Small Module

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Abstract

The main goal of the present article is to study basic properties of small modules. Let $R$ be a Noetherian ring, for all small module $A$ and index $I$, we get isomorphic $\text{Ext}^n(M, \coprod K_i) \cong \coprod \text{Ext}^n(M, K_i)$. Finally, we prove that if two modules of the sequence $0 \to A \to B \to C \to 0$ are small modules, so is the third.

Keywords: small modules; semisimple and noetherian ring

1. Introduction

Throughout the paper, a ring $R$ means an associative ring with unit, and a module means a right $R$-module over an arbitrary ring $R$. We will use the letter $R$ for a ring in all claims.

A module $M$ is called small if the functor $\text{Hom}_R(M, -)$ commutes with direct sums of all modules over a ring, that is for every index set $I$, then $\text{Hom}_R(M, \coprod K_i) \cong \coprod \text{Hom}_R(M, K_i)$. The notion of a compact object of a category, i.e., an object $c$ for which the covariant functor $\text{Hom}(c, -)$ commutes with all direct sums, has appeared as a natural tool in many branches of module theory. Small module, which are precisely compact objects of the category of all modules over a ring, have been useful in study of the structure theory of graded rings and almost free modules.

In this paper, we introduce and study the class of modules called small modules. The main results are in section 2.

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2. Small module

We begin with the following

**Definition 2.1** A module $M$ is called small if the functor $\text{Hom}_R(M, -)$ commutes with direct sums of all modules over a ring, that is for every index set $I$, then $\text{Hom}_R(M, \coprod K_i) \cong \coprod \text{Hom}_R(M, K_i)$.

**Lemma 2.2** Let $R$ is noetherian rings and $M$ be a small module, $(K_i \mid i \in I)$ be a system of module with index set $I$. Then $\operatorname{Ext}^1(M, \coprod K_i) \cong \coprod \operatorname{Ext}^1(M, K_i)$.

**Proof** There exists an exact sequence

$$0 \rightarrow K_i \rightarrow E_i \rightarrow A_i \rightarrow 0$$

where $E_i$ is injective, which induces an exact sequence

$$0 \rightarrow \coprod K_i \rightarrow \coprod E_i \rightarrow \coprod A_i \rightarrow 0$$

and $\coprod E_i$ is injective. Hence there is a commutative diagram with exact rows by long exact sequence lemma

$$\begin{array}{ccccccccc}
\text{Hom}(M, \coprod E_i) & \rightarrow & \text{Hom}(M, \coprod A_i) & \rightarrow & \text{Ext}^1(M, \coprod K_i) & \rightarrow & \text{Ext}^1(M, \coprod E_i) \\
\downarrow \cong & & \downarrow \cong & & \downarrow & & \downarrow \cong \\
\coprod \text{Hom}(M, E_i) & \rightarrow & \coprod \text{Hom}(M, A_i) & \rightarrow & \coprod \text{Ext}^1(M, K_i) & \rightarrow & \coprod \text{Ext}^1(M, E_i)
\end{array}$$

the last terms in each row are 0. By the five lemma, $\operatorname{Ext}^1(M, \coprod K_i) \cong \coprod \operatorname{Ext}^1(M, K_i)$.

**Theorem 2.3** Let $R$ is noetherian rings and $M$ be a small module. Then $\operatorname{Ext}^n(M, \coprod K_i) \cong \coprod \operatorname{Ext}^n(M, K_i)$ for all $n$.

**Proof** It follows by induction on $n$. By lemma 2.2, it is true for $n = 1$, if $n > 1$, by (1) and (2) in lemma 2.2 and long exact sequence lemma, there is a diagram

$$\begin{array}{ccccccccc}
\operatorname{Ext}^{n-1}(M, \coprod E_i) & \rightarrow & \operatorname{Ext}^{n-1}(M, \coprod A_i) & \delta & \rightarrow & \operatorname{Ext}^n(M, \coprod K_i) & \rightarrow & \operatorname{Ext}^n(M, \coprod E_i) \\
\downarrow \cong & & \downarrow \psi & & \downarrow & & \downarrow \cong \\
\coprod \operatorname{Ext}^{n-1}(M, E_i) & \rightarrow & \coprod \operatorname{Ext}^{n-1}(M, A_i) & \delta' & \rightarrow & \coprod \operatorname{Ext}^n(M, K_i) & \rightarrow & \coprod \operatorname{Ext}^n(M, E_i)
\end{array}$$

The first and last terms in each row are 0, so exactness gives $\delta$ and $\delta'$ isomorphism. By induction, there is an isomorphism $\psi$, and $\delta' \psi \delta^{-1} : \operatorname{Ext}^n(M, \coprod K_i) \cong \coprod \operatorname{Ext}^n(M, K_i)$ is an isomorphism.

**Proposition 2.4** Let $R$ and $S$ be commutative rings. If $A_R$ and $RB_S$ are small modules, then $A \otimes_R B$ is small module.
Proof Suppose $\prod K L_i$ be right $S$-modules with index set $K$, by adjoint isomorphism $\text{Hom}(A \otimes R B, \prod K L_i) \cong \text{Hom}(A, \text{Hom}(B, \prod K L_i))$, since $A_R$ and $R B_S$ are small module, hence $\text{Hom}(A \otimes R B, \prod K L_i) \cong \text{Hom}(A, \text{Hom}(B, \prod K L_i)) \cong \text{Hom}(A, \prod K)$.
$\text{Hom}(B, L_i)) \cong \prod K \text{Hom}(A, \text{Hom}(B, L_i)) \cong \prod K \text{Hom}(A \otimes R B, L_i)$. Then $A \otimes R B$ is small.

Corollary 2.5 Let $R$ and $S$ be commutative rings. If $R A$ and $S B_R$ are small modules, then $B \otimes_R A$ is small module.

Proposition 2.6 Let $R$ is noetherian rings and $0 \to A \to B \to C \to 0$ be a short exact sequence of modules. If $A$ and $C$ are small modules, then $B$ is small module.

Proof Since $\text{Hom}(-, \prod M_i)$ is left exact, there is a commutative diagram with exact rows

$$
\begin{array}{cccccc}
\text{Hom}(C, \prod M_i) & \longrightarrow & \text{Hom}(B, \prod M_i) & \longrightarrow & \text{Hom}(A, \prod M_i) & \longrightarrow \text{Ext}^1(C, \prod M_i) \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & \\
\prod \text{Hom}(C, M_i) & \longrightarrow & \prod \text{Hom}(B, M_i) & \longrightarrow & \prod \text{Hom}(A, M_i) & \longrightarrow \prod \text{Ext}^1(C, M_i) \\
\end{array}
$$

by five lemma, we conclude that $\text{Hom}(B, \prod M_i) \cong \prod \text{Hom}(B, M_i)$, thus $B$ is small module.

Proposition 2.7 Let $R$ is noetherian rings, if two modules of the sequence $0 \to A \to B \to C \to 0$ are small modules, so is the third.

Proposition 2.8 Let $R$ be a ring. If every $(M_i | i \in I)$ is small modules for every finite set $I$, then $\prod I M_i$ is small module.

Proof Let $(L_i | i \in K)$ be a system of module with index set $K$. Suppose $M_1$ and $M_2$ are small modules, we have $\text{Hom}(M_1 \oplus M_2, \prod K L_i) \cong \text{Hom}(M_1, \prod K L_i) \oplus \text{Hom}(M_2, \prod K L_i) \cong \prod K \text{Hom}(M_1, L_i) \oplus \prod K \text{Hom}(M_2, L_i) \cong \prod K \text{Hom}(M_1 \oplus M_2, L_i)$. Then $M_1 \oplus M_2$ is small.

Proposition 2.9 For a commutative artinian ring $R$, every small module of $R$ is finitely presented.

Proof First, note that every left artinian ring is left noetherian ring by Hopkins-Levitzki Theorem. By [3] Theorem 3.17 we have every self-small module is finitely generated, then every small module is finitely generated. Since $R$ is artinian ring, we get every finitely generated module is finitely presented.

Proposition 2.10 For a semisimple and noetherian ring $R$, every small module of $R$ is finitely presented.

Proof By [3] proposition 2.5 we have every self-small module is finitely generated, then every small module is finitely generated. Since $R$ is noetherian ring, we get every finitely generated module is finitely presented.
References


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