A Remark on Pullback Attractors
for the 2D Navier-Stokes Equations with
Weak Damping, Distributed and Continuous Delay

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Abstract
We shall show some results for the existence of pullback attractors of 2D Navier-Stokes equation with weak damping, distributed and continuous delay.

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1 Introduction

In this paper, we investigate the existence of pullback attractors for the 2D Navier-Stokes equations with weak damping, continuous and distributed delay:

\[
\begin{aligned}
&u_t - \nu \Delta u + (u \cdot \nabla)u + \alpha u + \nabla p = f(t - \rho(t), u(t - \rho(t))) \\
&\quad + \int_{-h}^{0} G(t, s, u(t + s))ds,
\end{aligned}
\]

\[
\begin{aligned}
div u = 0, &\quad (x, t) \in \Omega_{\tau}, \\
u = 0, &\quad (x, t) \in \partial \Omega_{\tau}, \\
u(\tau, x) = u_0(x), &\quad x \in \Omega, \\
u(t, x) = \phi(t, x), &\quad (x, t) \in \Omega_{\tau h},
\end{aligned}
\]
where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial \Omega$, $\Omega_t = \Omega \times (\tau, +\infty)$, $\Omega_{\tau h} = \Omega \times (\tau - h, \tau)$, $\tau \in \mathbb{R}$ is the initial time, $\nu$ is the kinematic viscosity of the fluid, $u = u(t, x) = (u_1(t, x), u_2(t, x))$ is the velocity vector field which is unknown, $p$ is the pressure, $\alpha > 0$ is positive constant, $\alpha u$ is the weak damping, $f(t - \rho(t), u(t - \rho(t)))$ is the external force term which contains memory effects during a fixed interval of time of length $h > 0$, $\rho(t)$ is an adequate given delay function, $\phi$ is the initial state of delay in $[\tau - h, \tau]$, $h > 0$ is a constant.

This paper will be organized as follows: in section 2, we shall give some preliminaries; in section 3, the existence and uniqueness of global weak and strong solutions will be derived; we shall prove the existence of pullback absorbing ball in section 4, the pullback attractors will be concluded in last section.

2 Preliminaries

Throughout this paper, $C$ will stand for a generic positive constant, depending on $\Omega$ and some constants, but independent of the choice of the initial time $\tau$ and $t$. We introduce the Hausdorff semidistance in $X$ from one set $B_1$ to another set $B_2$, i.e.,

$$\text{dist}_X(B_1, B_2) = \sup_{b_1 \in B_1} \inf_{b_2 \in B_2} \|b_1 - b_2\|_X.$$  

We set $E := \{u|u \in (C_0^\infty(\Omega))^2, \text{div}u = 0\}$, $H$ is the closure of the set $E$ in $(L^2(\Omega))^2$ topology, $W$ is the closure of the set $E$ in $(H^2(\Omega))^2$ topology, i.e.,

$$W = \{u \in W||u||_W = ||u||_{H^2}, u|_{\partial \Omega} = 0\}. \quad (2)$$

For each $t \in (\tau, T)$ with $T > \tau$, we define $u : (\tau - h, T) \rightarrow (L^2(\Omega))^2$, here $u_t$ is a function in $(-h, 0)$ defined by $u_t = u(t + s), s \in (-h, 0)$.

In the following section, we denote $C_H = C^0([-h, 0]; H)$ and $C_V = C^0([-h, 0]; V)$ as two Banach spaces equipped the norms

$$\|u\|_{C_H} = \sup_{\theta \in [-h, 0]} |u(t + \theta)| \quad (3)$$

and

$$\|u\|_{C_V} = \sup_{\theta \in [-h, 0]} \|u(t + \theta)\| \quad (4)$$

respectively, $L^2_H = L^2(-h, 0; H)$, $L^2_V = L^2(-h, 0; V)$.

Assume $\nu_0 \in H, \eta \in L^2_H$, then the problems (1) can be written as an abstract form

$$\frac{du}{dt} + \nu Au + \alpha u + B(u) = f(t - \rho(t), u(t - \rho(t))) + g(t, u_t), \quad (5)$$

$$u(\tau) = u_0, u(t) = \phi(t), t \in (\tau - h, \tau). \quad (6)$$
where \( g(t, u_t) = \int_{-h}^{0} G(t, s, u(t + s))ds, \) \( g(t, 0) = 0. \) In (1), the functions \( f : [-h, \infty) \times H \to H, \) \( g : [-h, 0] \times H \to H \) and \( \phi : [-h, 0] \to H \) are continuous which satisfy

(a) \( \rho : [0, \infty) \to [0, h], |\frac{d\rho}{dt}| \leq M < 1; \)

(b) there exist constants \( m_0(s), m_1(s) \) such that \( |G(t, s, u)| \leq m_0(s) + m_1(s) |u| \). Denote \( m_i = \int_{-h}^{0} m_i(s)ds, \) \( i = 1, 2. \)

(d) there exist constants \( a > 0, b > 0 \) such that \( |f(t, u)|^2 \leq a|u|^2 + b; \)

(e) \( (\nu \lambda_1)^2 > \frac{ae}{1-M} + \frac{1}{h}, \) \( \frac{ae}{(1-M)\nu \lambda_1} > 2\nu \lambda_1, \) where \( \lambda_1 \) is the first eigenvalue of \( A \) under the homogeneous Dirichlet boundary condition;

(f) from the assumption (d) (i.e., \( (\nu \lambda_1)^2 > \frac{ae}{1-M} + \frac{1}{h}, \) \( \frac{ae}{(1-M)\nu \lambda_1} > 2\nu \lambda_1, \) we have \(-\nu \lambda_1 + \frac{ae}{(1-M)\nu \lambda_1} < 0, \) so there exists \( \theta > 0, \) such that \( \theta - \nu \lambda_1 + \frac{ae}{(1-M)\nu \lambda_1} < 0. \) Noting \( \alpha > 0, \) we can deduce

\[
\theta - \nu \lambda_1 - 2\alpha + \frac{ae}{(1-M)\nu \lambda_1} < 0;
\]

(g) let \( p(\theta) = 2m_1 e^{\theta h} - \theta, \) it can be easily obtained \( p(0) > 0, \) \( p(\frac{1}{h}) = 2m_1 e - \frac{1}{h} < 0, \) hence, there exist \( \theta_1, \theta_2 > 0, \) when \( \theta_1 < \theta < \theta_2, \) such that \( p(\theta) < 0, \) i.e.,

\[
2m_1 e^{\theta h} < \theta;
\]

(h) from (c), there exist positive numbers \( L(\beta), L(\gamma) \) such that

\[
|f(t, u) - f(t, v)| \leq L(\beta) |u - v|, \quad |g(t, u) - g(t, v)| \leq L(\gamma) |u - v|.
\]

Moreover, there exist a constant \( C > 0, \) such that

\[
\int_{0}^{t} |g(s, u_s) - g(s, v_s)|^2ds \leq C \int_{-h}^{t} |u(r) - v(r)|^2dr.
\]

Here

\[
\int_{0}^{t} |g(s, u_s) - g(s, v_s)|^2ds \\
\leq L^2(\gamma) \int_{0}^{t} |u(s + \tau) - v(s + \tau)|^2ds \\
\leq L^2(\gamma) \int_{0}^{t} (\int_{-h}^{0} |u(s + \tau) - v(s + \tau)|^2ds)d\tau \\
\leq L^2(\gamma) \int_{-h}^{0} (\int_{-h}^{t} |u(s + \tau) - v(s + \tau)|^2d\tau)ds \\
\leq L^2(\gamma) \int_{-h}^{0} (\int_{s}^{t+s} |u(r) - v(r)|^2dr)ds \\
\leq C \int_{-h}^{t} |u(r) - v(r)|^2dr,
\]

where \( C = hL^2(\gamma). \)
3 Existence of Global Solutions

The existence of weak global solutions for (1) can be derived by similar methods as in [4]:

**Theorem 3.1** Let \( u_0 \in H, \phi \in L^2_H \) and the assumption \( f = f(t, u(t - \rho(t))) \) hold, then there exists a unique global weak solution of (1) that satisfies

\[
 u \in L^\infty(0, T; H) \bigcap L^2(0, T; V).
\]

Proof.

Step 1. Assume the orthogonal base in \( H \) of \( A \) is \( w_j \) such that \( Aw_j = \lambda_j w_j \), holds for \( j = 1, 2, \cdots, W_m = \text{span}\{w_1, w_2, \cdots, w_m\} \) is the subspace of \( H \). Constructing the approximation solution \( u_m(t) = \sum_{k=1}^{m} u_{mk}(t)w_k \) \((k = 1, 2, \cdots, m)\) of problem (1), where \( u_{mk}(t) \) is to be determined.

Step 2. We shall prove \( \frac{du_m}{dt} \) is uniformly bounded in \( L^2(0, T; V') \).

Step 3. We shall prove the uniqueness of global solution (see [4]).

The theorem 3.1 proves that for \( u_0 \in H, \phi \in L^2_H \), the problem (1) exists uniqueness solution \( u(\cdot; \tau, (u_0, \phi)) \), We can define the semi-processes for non-autonomous system \( \{U(t, \tau)\phi : C_H \to C_H\} \), which satisfies

\[
 U(t, \tau)\phi = u_t(\cdot; \tau, (\phi(0), \phi)), \forall \phi \in C_H, t \geq \tau,
\]

\[
 U(t, \tau)\phi = I_d.
\]

**Theorem 3.2** Let \( u_0 \in V, \phi \in L^2_V \), the assumption \( f = f(t, u(t - \rho(t))) \) holds, then there exists a unique global strong solution of (1), which satisfies

\[
 u \in L^\infty(0, T; V) \bigcap L^2(0, T; D(A)).
\]

Proof. see, e.g. [4].

**Theorem 3.3** Assume that the assumption \( f = f(t, u(t - \rho(t))) \) holds, \( u_0 \in H, \phi \in L^2_H \), the semi-processes \( \{U_f(t, \tau)|t \geq \tau\} \) defined by (7) is continuous for arbitrary \( t \geq \tau \).

Proof. see, e.g. [4].

4 Existence of Pullback Attractors in \( H \)

The uniqueness of the solution in Theorem 3.2 proves that the operator \( U(t, \tau)\phi \) is semi-processes.
However, we choose the skew-product flow in the space $H \times L^2_H = M^2_H$, and define a family of mappings $\tilde{U}(\cdot, \cdot) : M^2_H \to L^2_H$, as follows,

$$
\tilde{U}(t, \tau)(u_0, \phi) = u_t(\cdot; \tau, (u_0, \phi)), \ \forall (u_0, \phi) \in M^2_H, \ t \geq \tau,
$$

(8)

obviously,

$$
\tilde{U}(t, \tau)\phi = \tilde{U}(t, \tau)(\phi(0), \phi), \ t \geq \tau, \ \phi \in C_H.
$$

(9)

For arbitrary $(u_0, \phi) \in M^2_H$, the corresponding norm can be described as

$$
\|(u_0, \eta)\|_{M^2_H}^2 = |u_0|^2 + \int_{-h}^{0} |\phi(s)|^2 ds.
$$

(10)

**Lemma 4.1** Assume that $\{B(t)\}_{t \in \mathbb{R}}$ are a bounded sets in $C_H$, then the mapping $\tilde{U}(\cdot, \cdot)$ is attracting in $C_H$, such that $\{B(t)\}_{t \in \mathbb{R}}$ for the the semi-processes $\{U(\cdot, \cdot)\}$ is also attracting in $C_H$.

**Theorem 4.2** Assume that the assumption $f = f(t, u(t - \rho(t)))$ holds, $u_0 \in H, \phi \in L^2_H$, the semi-processes $\{U(t, \tau)\}$ possesses a bounded pullback absorbing set $B_0$ in $C_H$.

**Proof.** see, e.g. [4].

**Theorem 4.3** Assume that the assumptions in Theorem 4.1 hold, there exists a bounded pullback attracting set for the semi-processes $\{U(\cdot, \cdot)\}$ in $C_V$.

The main results in our paper can be stated as

**Theorem 4.4** Assume that the assumption $f = f(t, u(t - \rho(t)))$ holds, $u_0 \in H, \phi \in L^2_H$, there exists a pullback attractor $A$ of the problem (1) for the semi-processes $\{U_f(t, \tau)\} t \geq \tau$.

**Proof.** The Theorem 4.1 and 4.2 guarantee that there exists a bounded attracting set of the problem (1) in $C_H$ and $C_V$ respectively. If we can prove $u_t$ is compact in $C_H$, then the problem (1) possesses a pullback attractor, this is equivalent to prove the next two properties by the generalized Arzelà-Ascoli theorem:

(1) The embedding $V \subset\subset H$ is compact.
(2) $\{U(t, \tau)\}$ is equi-continuous.

From the fundamental theory of existence of pullback attractor, the problem (1) has a pullback attractor in $H$. □

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