commutes with $\rho$. We observe that each individual element of $D$ induces a cycle map on all elements of $D$; so by 2.2, the elements of $D$ mutually commute. Therefore $\varphi$ and $\psi$ commute, in particular, and so $(\pi, \varphi, \psi, \rho)$ is a commuting path in $\Delta_{R_4}$. Hence $d_{\Delta_{R_4}}(\pi, \rho) \leq 3$. We conclude that $\Delta_{R_4}$ has diameter $\leq 3$.

To see that the diameter is $\geq 3$, consider the pair $\sigma = (1 \ 2)$ and $\tau = (1 \ 3)$. Suppose that $\chi \in S_4$ commutes $\sigma$ and $\tau$. Then by 2.2, $\chi$ induces a cycle map on $\sigma$, and on $\tau$. Thus $\chi(\{1, 2\}) = \{1, 2\}$, and $\chi(\{1, 3\}) = \{1, 3\}$. Therefore $\chi(1) = 1$, and moreover, $\{1, 2, 3\} \subseteq Fix(\chi)$. This obviously implies that $\chi = 1_4$. Hence there is no commuting path $(\sigma, \chi, \tau)$ in $\Delta_{S_4 \setminus \{1_4\}}$; thus there is none in $\Delta_{R_4}$. It follows that $d_{\Delta_{R_4}}(\sigma, \tau) \geq 3$. □

Lemma 3.3. Suppose that $n > 4$, and let $\pi \in R_n$. Then $\pi$ is a product of two disjoint cycles of length $n/2$ each, or $\pi$ commutes with a nontrivial cycle of length $< n/2$. Either way, $\pi$ commutes with a nontrivial cycle of length $\leq n/2$.

Proof. First assume that $\pi$ is itself a cycle. Then, since $R_n$ includes no cycles of length $n - 1$ or $n$, by definition, $\pi$ fixes at least two elements, say $a, b \in \{1, 2, \ldots, n\}$. We observe that $\pi$ commutes with $(a \ b)$, by 2.8; and the length of $(a \ b)$, of course 2, is $< n/2$ because $n > 4$.

Next assume that $\pi$ is neither a cycle, nor a product of two disjoint cycles of length $n/2$ each. Let $\gamma$ be a cycle of minimum length over all nontrivial cycles of $\pi$. Then the length of $\gamma$ is $< n/2$, obviously, and $\gamma$ commutes with $\pi$, by 2.3.
Finally, if the cycle decomposition of $\pi$ consists of two cycles, each of length $n/2$, then in view of 2.3, we may say that $\pi$ commutes with a nontrivial cycle of length $\leq n/2$.

**Proposition 3.4.** For $n > 4$, the diameter of $\Delta_{R_n}$ is $\leq 4$.

*Proof.* Let $\pi, \rho \in R_n$. First suppose that at least one of $\pi$ and $\rho$, say $\pi$, commutes with a cycle $\gamma \in R_n$ of length $< n/2$. We observe that $\rho$ commutes with a cycle $\delta \in R_n$ of length $\leq n/2$, by 3.3. If $\gamma$ and $\delta$ are disjoint, on the one hand, then these cycles commute by 2.8. Thus $(\pi, \gamma, \delta, \rho)$ is a commuting path in $\Delta_{R_n}$, and so $d_{\Delta_{R_n}}(\pi, \rho) \leq 3$. On the other hand if $\text{Move}(\gamma) \cap \text{Move}(\delta) \neq \emptyset$, then we have

$$|\text{Move}(\gamma) \cup \text{Move}(\delta)| < |\text{Move}(\gamma)| + |\text{Move}(\delta)| < 2(n/2).$$

Therefore, $|\text{Move}(\gamma) \cup \text{Move}(\delta)| \leq n - 2$; hence $\text{Fix}(\gamma) \cap \text{Fix}(\delta)$ contains at least two elements, say $a$ and $b$. Define $\sigma = (a \ b)$, an element of $R_n$. We observe that $\sigma$ commutes with $\gamma$ and $\delta$ by 2.8. Furthermore, $(\pi, \gamma, \sigma, \delta, \rho)$ is a commuting path in $\Delta_{R_n}$. Thus, $d_{\Delta_{R_n}}(\pi, \rho) \leq 4$.

Now assume that neither $\pi$ nor $\rho$ commutes with a cycle of length $< n/2$. Then $\pi$ is a product of two disjoint cycles, each of length $n/2$, by 3.3. Likewise for $\rho$. Let us write

$$\pi = (a_1 \ a_2 \ \cdots \ a_{n/2}) \ (b_1 \ b_2 \ \cdots \ b_{n/2}), \ \rho = (c_1 \ c_2 \ \cdots \ c_{n/2}) \ (d_1 \ d_2 \ \cdots \ d_{n/2}).$$

Since $\text{Move}(\pi) = \text{Move}(\rho) = \{1, 2, \ldots, n\}$, we may take $a_1 = c_1$. We note that $a_1 \notin \{b_1, b_2, \ldots, b_{n/2}\}$, because the cycles of $\pi$ are disjoint, and $c_1 \notin \{d_1, d_2, \ldots, d_{n/2}\}$, by the same token. Hence, since $a_1 = c_1$, we realize that

$$\{b_1, b_2, \ldots, b_{n/2}\} \cup \{d_1, d_2, \ldots, d_{n/2}\} \subsetneq \{1, 2, \ldots, n\}.$$  

Therefore, $\{b_1, b_2, \ldots, b_{n/2}\} \cap \{d_1, d_2, \ldots, d_{n/2}\} \neq \emptyset$; let us assume that $b_1 = d_1$. Define

$$\varphi = (a_1 \ b_1)(a_2 \ b_2) \cdots (a_{n/2} \ b_{n/2}),$$

$$\tau = (a_1 \ b_1) = (c_1 \ d_1),$$

$$\psi = (c_1 \ d_1)(c_2 \ d_2) \cdots (c_{n/2} \ d_{n/2}).$$

Then $\varphi, \tau, \psi \in R_n$, clearly. We claim, furthermore, that $(\pi, \varphi, \tau, \psi, \rho)$ is a commuting path in $\Delta_{R_n}$. We observe:

$$(\varphi(a_1) \ \varphi(a_2) \ \cdots \ \varphi(a_{n/2})) = (b_1 \ b_2 \ \cdots \ b_{n/2}),$$

$$(\varphi(b_1) \ \varphi(b_2) \ \cdots \ \varphi(b_{n/2})) = (a_1 \ a_2 \ \cdots \ a_{n/2}).$$

Thus, $\varphi$ induces a cycle map on $\pi$, and hence $\varphi$ and $\pi$ are commuting elements by 2.2. Similarly, $\psi$ and $\rho$ commute. And $\tau$, being a cycle of both $\varphi$ and $\psi$, commutes with each of these elements by 2.3. Therefore we have our claim. It follows that $d_{\Delta_{R_n}}(\pi, \rho) \leq 4$.  

Proposition 3.5. Suppose \( n > 4 \). Let \( m \) be the odd element of \( \{ n - 1, n \} \), and define
\[
\pi = (1 \ 2 \ \cdots \ m - 2)(m - 1 \ m) \in R_n, \quad \rho = (2 \ 3 \ \cdots \ m - 1)(1 \ m) \in R_n.
\]
Then \( d_{\Delta S_n}(\pi, \rho) \geq 4 \).

Proof. Suppose that \( d_{\Delta S_n}(\pi, \rho) \leq 3 \). Let \( (\pi, \psi, \varphi, \rho) \) be a commuting path in \( \Delta S_n \). We observe that \( m \geq 5 \); hence the cycles of \( \pi \), including a trivial one if \( n \) is even, have distinct lengths. Therefore by 2.4, there exist positive integers \( r \) and \( s \) such that
\[
\psi = (1 \ 2 \ \cdots \ m - 2)^r(m - 1 \ m)^s.
\]
Similarly, there exist \( t, u \in \mathbb{N} \) such that
\[
\varphi = (2 \ 3 \ \cdots \ m - 1)^t(1 \ m)^u.
\]
We also note that \( \varphi \) and \( \psi \) are nontrivial, being elements of \( S_n \setminus \{1_n\} \). Moreover they are commuting elements, by our assumption that they appear consecutively in a commuting path.

Since \( m - 2 \) is odd, the cycle decomposition of \( (1 \ 2 \ \cdots \ m - 2)^r \) does not contain a transposition. Therefore \( (m - 1 \ m) \) is the only potential transposition among the cycles of \( \varphi \). So, if \( (m - 1 \ m) \) is a cycle of \( \varphi \), then \( (\psi(m - 1) \ \psi(m)) = (m - 1 \ m) \) by 2.2. But clearly \( \psi(m - 1) \neq m \). Thus \( m - 1 \) and \( m \) are fixed by \( \psi \). However, this implies that \( \psi \) is the identity on all of \( \{1, 2, \ldots, n\} \), contradicting that \( \psi \) is nontrivial. Hence \( (m - 1 \ m) \) must not be a cycle of \( \varphi \), and we therefore conclude that \( m - 1 \) and \( m \) are fixed by \( \varphi \). By a parallel argument, \( \psi \) fixes the elements 1 and \( m \). Thus
\[
\varphi = (1 \ 2 \ \cdots \ m - 2)^r, \quad \psi = (2 \ 3 \ \cdots \ m - 1)^t.
\]

Now on the one hand, we observe that \( \varphi(1) \in \{2, 3, \ldots, m - 2\} \), because \( \varphi \) is nontrivial. On the other hand, since \( \varphi \) and \( \psi \) commute, and \( \psi^{-1} \) fixes 1, we have \( \varphi(1) = (\psi \varphi \psi^{-1})(1) = \psi(\varphi(1)) \). Therefore \( \psi \) fixes an element of \( \{2, 3, \ldots, m - 2\} \). But then \( \psi = \text{id} \), a contradiction.

We may now give an argument for Theorem 3.1.

Proof. We obtain the result by combining 3.2, 3.4, and 3.5. We note that 3.5 implies that \( \pi \) and \( \rho \) are at distance \( \geq 4 \) in \( \Delta R_n \), a subgraph of \( \Delta S_n \). \( \square \)
4 The commuting graph $\Delta_{R_n} \cup C_n^2$ for even $n$

We shall prove that if $n$ is even, the result of Theorem 3.1 applies to the larger graph $\Delta_{R_n} \cup C_n^2$.

**Theorem 4.1.** Suppose that $n$ is even, $n \geq 4$. Then $\Delta_{R_n} \cup C_n^2$ is connected. Moreover the diameter of the commuting graph is 3 or 4, according as $n = 4$ or $n > 4$.

Once again, the theorem is realized through several propositions.

**Proposition 4.2.** The diameter of $\Delta_{R_4} \cup C_4^4$ is 3.

*Proof.* From the proof of 3.2, we recall that $D$ denotes the set of all double transpositions in $S_4$. In the proof we argued that $\Delta_{R_4}$ has diameter $\leq 3$ through two observations. In particular, the elements of $D$ mutually commute, and each element of $R_4$ commutes with an element of $D$. The second of these observations extends to include $C_4^4$. Indeed, given $\gamma = (a \ b \ c \ d) \in C_4^4$, we observe that $\gamma$ commutes with $\gamma^2 = (a \ c)(b \ d) \in D$. Thus, as in the proof of 3.2, we conclude that $\Delta_{R_4} \cup C_4^4$ has diameter $\leq 3$.

Now, in demonstrating that the inequality of 3.2 is sharp, we defined $\sigma = (1 \ 2)$ and $\tau = (2 \ 3)$, elements of $R_4$, and argued that $d_{\Delta_{S_4}}(\sigma, \tau) \geq 3$. Thus the distance between $\sigma$ and $\tau$ is $\geq 3$ in $\Delta_{R_4} \cup C_4^4$. We conclude that $\Delta_{R_4} \cup C_4^4$ has diameter $\geq 3$. \hfill \Box

**Lemma 4.3.** Suppose that $n \geq 4$. Let $\pi \in R_n$ be a product of $i \geq 2$ disjoint nontrivial cycles of a common length. Let $\gamma \in R_n$ be a nontrivial cycle of length $j$, where $j \leq i$. Then there exists an element $\rho \in R_n$ that commutes with $\pi$ and $\gamma$.

*Proof.* We consider two possibilities. First assume that there exists a nontrivial cycle $\delta$ in the decomposition of $\pi$ that is disjoint from $\gamma$. Then $\pi$ and $\delta$ commute, by 2.3, and $\gamma$ and $\delta$ commute as well, by 2.8. We observe that $\delta$ is an element of $R_n$, because its length is at most $n/i \leq n/2 \leq n - 2$. Thus $\delta$ may serve as $\rho$.

Now let us assume that for each nontrivial cycle $\delta$ in the decomposition of $\pi$, $\text{Move}(\gamma) \cap \text{Move}(\delta)$ is nonempty. Then the number of nontrivial cycles of $\pi$ does not exceed the length of $\gamma$. In other words $i \leq j$, and hence $i = j$, because the reverse inequality is being assumed. Let $m$ denote the common length of the cycles of $\pi$, and suppose that in decomposed form we have

$$\pi = \prod_{k=0}^{i-1} (a_{k,0} \ a_{k,1} \ a_{k,2} \cdots \ a_{k,m-1}).$$
The commuting graph of the symmetric group $S_n$

Also suppose that $\gamma = (b_0 \ b_1 \ b_2 \ \cdots \ b_{i-1})$, and with no loss in generality, take $a_{k,0} = b_k$ for $0 \leq k < i$. Define

$$\rho = \prod_{l=0}^{m-1} (a_{0,l} \ a_{1,l} \ a_{2,l} \ \cdots \ a_{i-1,l}).$$

We observe that $\rho$ is nontrivial, because $i \geq 2$, and not itself a cycle, because $m \geq 2$. Hence $\rho \in R_n$. Also, $\rho$ commutes with $(a_{0,0} \ a_{1,0} \ a_{2,0} \ \cdots \ a_{i-1,0})$, one of its cycles, by 2.3. Thus $\rho$ commutes with $\gamma$. Furthermore, for each $k \in \{0,1,\ldots,i-1\}$, and each $l \in \{0,1,\ldots,m-1\}$,

$$\left(\rho \pi\right)(a_{k,l}) = \rho(a_{k,(l+1) \mod m}) = a_{(k+1) \mod i,(l+1) \mod m},$$

$$\left(\pi \rho\right)(a_{k,l}) = \pi(a_{(k+1) \mod i,l}) = a_{(k+1) \mod i,(l+1) \mod m}.$$

Hence $\rho$ commutes with $\pi$ as well, because both of $\pi$ and $\rho$ fix each element of the set

$$\{1,2,\ldots,n\} \setminus \{a_{k,l} | 0 \leq k < i, 0 \leq l < m\}.$$

Proof. Assume that $\gamma = (a_1 \ a_2 \ \cdots \ a_n)$. Since $n$ is even, we have

$$\gamma^{n/2} = (a_1 \ a_{n/2+1})(a_2 \ a_{n/2+2}) \ \cdots \ (a_{n/2} \ a_n) \in R_n.$$

We observe that $\gamma$ commutes with $\gamma^{n/2}$, because a group element commutes with each of its powers. By 3.3, there exists a nontrivial cycle $\delta \in R_n$ of length $\leq n/2$ that commutes with $\pi$, because $\pi \in R_n$. Furthermore by 4.3, there exists $\rho \in R_n$ commuting with $\gamma^{n/2}$ and $\delta$. Hence $(\pi,\delta,\rho,\gamma^{n/2},\gamma)$ is a commuting path in $\Delta_{R_n \cup C^n}$. Therefore we have the proposition. \qed

Lemma 4.5. Suppose that $n$ is even, and let $\pi \in S_n$ be a product of $n/2$ disjoint transpositions. Then the order of the centralizer of $\pi$ in $S_n$ is given by

$$(n/2)! \cdot 2^{n/2} = (2)(4)(6)(8) \cdots (n).$$

Proof. In view of 2.2, we must show that the number of elements of $S_n$ that induce a cycle map on $\pi$ is $(n/2)! \cdot 2^{n/2}$. Suppose that we have the decomposition

$$\pi = (a_1 \ b_1)(a_2 \ b_2) \ \cdots \ (a_{n/2} \ b_{n/2}).$$

Consider a general element of $S_n$, written in table form as follows:

$$\rho = \begin{pmatrix} a_1 & b_1 & a_2 & b_2 & \cdots & a_{n/2} & b_{n/2} \\ x_1 & y_1 & x_2 & y_2 & \cdots & x_{n/2} & y_{n/2} \end{pmatrix}$$
We observe that \( \rho \) induces a cycle map on \( \pi \) if and only if
\[
\begin{align*}
\{ \{ x_1, y_1 \}, \{ x_2, y_2 \}, \ldots, \{ x_{n/2}, y_{n/2} \} \} &= \{ \{ a_1, b_1 \}, \{ a_2, b_2 \}, \ldots, \{ a_{n/2}, b_{n/2} \} \}.
\end{align*}
\]
To produce a table \( \rho \) that satisfies this condition, we may first arrange the sets \( \{ a_i, b_i \} \), \( 1 \leq i \leq n/2 \), into a sequence, then arbitrarily order the pair of elements within each set. The number of ways to complete this process is
\[
(n/2)! \cdot 2^{n/2} = (1 \cdot 2)(2 \cdot 2)(3 \cdot 2) \cdots ((n/2) \cdot 2) = (2)(4)(6) \cdots (n).
\]

**Proposition 4.6.** Suppose that \( n \) is even, \( n > 4 \). For \( \gamma, \delta \in C_n^m \), the distance between \( \gamma \) and \( \delta \) in \( \Delta_{R_n \cup C_n^m} \) is \( \leq 4 \).

**Proof.** Let \( \pi = \gamma^{n/2} \), and \( \rho = \delta^{n/2} \). We observe that \( \gamma \) commutes with \( \pi \), because a group element commutes with each of its powers. We also note that \( \pi \) is a product of \( n/2 \) disjoint transpositions, and in particular, \( \pi \in R_n \). Likewise, \( \delta \) commutes with \( \rho \in R_n \), a product of \( n/2 \) disjoint transpositions. Let \( H \) and \( K \) be the centralizers of \( \pi \) and \( \rho \) in \( S_n \), respectively. By a well-known result of finite group theory, we have
\[
|HK| = \frac{|H| \cdot |K|}{|H \cap K|}.
\]
(Refer to [3, p.39].) It is also well known that \( |S_n| = n! \). Therefore \( |HK| \leq n! \), because \( HK \subseteq S_n \). But by 4.5,
\[
|H| \cdot |K| = [(2)(4)(6) \cdots (n)]^2 > n!.
\]
Hence \( |H \cap K| > 1 \). Let \( \varphi \in (H \cap K) \setminus \{\text{id}\} \), and note that \( \varphi \) commutes with \( \pi \) and \( \rho \). Suppose that \( \varphi \in C_n^{m-1} \). Then \( \text{Fix}(\varphi) \) contains precisely one element, say \( a \). So by 2.6, \( a \in \text{Fix}(\pi) \cap \text{Fix}(\rho) \). But \( \text{Fix}(\pi) = \text{Fix}(\rho) = \emptyset \), obviously. Thus \( \varphi \notin C_n^{m-1} \), and so \( \varphi \in R_n \cup C_n^m \). We now realize that \( (\gamma, \pi, \varphi, \rho, \delta) \) is a commuting path in \( \Delta_{R_n \cup C_n^m} \). The proposition follows. \( \square \)

We may now provide an argument for Theorem 4.1.

**Proof.** The assertion for \( n = 4 \) is handled in Proposition 4.2. For \( n > 4 \) and \( n \) even, we combine the results of 3.4, 3.5, 4.4, and 4.6. \( \square \)

We conclude the section by developing a second argument for Proposition 4.6, one which is more enlightening but also more technical. We shall illustrate the construction of a particular element \( \varphi \), based on \( \pi = \gamma^{n/2} \) and \( \rho = \delta^{n/2} \). Especially noteworthy is that the element \( \varphi \), like \( \pi \) and \( \rho \), will be a product of \( n/2 \) disjoint transpositions.
Lemma 4.7. Suppose that $n$ is even, $n \geq 4$. Let $\pi, \rho \in S_n$. Furthermore assume that each of $\pi$ and $\rho$ is a product of $n/2$ disjoint transpositions. Then for all nonnegative integers $j$,
\[ \text{Fix} [\pi(\rho\pi)^j] = \text{Fix} [\rho(\pi\rho)^j] = \emptyset. \] (4.2)

Proof. We proceed by induction on $j$. Clearly we have Fix($\pi$) = Fix($\rho$) = $\emptyset$, thus (4.2) holds for $j = 0$. Let $j$ be a nonnegative integer, and inductively assume that (4.2) holds for this particular $j$. However, suppose that $\text{Fix} [\pi(\rho\pi)^{j+1}] \neq \emptyset$. Then there exists an element $a \in \{1, 2, \ldots, n\}$ such that $[\pi(\rho\pi)^{j+1}] (a) = a$. We note that $\pi^{-1} = \pi$, because $\pi$ is a product of disjoint transpositions. Therefore $[\rho(\rho\pi)^{j}] (\pi(a)) = \pi(a)$, and hence $\text{Fix} [\rho(\pi\rho)^j]$ is nonempty. This is a contradiction. Thus $\text{Fix} [\pi(\rho\pi)^{j+1}] = \emptyset$. And by a parallel argument, $\text{Fix} [\rho(\pi\rho)^{j+1}] = \emptyset$. Hence we have the lemma, by induction. □

Lemma 4.8. Let $n, \pi$, and $\rho$ be as in Lemma 4.7. Let $H$ be the subgroup of $S_n$ generated by $\pi$ and $\rho$, and let $A \subseteq \{1, 2, \ldots, n\}$ be an orbit under the natural action of $H$ on $\{1, 2, \ldots, n\}$. Then the elements of $A$ may be arranged into a sequence $(x_0, x_1, \ldots, x_{2k-1})$ such that $\pi(x_{2j}) = x_{2j+1}$ and $\rho(x_{2j+1}) = x_{(2j+2) \mod 2k}$ for $0 \leq j < k$. In particular, $A$ has even order.

Proof. Let $x_0$ be an arbitrary element of $A$. For each integer $j \in \mathbb{Z}$, we define
\[ x_{2j} = (\rho \pi)^j(x_0), \quad x_{2j+1} = [\pi(\rho \pi)^j] (x_0). \]
Then we have $x_{2j+1} = \pi(x_{2j})$, and $x_{2j+2} = (\rho \pi)(x_{2j}) = \rho(x_{2j+1})$. We note that each $x_i$ is an element of $\{1, 2, \ldots, n\}$. Thus there exists a repeated value in the sequence $(x_0, x_1, x_2, \ldots)$. Let $m$ be the minimum positive index such that $x_m$ is an element of $\{x_0, x_1, \ldots, x_{m-1}\}$. In particular suppose that $x_m = x_l$, where $l \in \{0, 1, \ldots, m - 1\}$.

We claim that $m$ and $l$ have the same parity. If $l$ is even, say $l = 2i$, then for an arbitrary nonnegative integer $j$,
\[ x_{l+2j+1} = [\pi(\rho \pi)^{i+j}] (x_0) = [\pi(\rho \pi)^j(\rho \pi)^i] (x_0) = [\pi(\rho \pi)^j] (x_l). \]
And if $l = 2i + 1$,
\[ x_{l+2j+1} = (\rho \pi)^{i+j+1}(x_0) = [\rho(\pi \rho)^j(\rho \pi)^i] (x_0) = [\rho(\pi \rho)^j] (x_l). \]
But regardless of the parity of $l$, we see that $x_{l+2j+1} \neq x_l$, by 4.7. Thus we have our claim, because $x_m = x_l$. Assume that $m = l + 2k$, where $k$ is a positive integer. Since $m - 1$ and $l - 1$ have the same parity, $x_m = \pi(x_{m-1})$ and $x_l = \pi(x_{l-1})$, or $x_m = \rho(x_{m-1})$ and $x_l = \rho(x_{l-1})$. Whichever the case, $x_{m-1} = x_{l-1}$, because $\pi$ and $\rho$ are injective. It follows that $l = 0$, because of the minimum condition that we imposed on $m$. Hence $m = 2k$. 

The commuting graph of the symmetric group $S_n$ 297
Now for an arbitrary integer $j$, we have
\[ x_{2j+2k} = (\rho \pi)^{j+k}(x_0) = (\rho \pi)^j(x_{2k}) = (\rho \pi)^j(x_0) = x_{2j}. \]

Therefore,
\[ x_{(2j+1)+2k} = x_{2(j+k)+1} = \pi(x_{2j+2k}) = \pi(x_{2j}) = x_{2j+1}. \]

Hence the function $i \mapsto x_i$, defined on $\mathbb{Z}$, has period $2k$.

We observe that $A = \{ \varphi(x_0) \mid \varphi \in H \}$. Thus \{ $x_i \mid i \in \mathbb{Z}$ $\} \subseteq A$. We complete the proof by demonstrating the reverse inclusion. Let $\varphi$ be an element of $H$. Then for some nonnegative integer $t$, there exist elements $\psi_s \in \{ \pi, \pi^{-1}, \rho, \rho^{-1} \}$, $1 \leq s \leq t$, such that $\varphi = \psi_1\psi_2\psi_3 \cdots \psi_t$. (See [2, p.62].) However, $\pi^{-1} = \pi$, or equivalently $\pi^2 = \operatorname{id}$, because $\pi$ is a product of disjoint transpositions. Likewise, $\rho^{-1} = \rho$. Thus by canceling successive factors of $\psi_1\psi_2\psi_3 \cdots \psi_t$ as long as possible, we obtain
\[ \varphi \in \{ (\rho \pi)^j, \pi(\rho \pi)^j, (\pi \rho)^j, \rho(\pi \rho)^j \}, \]
for some nonnegative integer $j$. We have $(\rho \pi)^j(x_0) = x_{2j}$, and $[\pi(\rho \pi)^j](x_0) = x_{2j+1}$. Furthermore, we observe
\[
(\pi \rho)^j(x_0) = \left( (\rho^{-1} \pi^{-1})^{-j} \right) (x_0) = \left( (\rho \pi)^{-j} \right) (x_0) = x_{-2j},
\]
\[
[\rho(\pi \rho)^j](x_0) = \left[ \pi(\pi \rho)^{j+1} \right] (x_0) = \pi(x_{-2(j+1)}) = x_{-2(j+1)+1}.
\]

Therefore $\varphi(x_0) \in \{ x_i \mid i \in \mathbb{Z} \}$. We conclude that $A \subseteq \{ x_i \mid i \in \mathbb{Z} \}$, as desired. \hfill \Box

**Proposition 4.9.** Let $n$, $\pi$, and $\rho$ be as in 4.7. Then there exists $\varphi \in S_n$, also a product of $n/2$ disjoint transpositions, commuting with $\pi$ and $\rho$.

**Proof.** Let $H$ be the subgroup of $S_n$ generated by $\pi$ and $\rho$. Let $A$ be an arbitrary orbit under the natural action of the subgroup $H$ on $\{1, 2, \ldots, n\}$. Then each of the elements $\pi$ and $\rho$ maps $A$ to itself, bijectively. Let $\pi_A$ and $\rho_A$ denote the restrictions of $\pi$ and $\rho$ to $A$, respectively. Let $(x_0, x_1, x_2, \ldots, x_{2k-1})$ be an arrangement of the elements of $A$, as in 4.8. For $0 \leq i < k$, define
\[ \varphi_A(x_i) = x_{(2k-1)-i}, \quad \varphi_A(x_{(2k-1)-i}) = x_i. \]

Since $\pi$ is a product of disjoint transpositions, and $\pi(x_{2j}) = x_{2j+1}$ for $0 \leq j < k$, by design, we see that
\[ \pi_A = (x_0 \ x_1)(x_2 \ x_3)(x_4 \ x_5) \cdots (x_{2k-2} \ x_{2k-1}). \]
The commuting graph of the symmetric group $S_n$

Applying $\varphi_A$ to the elements $x_i$ within the respective transpositions here, we reverse the sequence of indices and obtain the product

$$(x_{2k-1} x_{2k-2})(x_{2k-3} x_{2k-4})(x_{2k-5} x_{2k-6}) \cdots (x_1 x_0) = \pi_A.$$ 

Hence $\varphi_A$ induces a cycle map on $\pi_A$. Furthermore, defining $\varphi = \prod_A \varphi_A$, we realize that $\varphi$ induces a cycle map on $\prod_A \pi_A = \pi$. Therefore $\varphi$ and $\pi$ are commuting elements, by 2.2.

Now, since $\rho$ is a product of disjoint transpositions, and furthermore, $\rho(x_{2j+1}) = x_{(2j+2) \text{mod} 2k}$, we have

$$\rho_A = (x_1 x_2)(x_3 x_4)(x_5 x_6) \cdots (x_{2k-3} x_{2k-2})(x_{2k-1} x_0).$$

Applying $\varphi_A$ to the respective $x_i$ here yields

$$(x_{2k-2} x_{2k-3})(x_{2k-4} x_{2k-5})(x_{2k-6} x_{2k-7}) \cdots (x_2 x_1)(x_0 x_{2k-1}) = \rho_A.$$ 

Therefore, we see that $\varphi$ induces a cycle map on $\rho$, so $\varphi$ and $\rho$ are commuting elements, by 2.2. And $\varphi$ is obviously a product of disjoint transpositions, fixing no element of $\{1, 2, \ldots, n\}$. Thus the proof is complete.

We have now realized our goal of a more constructive route to Proposition 4.6. But furthermore, Proposition 4.9 yields a new proof of a result from [1, p.139].

**Theorem 4.10.** Suppose that $n$ is even, $n > 4$. Let $X$ be the set of all products of $n/2$ disjoint transpositions in $S_n$. Then the commuting graph $\Delta_X$ has diameter 2.

**Proof.** The diameter of $\Delta_X$ is $\leq 2$ by 4.9. Define

$$\pi = (1 \ 2)(3 \ 4)(5 \ 6) \cdots (n - 1 \ n),$$

$$\rho = (2 \ 3)(4 \ 5)(6 \ 7) \cdots (n - 2 \ n - 1)(1 \ n),$$

a particular pair of elements of $X$. We observe that $(\rho\pi)(1) = \rho(2) = 3$, while $(\pi\rho)(1) = \pi(n) = n - 1$. Therefore $(\rho\pi)(1) \neq (\pi\rho)(1)$, because $n > 4$. Hence $\pi$ and $\rho$ are not commuting elements. We conclude that the diameter of $\Delta_X$ is $> 1$.

We remark that in the terminology of [1], $\Delta_X$ is referred to as a commuting involution graph.
5 An upper bound on the diameter of $\Delta_{S_n}$ for composite $n$ and $n-1$

Theorem 5.1. Suppose that each of $n$ and $n-1$ is a composite number. Then the diameter $\Delta_{S_n}$ is $\leq 5$.

We remark that $n \geq 9$ here, implicitly. We obtain the theorem through two propositions, that shall accompany Theorem 3.1.

Proposition 5.2. Suppose that $n > 4$, and $l \in \{n-1, n\}$ is a composite number. Let $\pi \in R_n$ and $\gamma \in C^l_n$. Then the distance between $\pi$ and $\gamma$ in $\Delta_{R_n \cup C^l_n}$ is $\leq 5$.

Proof. By 3.3, there exists a cycle $\delta \in R_n$ of length $\leq n/2$ that commutes with $\pi$. We have $|\text{Fix}(\delta)| \geq n/2$, thus $|\text{Fix}(\delta)| \geq 3$ because $n > 4$. Choose $a, b \in \text{Fix}(\delta)$, and define $\tau = (a b) \in R_n$. Then $\tau$ is disjoint from $\delta$; hence $\tau$ and $\delta$ are commuting elements, by 2.8.

Now assume that $\gamma = (c_1 c_2 c_3 \cdots c_l)$. Since $l$ is composite, there exists a positive integer $i \in (1, l)$ that divides $l$. We observe that $\gamma$ commutes with $\gamma^i$, because a group element commutes with each of its powers. In decomposed form, we have

$$\gamma^i = \prod_{j=1}^{i} (a_{j+i} a_{j+2i} \cdots a_{j+(i/i-1)i}).$$

In particular, $\gamma^i$ is a product of $i \geq 2$ disjoint cycles, each of length $l/i \geq 2$. Hence $\gamma^i \in R_n$. Moreover since $\tau$ is a cycle of length $\leq i$, there exists an element $\rho \in R_n$ commuting with $\gamma^i$ and $\tau$, by 4.3. Therefore, $(\pi, \delta, \tau, \rho, \gamma^i, \gamma)$ is a commuting path in $\Delta_{R_n \cup C^l_n}$. The proposition follows.

Proposition 5.3. Let $l$ and $m$ be elements of the set $\{n-1, n\}$. Assume that $l \leq m$, and that each of $l$ and $m$ is a composite number. Let $\gamma \in C^l_n$, and $\delta \in C^m_n$. Then the distance between $\gamma$ and $\delta$ in $\Delta_{R_n \cup C^l_n \cup C^m_n}$ is $\leq 5$.

Proof. Let $D_l \subseteq \{2, 3, \ldots, l-1\}$ be the set of all proper nontrivial divisors of $l$. Since $l$ is composite, $D_l$ is nonempty. Let $i = \max(D_l)$. Since $i \in D_l$, we have $l/i \in D_l$ as well. Therefore $l/i \leq i$, because $i$ is maximal; so $\sqrt{l} \leq i$. Analogously, we let $D_m \subseteq \{2, 3, \ldots, m-1\}$ be the collection of all proper nontrivial divisors of $m$, nonempty because $m$ is composite, and we let $j = \max(D_m)$. We then note that $m/j \in D_m$, and deduce that $\sqrt{m} \leq j$. Moreover we have $\sqrt{l} \leq j$, because $l \leq m$. Hence $l \leq ij$, and so $l/i \leq j$.

Now assume that

$$\gamma = (a_1 a_2 a_3 \cdots a_l), \quad \delta = (b_1 b_2 b_3 \cdots b_m).$$
Then we have, in decomposed form,

\[ \gamma^i = \prod_{k=1}^{i} \left(a_{k+i} a_{k+2i} \ldots a_{k+[l/i]-1} \right); \]

\[ \delta^j = \prod_{k=1}^{j} \left(b_{k+j} b_{k+2j} \ldots b_{k+[m/j]-1} \right). \]

We observe that \( \gamma^i \) consists of \( i \) nontrivial cycles, each of length \( l/i \), and \( \delta^j \) consists of \( j \) nontrivial cycles, each of length \( m/j \). So obviously, \( \gamma^i, \delta^j \in R_n \).

Let \( \sigma \) be any nontrivial cycle in the decomposition of \( \gamma^i \). Then \( \sigma \in R_n \), because \( \gamma^i \) has multiple nontrivial cycles. By 2.3, \( \sigma \) commutes with \( \gamma^i \). Furthermore since \( l/i \leq j \), there exists an element \( \rho \in R_n \) that commutes with \( \sigma \) and \( \delta^j \), by 4.3. Hence \( \langle \gamma, \gamma^i, \sigma, \rho, \delta^j \rangle \) is a commuting path in \( \Delta_{R_n} \cup C_n \), because \( \gamma \) and \( \delta \) commute with \( \gamma^i \) and \( \delta^j \), respectively. Thus we have the proposition. \( \square \)

We finish the section with an argument for Theorem 5.1.

Proof. As noted earlier, we have \( n \geq 9 \), because \( n \) and \( n - 1 \) are composite numbers. The theorem is realized immediately by combining the results of Theorem 3.1, and Propositions 5.2 and 5.3. \( \square \)

6 The existence of elements at distance 5 in \( \Delta S_n \)

We exhibit two pairs of elements at distance \( \geq 5 \) in the commuting graph \( \Delta S_n \). In each of our constructions, we shall require the following standard result.

Lemma 6.1. Let \( G \) be a group. Suppose that \( g \in G \) has finite order \( i \). Then for all positive integers \( j \), \( \langle g^j \rangle = \langle g^{\gcd(i,j)} \rangle \).

Proof. Assume that \( j = k \cdot \gcd(i,j) \), where \( k \) is a positive integer that is relatively prime to \( i \). On the one hand, we observe that \( (g^{\gcd(i,j)})^k = \gamma^j \). Thus \( \gamma^j \in \langle \gamma^{\gcd(i,j)} \rangle \), and so \( \langle \gamma^j \rangle \subseteq \langle \gamma^{\gcd(i,j)} \rangle \). On the other hand, by a well-known result of number theory, there exist integers \( x \) and \( y \) such that \( ix + ky = \gcd(i,k) = 1 \). (See [3, p.11].) Therefore, we have

\[ (\gamma^j)^y = (\gamma^{\gcd(i,j)})^{ky} = (\gamma^{\gcd(i,j)})^{1-ix} = \gamma^{\gcd(i,j)} (\gamma^i)^{-(x \cdot \gcd(i,j))} = \gamma^{\gcd(i,j)}. \]

Hence \( \gamma^{\gcd(i,j)} \in \langle \gamma^j \rangle \), and thus \( \langle \gamma^{\gcd(i,j)} \rangle \subseteq \langle \gamma^j \rangle \). \( \square \)

Proposition 6.2. Suppose that \( n \) is a positive integer, \( n \geq 3 \). Let

\[ \gamma = (1 \ 2 \ 3 \ \ldots \ n - 1) \in C_n^{m-1}, \quad \delta = (1 \ 2 \ 3 \ \ldots \ n) \in C_n^m. \]

Then the distance between \( \gamma \) and \( \delta \) in \( \Delta S_n \) is at least 5.
Proof. We note that $\gamma$ and $\delta$ are in fact elements of $S_n \setminus \{1_n\}$, because $n \geq 3$. Suppose that $d_{\Delta_n}(\gamma, \delta) \leq 4$. In particular, assume that $(\gamma, \varphi, \chi, \psi, \delta)$ is a commuting path in $\Delta_n$. Since $\varphi$ and $\psi$ commute with $\gamma$ and $\delta$, respectively, there exist positive integers $s$ and $t$ such that $\varphi = \gamma^s$ and $\psi = \delta^t$, by 2.4. Let $u = \gcd(s, n - 1)$ and $v = \gcd(t, n)$; and note that $u$ and $v$ are proper divisors of $n - 1$ and $n$, respectively, because $\varphi$ and $\psi$ are nontrivial elements. Let $\pi = \gamma^u$, $\rho = \delta^v$, and $H = \langle \pi, \rho \rangle$. By 6.1, we have $\langle \varphi \rangle = \langle \pi \rangle$ and $\langle \psi \rangle = \langle \rho \rangle$. Hence each subgroup of $S_n$ that contains $\varphi$ and $\psi$ will also contain $\pi$ and $\rho$, and vice-versa. Therefore, $H = \langle \varphi, \psi \rangle$.

We observe that precisely one of the integers $n - 1$ and $n$ is divisible by 2; so $u + v < (n - 1)/2 + n/2$. Thus $u + v$, itself an integer, must be $\leq n - 1$. Let $m = \min(u, v)$, and let $a \in \{n - m, n - m + 1, n - m + 2, \ldots, n - 1\}$. We observe that $n - 1$ is strictly less than $a + u$ and $a + v$, but $a + u + v \leq 2(n - 1)$. Therefore,

$$
(r \pi)(a) = \rho[a + u - (n - 1)] = a + u + v - (n - 1),
$$

$$
(\pi \rho)(a + 1) = \pi[(a + 1) + v - n] = \pi[a + v - (n - 1)]
= a + u + v - (n - 1).
$$

Thus $(r \pi)(a) = (\pi \rho)(a + 1)$, and so $(\rho^{-1} \pi^{-1} r \pi)(a) = a + 1$. Hence $a + 1 \in [a]_H$, because $\rho^{-1} \pi^{-1} r \pi \in H$. Moreover $[a]_H = [a + 1]_H$, by 2.10. Therefore, we conclude that

$$
[n - m]_H = [n - m + 1]_H = \cdots = [n - 1]_H = [n]_H. \quad (6.1)
$$

Suppose that $b \in \{1, 2, \ldots, n - m - 1\}$. We observe that $i = 0$ is a solution to $b + im < n - m$; thus there exists a maximum nonnegative integer $i$ for which the inequality holds. For this $i$, we have

$$
b + (i + 1)m \in \{n - m, n - m + 1, n - m + 2, \ldots, n - 1\}.
$$

We observe that if $m = u$, then $\pi^{i+1}(b) = b + (i + 1)m$. And if $m = v$, then $\rho^{i+1}(b) = b + (i + 1)m$. Either way, we have $[b]_H = [b + (i + 1)m]_H$, because each of $\pi^{i+1}$ and $\rho^{i+1}$ is an element of $H$. Together with (6.1), this implies that

$$
[1]_H = [2]_H = \cdots = [n - 1]_H = [n]_H.
$$

Hence $[n]_H = \{1, 2, \ldots, n\}$.

Define $K = \langle \varphi, \chi, \psi \rangle$. Then $K$ may be explicitly described as the set of all products of the form $\eta_1 \eta_2 \cdots \eta_w$, where $w$ is a positive integer, and each $\eta_j$ is an element of $\{\varphi, \varphi^{-1}, \chi, \chi^{-1}, \psi, \psi^{-1}\}$. (See [2, p.62].) We recall that $\chi$ sits between $\varphi$ and $\psi$ in our commuting path; hence $\chi$ commutes with $\varphi$ and $\psi$. Therefore $\chi$ commutes with $\varphi^{-1}$ and $\psi^{-1}$ as well. And of course $\chi$ commutes
The commuting graph of the symmetric group $S_n$ with itself and its inverse. Thus $\chi$ commutes with all products $\eta_1\eta_2\cdots\eta_w$. In other words, $\chi$ is a member of the center of $K$.

Since $\varphi \in \langle \gamma \rangle \setminus \{\text{id}\}$, we see that $Fix(\varphi) = \{n\}$. Therefore by 2.6, $n \in Fix(\chi)$, because $\varphi$ and $\chi$ are commuting elements. Moreover since $\chi$ is an element of the center of $K$, we have $[n]_K \subseteq Fix(\chi)$, by 2.11. But $H$ is a subgroup of $K$, because $H = \langle \varphi, \psi \rangle$ and $\varphi, \psi \in K$. Thus $[n]_H \subseteq [n]_K$, and so $[n]_K = \{1, 2, \ldots, n\}$. We conclude that $Fix(\chi) = \{1, 2, \ldots, n\}$, which implies that $\chi$ is the identity element of $S_n$. This is a contradiction. Hence we have the proposition.

To prove the main result of the paper, we must still demonstrate the existence of a pair of elements at distance 5 in $\Delta_{R_n \cup C_n^{n-1}}$, when $n-1$ is composite. For the remainder of the section, the following setup shall apply.

- Let $n$ be a positive integer such that $n - 1$ is a composite number.
- Let $M$ be the maximum proper divisor of $n - 1$.
- Define the following elements of $C_n^{n-1}$:
  \begin{align*}
  \gamma &= (1 2 \cdots n-1); \\
  \delta &= (1 2 \cdots M \ n \ M+1 \ M+2 \cdots n-2). \tag{6.2} \tag{6.3}
  \end{align*}
- Let $p$ and $q$ be arbitrary prime divisors of $n - 1$, possibly alike.
- Let $r = (n - 1)/p$, $s = (n - 1)/q$, and $m = \min(r, s)$.
- Define $H = \langle \gamma^r, \delta^s \rangle$.

We note that $n \geq 5$, $M > 1$, and $m > 1$, because $n - 1$ is composite. We also point out that $r$ and $s$ are proper divisors of $n - 1$, hence $r, s \leq M$.

**Lemma 6.3.** Suppose that $a$ is an integer such that $M + 1 - m \leq a \leq M - 2$. Then $[a]_H = [a + 2]_H$.

**Proof.** First assume that $r \leq s$. We observe that

$$M + 1 \leq a + r \leq 2M - 2 \leq n - 3.$$  

Let $i$ be the maximum positive integer such that $M + 1 \leq a + ir \leq n - 2$. Then since $r \leq s$, we have $n - 2 < a + ir + s \leq (n - 2) + M$. Therefore

$$\delta^{ir}\gamma^r(a) = \delta^s(a + ir) = a + ir + s - (n - 2).$$

But we have $M + 1 \leq a + s \leq n - 3$ as well, so

$$\gamma^{ir}\delta^s(a + 2) = \gamma^{ir}(a + s + 1) = a + ir + s + 1 - (n - 1).$$
Hence $(\delta^s \gamma^r)(a) = (\gamma^r \delta^s)(a + 2)$, and thus $(\delta^{-s} \gamma^{-r} \delta^s \gamma^r)(a) = a + 2$. It follows that $[a]_H = [a + 2]_H$, because $\delta^{-s} \gamma^{-r} \delta^s \gamma^r \in H$.

Now let us assume that $s < r$. Let $j$ be the maximum positive integer such that $M + 1 \leq a + js \leq n - 3$. Then $n - 2 < a + r + js \leq (n - 3) + M$, because $s$ is strictly less than $r$. Thus

$$(\gamma^r \delta^j)(a + 2) = \gamma^r(a + js + 1) = a + r + js + 1 - (n - 1).$$

But on the other hand,

$$(\delta^j \gamma^r)(a) = \delta^j(a + r) = a + r + js - (n - 2).$$

Thus $(\gamma^r \delta^j)(a + 2) = (\delta^j \gamma^r)(a)$. So once again, $[a]_H = [a + 2]_H$. \hfill \Box

**Lemma 6.4.** Suppose that $a$ and $b$ are integers such that $M - m + 1 \leq a, b \leq M$. Assume that $a$ and $b$ have opposite parity. Then $[a]_H \cup [b]_H = \{1, 2, \ldots, n\}$.

**Proof.** Let $i$ and $j$ be the odd and even elements of the set $\{m - 1, m\}$, respectively. Define

$$C = \{M - m + 1, M - m + 3, M - m + 5, \ldots, M - m + i\},$$

$$D = \{M - m + 2, M - m + 4, M - m + 6, \ldots, M - m + j\}.$$

We observe that $C \cup D = \{M - m + 1, M - m + 2, \ldots, M - 1, M\}$, so $a, b \in C \cup D$. Also, either the elements of $C$ are strictly even and those of $D$ are strictly odd, or vice versa. Thus one of the elements $a$ and $b$ is a member of $C$, and the other is a member of $D$. But by 6.3,

$$[M - m + 1]_H = [M - m + 3]_H = [M - m + 5]_H = \cdots = [M - m + i]_H,$$

$$[M - m + 2]_H = [M - m + 4]_H = [M - m + 6]_H = \cdots = [M - m + j]_H.$$

Therefore $C \cup D \subseteq [a]_H \cup [b]_H$.

Suppose $1 \leq x \leq M - m$. Let $k$ be the maximum nonnegative integer such that $x + km \leq M - m$, and let $y = x + (k + 1)m$. Then $y \in C \cup D$, because $C \cup D$ consists of $m$ consecutive integers. We observe that $\gamma^{(k+1)m}(x) = \delta^{(k+1)m}(x) = y$. Also, if $m = r$ then $\gamma^{(k+1)m} = (\gamma^r)^{k+1} \in H$, and if $m = s$ then $\delta^{(k+1)m} = (\delta^s)^{k+1} \in H$. Therefore, $[x]_H = [y]_H$. But $y \in [a]_H$ or $y \in [b]_H$, because $y \in C \cup D$. Hence $x \in [a]_H$ or $x \in [b]_H$. In other words, $x \in [a]_H \cup [b]_H$.

Now assume that $M + 1 \leq z \leq n - 1$. Let $l$ be the maximum nonnegative integer such that $z - lr \geq M + 1$. Then $z - (l+1)r \in \{1, 2, \ldots, M\}$, since $r \leq M$. And we have $\gamma^{-(l+1)r}(z) = z - (l+1)r$. Therefore $[z]_H = [z - (l+1)r]_H$, because $\gamma^{-(l+1)r} = (\gamma^r)^{-(l+1)} \in H$. But we have already shown that $\{1, 2, \ldots, M\} \subseteq [a]_H \cup [b]_H$. Thus $z \in [a]_H \cup [b]_H$.

Finally, we observe that $\delta^{-s}(n) = M - s + 1$. Hence $[n]_H = [M - s + 1]_H$. Therefore $n \in [a]_H \cup [b]_H$, because $M - s + 1 \in \{1, \ldots, M - 1\} \subseteq [a]_H \cup [b]_H$. \hfill \Box
Proposition 6.5. Suppose \( p = 2 \) or \( q = 2 \), but \( p \neq q \). Then

\[
[n - 1]_H \cup [n]_H = \{1, 2, \ldots, n\}.
\]

Proof. Since \( 2 \in \{p, q\} \), and \( p \) and \( q \) are divisors of \( n - 1 \), we realize that \( n - 1 \) is even. Therefore \( M = (n - 1)/2 \).

Assume that \( p = 2 \). Then we have \( r = M \) and \( s = m \); thus \( \gamma^r(n - 1) = M \) and \( \delta^{-s}(n) = M - m + 1 \). Therefore \([n - 1]_H = [M]_H\), and \([n]_H = [M - m + 1]_H\).

Since \( p \neq q \), \( q \) is an odd prime. Hence \( s \) is even, because \( n - 1 \) is even, and so \( M \) and \( M - m + 1 \) have opposite parity, because \( s = m \). Thus by 6.4, we have \([M]_H \cup [M - m + 1]_H = \{1, 2, \ldots, n\}\), and therefore \([n - 1]_H \cup [n]_H = \{1, 2, \ldots, n\}\).

Now suppose that \( q = 2 \). Then by analogy to the above case, \( r = m \) and \( s = M \), and \( r \) is even. We observe that the set \( \{M - m + 1, M - m + 2, \ldots, M\} \) consists of \( r \) consecutive integers. Therefore the set includes elements \( a \) and \( b \) such that \( a \equiv 0 \pmod{r} \) and \( b \equiv 1 \pmod{r} \). Let \( i \) and \( j \) be integers such that \( a = ir \) and \( b = jr + 1 \). Then \( \gamma^i(n - 1) = a \), and \( (\gamma^j\delta^{-s})(n) = \gamma^j(1) = b \), because \( s = M \). Hence \([n - 1]_H = [a]_H\), and \([n]_H = [b]_H\). But \( a \) and \( b \) have opposite parity, because \( r \) is even. Thus by 6.4 once again, we have \([n - 1]_H \cup [n]_H = \{1, 2, \ldots, n\}\).

We point out that \([n - 1]_H \neq [n]_H\) is a possibility under the hypotheses of 6.5. For example if \( n - 1 = 6 \), then

\[
\gamma = (1 \ 2 \ 3 \ 4 \ 5 \ 6), \quad \delta = (1 \ 2 \ 3 \ 7 \ 4 \ 5).
\]

If \( p = 2 \) and \( q = 3 \), we have \( r = 3 \) and \( s = 2 \). Therefore \( \gamma^r = (1 \ 4)(2 \ 5)(3 \ 6) \), and \( \delta^s = (1 \ 3 \ 4)(2 \ 7 \ 5) \). Thus

\[
[6]_H = \{1, 3, 4, 6\}, \quad [7]_H = \{2, 5, 7\}.
\]

Definition 6.6. If the natural action of \( H \) on \( \{1, 2, \ldots, n\} \) has precisely one orbit, then we shall say that \( H \) is transitive.

Proposition 6.7. If \( p = q = 2 \), then \( H \) is transitive.

Proof. We observe that \( r = s = M = (n - 1)/2 \). Thus for \( 1 \leq a \leq M - 1 \), we have

\[
(\delta^{-s}\gamma^r)(a) = (\delta^{-M}\gamma^M)(a) = \delta^{-M}(a + M) = a + 1.
\]

Therefore \([a]_H = [a + 1]_H\), and furthermore, \([1]_H = [2]_H = \cdots = [M]_H\).

Now, for \( M + 2 \leq a \leq n - 1 \),

\[
(\delta^s\gamma^{-r})(a) = \delta^M(a - M) = a - 1.
\]

Hence \([a]_H = [a - 1]_H\), and so \([M + 1]_H = [M + 2]_H = \cdots = [n - 1]_H\).
Finally, we notice that $\gamma^r(n-1) = M$, and $\delta^{-s}(n) = 1$. Therefore we have $[n-1]_H = [M]_H$, and $[n]_H = [1]_H$. So we conclude that

$$[1]_H = [2]_H = \cdots = [n-1]_H = [n]_H.$$  

**Lemma 6.8.** If each of $r$ and $s$ is $\leq n - M - 3$, then $H$ is transitive.

*Proof.* Let $a = M - m + 1$. Since $m \geq 2$, the set $\{M-m+1, M-m+2, \ldots, M\}$ contains at least two elements; thus $\{a, a+1\}$ is a subset. We point out that each of $a+r$ and $a+s$ is $\geq M+1$, but that $a+r+s = M + \max(r, s) + 1 \leq M + (n - M - 3) + 1 = n - 2$. Therefore,

$$\begin{align*}
(\delta^s \gamma^r)(a) &= \delta^s(a+r) = a+r+s, \\
(\gamma^r \delta^s)(a+1) &= \gamma^r(a+s) = a+r+s.
\end{align*}$$

Hence $(\delta^{-s} \gamma^{-r} \delta^s \gamma^r)(a) = a + 1$, and so $[a]_H = [a+1]_H$. But by 6.4, we have $[a]_H \cup [a+1]_H = \{1, 2, \ldots, n\}$. Thus the lemma follows. 

**Proposition 6.9.** If $p \neq 2$ and $q \neq 2$, then $H$ is transitive.

*Proof.* We observe that $n-1$, being divisible by an odd prime, is not a power of 2. In the case of $n-1 = 6$ and $p = q = 3$, $\gamma$ and $\delta$ are as in equation (6.4), and $r = s = 2$. Therefore $\gamma^r = (1 3 5)(2 4 6)$, and $\delta^s = (1 3 4)(2 7 5)$. Thus we obviously have

$$[1]_H = [3]_H = [5]_H, \quad [2]_H = [4]_H = [6]_H;$$

$$[1]_H = [3]_H = [4]_H, \quad [2]_H = [7]_H = [5]_H.$$ 

Hence we see that $[1]_H = [2]_H = \cdots = [7]_H$, and so $H$ is transitive.

For $n-1 = 9$, we have $p = q = r = s = M = 3$. And for $n-1 = 10$, $p = q = M = 5$ and $r = s = 2$. But in either case, each of $r$ and $s$ is less than $n - M - 3$. Therefore $H$ is transitive, by 6.8.

Now assume that $n-1 \geq 12$. Since $p$ and $q$ are both odd primes, each of $r$ and $s$ is $\leq (n-1)/3$. Therefore

$$M + \max(r, s) \leq \frac{n-1}{2} + \frac{n-1}{3} = \frac{n - n + 5}{6} \leq n - 3.$$ 

Thus by 6.8, $H$ is transitive once again.

In the proof of the culminating result of the current section, as follows, the prime numbers $p$ and $q$ that have been under consideration, and thus the group $H$, shall arise. In stating the proposition, we keep our assumptions that $n - 1$ is composite, and $M$ is the maximum proper divisor of $n - 1$. The definitions of $\gamma$ and $\delta$, as in (6.2) and (6.3), remain as well. Our argument here revisits many of the techniques that we applied in the proof of 6.2.
Proposition 6.10. The distance between $\gamma$ and $\delta$ in $\Delta_{S_n}$ is at least 5.

Proof. Suppose that the distance between $\gamma$ and $\delta$ in $\Delta_{S_n}$ is $\leq 4$. In particular, assume that $(\gamma, \varphi, \chi, \psi, \delta)$ is a commuting path in $\Delta_{S_n}$. Since $\varphi$ and $\psi$ commute with $\gamma$ and $\delta$, respectively, there exist positive integers $t$ and $u$ such that $\varphi = \gamma^t$ and $\psi = \delta^u$, by 2.4. We observe that $\gamma^{gcd(t,n-1)} \in \langle \varphi \rangle$, and $\delta^{gcd(u,n-1)} \in \langle \psi \rangle$, by 6.1. Also, $gcd(t,n-1)$ and $gcd(u,n-1)$ are proper divisors of $n-1$, because $\varphi$ and $\psi$ are nontrivial elements. Suppose that $n-1 = p \cdot j \cdot gcd(t,n-1) = q \cdot k \cdot gcd(u,n-1)$, where $p$ and $q$ are prime numbers, and $j$ and $k$ are positive integers. Since $\gamma^{gcd(t,n-1)} \in \langle \varphi \rangle$, we have $(\gamma^{gcd(t,n-1)})^j = \gamma^{(n-1)/p} \in \langle \varphi \rangle$ as well. Similarly, $\delta^{(n-1)/q} \in \langle \psi \rangle$. And we note that $\gamma^{(n-1)/p}$ and $\delta^{(n-1)/q}$ are nontrivial elements, because each of $(n-1)/p$ and $(n-1)/q$ is strictly less than $n-1$.

Now given the structure of our commuting path, we see that $\chi$ commutes with $\varphi$ and $\psi$. So furthermore, $\chi$ commutes with each element of $\langle \varphi \rangle$, and each of $\langle \psi \rangle$. Hence $\chi$ commutes with $\gamma^{(n-1)/p}$ and $\delta^{(n-1)/q}$.

Let us define $K = \langle \gamma^{(n-1)/p}, \chi, \delta^{(n-1)/q} \rangle$.

We observe that $\chi$ is a member of the center of $K$, as in the proof of 6.2. Also, by 2.6, $n-1$ and $n$ are fixed by $\chi$, because we obviously have $Fix(\gamma^{(n-1)/p}) = \{n\}$ and $Fix(\delta^{(n-1)/q}) = \{n-1\}$. Moreover we have $[n-1]_K \cup [n]_K \subseteq Fix(\chi)$, by 2.11. But letting $H = \langle \gamma^{(n-1)/p}, \delta^{(n-1)/q} \rangle$, $[n-1]_H \cup [n]_H = \{1, 2, \ldots, n\}$, in view of Propositions 6.5, 6.7, and 6.9. Hence $[n-1]_K \cup [n]_K = \{1, 2, \ldots, n\}$, because $H$ is a subgroup of $K$. We conclude that $Fix(\chi) = \{1, 2, \ldots, n\}$. In other words, $\chi$ is the identity element of $S_n$. This is a contradiction, so the proof is complete. 

\[ \square \]

7 Proof of main result

In order to realize the main result of the paper, we prove one further proposition.

Proposition 7.1. Suppose that $p \in \{n-1, n\}$ is a prime number. Let $\gamma$ be an element of $C_n^p$. Then $\Delta_{\langle \gamma \rangle}$ is a connected component of $\Delta_{S_n}$. Furthermore, the diameter of $\Delta_{\langle \gamma \rangle}$ is 0 or 1, according as $p = 2$ or $p > 2$.

Proof. We observe that $\langle \gamma \rangle$ is an abelian group, containing $p-1$ nontrivial elements. Thus we see that the diameter of $\Delta_{\langle \gamma \rangle}$ is equal to 0 if $p = 2$, but equal to 1 if $p > 2$.

Suppose that the connected component of $\gamma$ in $\Delta_{S_n}$ strictly contains $\Delta_{\langle \gamma \rangle}$. Then there exists a commuting path $(\gamma, \pi, \rho)$ in $\Delta_{S_n}$ such that $\gamma$ and $\rho$ are noncommuting elements. Since $\pi$ commutes with $\gamma$, we have $\pi \in \langle \gamma \rangle$, by 2.4. Therefore the order of $\pi$ is a divisor of $p$, the order of $\gamma$, by Lagrange’s Theorem.
Thus \( \pi \) has order \( p \), because \( p \) is a prime and \( \pi \) is nontrivial. It follows that \( \langle \pi \rangle = \langle \gamma \rangle \). Furthermore, we claim that \( \pi \) is itself a cycle of length \( p \). We observe that if \( l \) is a positive integer, and \( l \notin \{1, p\} \), then the decomposition of \( \pi \) cannot contain a cycle of length \( l \), because the order of \( \pi \) is not divisible by \( l \). Also, the decomposition of \( \pi \) cannot have two disjoint cycles of length \( p \), since \( 2p > n \). Hence we have our claim. Therefore by 2.4, \( \rho \in \langle \pi \rangle \), because \( \rho \) and \( \pi \), being adjacent in our commuting path, are commuting elements. Thus \( \rho \in \langle \gamma \rangle \). In particular, we conclude that \( \rho \) and \( \gamma \) are commuting elements, which is a contradiction. Thus we have the proposition.

We may now give arguments to obtain Theorem 1.1.

**Proof.** We consider the three cases separately.

\textbf{a.} This follows at once from Theorem 5.1 and Proposition 6.2.

\textbf{b.} First consider the case of \( n = 3 \). We observe that \( R_3 = \emptyset \), and

\[ C_3^3 = \{(1\ 2\ 3), (1\ 3\ 2)\} = \langle(1\ 2\ 3)\rangle \setminus \{13\}. \]

Thus \( \Delta_{R_3 \cup C_3^3} \) is a connected component of \( \Delta_{S_3} \), of diameter 1, by 7.1. Also, for \( \gamma \in C_3^2 \), \( \Delta_{\langle \gamma \rangle} \) is a component of \( \Delta_{S_n} \), of diameter 0, by 7.1.

Regarding the \( n = 4 \) case, we observe that for each \( \gamma \in C_4^3 \), \( \Delta_{\langle \gamma \rangle} \) is a connected component of \( \Delta_{S_4} \), of diameter 1, by 7.1. And by 4.1, \( \Delta_{R_4 \cup C_4^3} \) is connected of diameter 3. Thus we see that \( \Delta_{R_4 \cup C_4^4} \) is a component of \( \Delta_{S_4} \), in particular.

For \( n > 4 \), we note that \( n \) is even, because \( n - 1 \) is a prime. Therefore by 4.1 and 7.1, once again, we realize that the connected components of \( \Delta_{S_n} \) are \( \Delta_{R_n \cup C_n^4} \), of diameter 4, and \( \Delta_{\langle \gamma \rangle} \) for \( \gamma \in C_{n-1}^n \), each of diameter 1.

\textbf{c.} By 7.1, \( \Delta_{\langle \gamma \rangle} \) is a connected component of \( \Delta_{S_n} \), of diameter 1, for each \( \gamma \in C_{n-1}^n \). We observe that \( n - 1 \) is even and \( > 2 \), because \( n \) is a prime and \( > 3 \). Hence \( n - 1 \) is composite. Therefore by 3.4, 5.2, 5.3, and 6.10, \( \Delta_{R_n \cup C_{n-1}^n} \) is connected of diameter 5. Thus we see that \( \Delta_{R_n \cup C_{n-1}^n} \) is a component of \( \Delta_{S_n} \).

\[ \Box \]

**References**

http://dx.doi.org/10.1016/s0021-8693(03)00302-8


Received: May 1, 2014; Published: August 14, 2015