On the Limit Cycles for Liénard Equation

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Abstract

In this work, we present some new criteria on the non-existence and uniqueness of limit cycles for the Liénard equation.

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1 Introduction

We consider the Liénard equation given by

\[
\begin{cases}
\dot{x}_1 = x_2, \\
\dot{x}_2 = -g(x_1) - f(x_1)x_2,
\end{cases}
\]
It is well-known the relevance of the Liénard equation in the qualitative theory of differential equations, which models several oscillatory phenomena. We are concerning on the non-existence and the uniqueness of limit cycles, for this we use of the extended Bendixson-Dulac criterion (see [1] and [2]).

Given an open set $\Omega \subset \mathbb{R}^2$ we consider

$$\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2), \\
\dot{x}_2 &= f_2(x_1, x_2), \quad (x_1, x_2) \in \Omega,
\end{align*}$$

(2)

where $f_1, f_2$ are $C^1$-functions on $\Omega$, the associated vector field is $F(x_1, x_2) = (f_1(x_1, x_2), f_2(x_1, x_2))$. As usual its divergence is $\text{div}(F) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$.

We consider the sets

$$\mathcal{F}^\pm_\Omega := \{f \in C^0(\Omega, \mathbb{R}^\pm \cup \{0\}) : \text{vanishes only on a measure zero set}\},$$

and $\mathcal{F}_\Omega := \mathcal{F}^-_\Omega \cup \mathcal{F}^+_\Omega$, along the paper we will use the Lebesgue measure.

Recall that an open subset $\Omega \subset \mathbb{R}^2$ intuitively is said to be $l$-connected if it has $l$-holes, i.e., if its first fundamental group is a free group with $l$-generators, we denote $l(\Omega) = l$.

For $h : \Omega \to \mathbb{R}$ a continuous function, let $Z(h) := \{x \in \Omega : h(x) = 0\}$ be the set of zeros of $h$.

Following ([2]) we denote by $l(\Omega, h)$ the sum of the quantities $l(U)$ over all the connected components $U$ of $\Omega \setminus Z(h)$, also denote by $\text{co}(h)$ the numbers of closed ovals of $Z(h)$ contained in $\Omega$.

Our results are established with the help of the techniques developed in [4], let us recall the following result

**Proposition 1.1** ([4], prop. 2) Let $\Omega \subset \mathbb{R}^2$ be an open set with regular boundary. Suppose that there is $s \in \mathbb{R}$ and a function $c : \Omega \to \mathbb{R}$ such that

$$\langle \nabla h, F \rangle + sh\text{div}(F) = ch,$$

(3)

admits an analytic solution $h$, with $ch$ does not change sign and vanishes only on a null measure subset, then $h$ is a Dulac function. Therefore the limit cycles of system (2) are either totally contained in $Z(h)$, or do not intersect $Z(h)$. Moreover, the number of limit cycles contained in $Z(h)$ is at most $\text{co}(h)$ and the number $N$ of limit cycles that do not intersect $Z(h)$ satisfies

$$N \leq \begin{cases} 
  l(\Omega) & \text{if } s > 0, \\
  0 & \text{if } s = 0, \\
  l(\Omega, h) & \text{if } s < 0.
\end{cases}$$

(4)
2 Results and applications

Our first result gives a criterion for the non existence of limit cycles for (1)

**Proposition 2.1** If there are $c_1, c_2$ constants, such that $c_1 g(x_1) + c_2 f(x_1) \in \mathcal{F}_{\mathbb{R}^2}$, then the Liénard system has not periodic orbits.

**Proof:** Seeking for a function of Dulac $h = h(z)$, depending only on $z = z(x_1, x_2)$, the associated equation (3) becomes

$$
\left[ x_2 \frac{\partial z}{\partial x_1} + (-g(x_1) - f(x_1)x_2) \frac{\partial z}{\partial x_2} \right] \frac{d\ln h}{dz} = c + sf(x_1),
$$

taking $z$ such that $\frac{\partial z}{\partial x_1} - f(x_1) \frac{\partial z}{\partial x_2} = 0$; hence we obtain $z = \int x_1 f(s)ds + x_2$ and

$$
\frac{d\ln h}{dz} = \frac{c + sf(x_1)}{-g(x_1)},
$$

(5)

where the right side depends on $z$, denoted this by $\phi(z)$; so taking

$$
c = c_1 f(x_1) + c_2 g(x_1) \text{ and } s = -c_1,
$$

then $\phi(z) = -c_2$ and the equation (5) is written as $\frac{d\ln h}{dz} = -c_2$, whose solution is

$$
h(x_1, x_2) = e^{c_2} = \exp[ c_2(\int x_1 f(s)ds + x_2) ].
$$

Note that $Z(h) = \emptyset$ contains no ovals. In particular, $co(h) = 0$.

Since $l(\mathbb{R}^2, h) = l(\mathbb{R}^2) = 0$, then by Proposition 1.1, the equation (1) has no limit cycles, so the result follows. \qed

**Corollary 2.2** If any of the following conditions holds

1. $g(x_1)$ or $f(x_1)$ belong to $\mathcal{F}_{\mathbb{R}^2}$,

2. $kf(x_1) > g(x_1)$, for some $k$ constant,

then the Liénard system has not periodic orbits.

**Remark:** The condition 1 is a slight generalization of Criterion 1, (C1) in [3].

**Theorem 2.3** Assume $g$ polynomial. If for some $a \in \mathbb{R}$ is satisfied that $f(x_1)(G(x_1) + a) \in \mathcal{F}_{\mathbb{R}^2}$ where $G(x_1) = \int_0^{x_1} g(s)ds$, then the Liénard equation has at most one limit cycle.
If $g(x_1) = x_1$, and there exists $\delta > 0$ such that $f(x_1) < 0$ for $x_1 \in (-\delta, \delta)$, $f(x_1) > 0$ in $(-\infty, -\delta) \cup (\delta, +\infty)$ also $\int_{x_1}^{\pm\infty} f(s) ds = \pm \infty$, then system (1) has at most one limit cycle.

Proof: Indeed, taking $G(x_1) = \int_{x_1}^{x_1} g(s) ds = \frac{x_1^2}{2}$ and $a = -\frac{x_1^2}{2}$, the hypotheses of Theorem 2.3 are fulfilled, therefore (1) has at most one limit cycle.

Proposition 2.5 If there is $m \in \mathbb{R}$ such that $f(x_1)(2F(x_1) + m) - g(x_1) \in \mathcal{F}_{\mathbb{R}^2}$ with $F(x_1) = \int_{x_1}^{x_1} f(s) ds$, then the Liénard equation has not limit cycles.

Proof: Consider the equation (6) and taking $z$ such that $\frac{\partial z}{\partial x_1} - f(x_1) \frac{\partial z}{\partial x_2} = f(x_1)$; hence we get $z = x_2 + 2\int_{x_1}^{x_1} f(s) ds + a$ for some constant $a$ and (6) becomes

$$\frac{d \ln h}{dz} = \frac{c + sf(x_1)}{|x_2 f(x_1) - g(x_1)|},$$

where the right side depends on $z$, denoted this by $\eta$, thus $c = -\eta(z) (f(x_1)x_2^2 - sf(x_1))$, we choose $s = -2$ and $\eta(z) = \frac{1}{z}$; simplifying we have $c = -f(x_1)\frac{2}{z} \int_{x_1}^{x_1} g(s) ds + a$, taking $c$ in this way, the equation (7) is written as $\frac{d \ln h}{dz} = \frac{d \ln z}{dz}$ whose solution is

$$h(x_1, x_2) = z = \frac{x_2^2}{2} + \int_{x_1}^{x_1} g(s) ds + a.$$
where the right side depends on $z$, we take this equal to $\frac{1}{z}$; so
\[ c = \frac{x_2 f(x_1) - g(x_1) + zsf(x_1)}{z} \]
then, replacing $z$ in the numerator and choosing $s = -1$, $a = -m$, we get
\[ c = \frac{-g(x_1) + f(x_1)[2F(x_1) + m]}{z} \]
whose solution is $h(x_1, x_2) = z = 2 \int_{x_1}^{x_2} f(s)ds + x_2 - m$.

Note that $Z(h)$ is a submanifold of $\mathbb{R}^2$ homeomorphic to $\mathbb{R}$ therefore no limit cycles are contained in $Z(h)$.

Since $l(\mathbb{R}^2, h) = 0$, then by Proposition 1.1, (1) has not limit cycles. \hfill \Box

**Proposition 2.6** If there exist a function $\psi(x_1)$ strictly monotonic such that it satisfies the inequality
\[ \psi^2(x_1)f^2(x_1) + 4\psi'(x_1)\psi(x_1)g(x_1) \in \mathcal{F}_{\mathbb{R}^2}^-, \]
then the Lienard system has not periodic orbits.

**Proof:** Suppose the function $h$ has the form $h = \psi(x_1)x_2$, then the associated equation becomes
\[ x_2 \frac{d\psi(x_1)}{dx_1} - (g(x_1) + f(x_1)x_2)\psi(x_1) = h[c + s\psi(x_1)]. \]

Taking $s = 0$ and as by conditions of Proposition 1.1 $ch$ must be in $\mathcal{F}_{\mathbb{R}^2}$, we need that
\[ x_2^2 \psi_{x_1}(x_1) - \psi(x_1)f(x_1)x_2 - \psi(x_1)g(x_1) \in \mathcal{F}_{\mathbb{R}^2}. \]
Now this inequality holds if $\psi' \geq 0$, ($\psi' \leq 0$) and the discriminant
\[ \psi^2(x_1)f^2(x_1) + 4\psi_{x_1}\psi(x_1)g(x_1) \in \mathcal{F}_{\mathbb{R}^2}^- \]
But it holds by hypothesis. \hfill \Box

**Corollary 2.7** If $g(x_1)$ is strictly decreasing and satisfies $f^2(x_1) + 4g_{x_1} \in \mathcal{F}_{\mathbb{R}^2}^-$, then the Lienard system has not periodic orbits.

**References**


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