On Solving Boundary Value Problems Using

Adomian Decomposition Method

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Abstract

This research work employed the Adomian Decomposition method to solve the various stages of dynamical damped equations. Comparison of the numerical solutions obtained with the analytical solutions at each stage indicates minimal errors, which confirmed the suitability and the efficiency of the method.

Keywords: Adomian decomposition method, Adomian polynomial, boundary value problems, damped equations

1 Introduction

They have been a lot of research interest in the application of Adomian decomposition method to solving wide variety of stochastic and deterministic problems. This has resulted in the development of various Adomian numerical approaches for solving linear/non-linear boundary value problems of differential equations (see, [2], [3], [4], [5], [7]).

The focus of this work is on a second order linear dynamic system model equation, which has not been critically treated by the users of the Adomian decom-
position method. Therefore, the aim of this work is to employ Adomian decomposition method to approximate the solution of the spring/mass dynamic system at different stages, and comparing the obtained results with the analytical solutions at the various stages. A mathematical illustration confirmed the suitability of the results.

2 The Concept of Adomian Decomposition Method

A reviewed of the basic mathematical methodology of ([2], [6]), the general boundary problem of a differential equation for a non-linear function 
\[ f : R \times R^d \rightarrow R^d, \]
is presented as

\[ y' = f(t, y) \]
\[ y(0) = y_0. \]  

(2.1)

Assume that \( f \) is analytic near \( y(0) = y_0 \), then the integral solution of equation (2.1) is

\[ y(t) = y_0 + \int_0^t f(t, y(t))dt. \]  

(2.2)

By the Adomian method ([4], [5]), \( y \) is consider in the series form as

\[ y = y_0 + \sum_{n=0}^{\infty} y_n, \]  

(2.3)

such that the nonlinear function \( f(t, y) \) can also be express as

\[ f(t, y) = \sum_{n=0}^{\infty} A_n(t, y_0, y_1, \ldots, y_n), \]  

(2.4)

and \( A_n \) is obtained as

\[ A_n = \sum_{n=0}^{\infty} \frac{d^n}{n!dt^n} f(t, y_n), \quad n = 0, 1, 2, \ldots \]  

(2.5)

where \( A_n \) is a polynomial in \( y_n \), called the Adomian polynomial.

Evaluating equation (2.5) at \( n = 1, 2, \ldots \) then the first \( n \)th term of the Adomian polynomials are:
Solving boundary value problems

\[ A_0 = f(t, y_0) \]
\[ A_1 = y_1 f'(t, y_0) \]
\[ A_2 = y_2 f'(t, y_0) + \frac{1}{2} y_1^2 f''(t, y_0) \]
\[ A_3 = y_3 f'(t, y_0) + y_1 y_2 f''(t, y_0) + \frac{1}{6} y_1^3 f'''(t, y_0) \]
\[ \vdots \]
\[ A_n = y_n \left( y_0 + \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(t, y_0) \right) \]

3 Application of Adomian Decomposition Method in Dynamical system

It is observed that most natural phenomenon’s are dynamical in nature, and thus are sometime represented by the state equation that evolves with the flow of time [3]. The input and the current state of the dynamical system is a determinant of the evolution of the system [8]

Considering the spring/mass system with a flexible wire suspended vertically from a rigid body and a mass \( m \) attached to its free end, by Hooke’s law the amount of stretch \( s \) on the spring is directly proportional to the mass. That is

\[ F = ks, \quad (3.0) \]

where \( k \) is the constant of proportionality called the spring constant. At equilibrium, the weight of the body is balance by the restoring force \( ks \), that is

\[ mg - ks = 0. \quad (3.1) \]

If the mass \( m \) is displaced by an amount \( x \) from its equilibrium position, the restoring force of the spring is thus \( k(y + s) \). Assuming that there are no retarding forces acting on the system and the mass vibrates free of other external forces, then by Newton’s second law of motion, the net or resultant force of the restoring force and the weight is written;

\[ \frac{md^2 y}{dt^2} = -k(y + s) + mg. \quad (3.2) \]

3.1 The Mathematical Solutions of the various stages of Damped motion

(i). **Free undamped motion**: Considering the mass/string system with an initial displacement and velocity \( y(0) = y_1 \) and \( \dot{y}(0) = y_2 \) respectively, then by equation (3.2) the model equation is
\[ \frac{md^2 y}{dt^2} + w^2 y = 0, \quad (3.3) \]

where \( w^2 = k/m \). The analytical solution is,

\[ y(t) = c_1 \cos wt + c_2 \sin wt. \quad (3.4) \]

Example;

\[ y(t) = -64 y, \quad y(0) = \frac{2}{3} \quad \text{and} \quad y'(0) = -\frac{4}{3}, \]

has analytical solution

\[ y(t) = \frac{2}{3} \cos 8t - \frac{1}{6} \sin 8t. \quad (3.5) \]

The numerical solution by the Adomian decomposition method of (2.4 – 2.5) for

\[ y_0(t) = \frac{2}{3} - \frac{4}{3} t \]
\[ y_1(t) = \frac{64}{9} t^2 (-3 + 2t) \]
\[ y_2(t) = -\frac{1024}{45} t^4 (-5 + 2t) \]
\[ y_3(t) = \frac{32768}{945} t^6 (-7 + 2t) \]
\[ y_4(t) = -\frac{262144}{8505} t^8 (-9 + 2t) \]
\[ y_5(t) = \frac{8388608}{467775} t^{10} (-11 + 2t) \]
\[ y_6(t) = -\frac{134217728}{18243225} t^{12} (-13 + 2t) \]
\[ y_7(t) = \frac{4294967296}{1915538625} t^{14} (-15 + 2t) \]
\[ y_8(t) = -\frac{17179869184}{32564156625} t^{16} (-17 + 2t) \]
\[ y_9(t) = \frac{549755813888}{5568476782875} t^{18} (-19 + 2t) \]

the first 11\textsuperscript{th} terms are;

\[ y_{10}(t) = -\frac{8796093022208}{584689432201875} t^{20} (-21 + 2t). \]

By equation (2.3),

\[ \sum_{n=0}^{10} y_n(t) = y_0 + y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9 + y_{10} + ... + O(h), \quad (3.6) \]
Thus

\[
\sum_{n=0}^{10} y_n(t) = \frac{2}{3} - \frac{4}{3} t + \frac{64}{9} t^3 (-3 + 2t) - \frac{1024}{45} t^4 (-5 + 2t) + \frac{32768}{945} t^5 (-7 + 2t) - \frac{262144}{8505} t^6 (-9 + 2t) + \frac{8388608}{467775} t^{10} (-11 + 2t) - \frac{134217728}{18243225} t^{12} (-13 + 2t) + \frac{4294967296}{1915538625} t^{14} (-15 + 2t) - \frac{17179869184}{32564156625} t^{16} (-17 + 2t) + \frac{549755813888}{556847678275} t^{18} (-19 + 2t) - \frac{879609302208}{584689432201875} t^{20} (-21 + 2t) + \ldots O(h)
\]

### Table 3.1 Comparison Results \((h = 0.01)\)

<table>
<thead>
<tr>
<th>(t)</th>
<th>Solution by Adomian</th>
<th>Analytical solution</th>
<th>Error</th>
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</thead>
<tbody>
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<td>0.0000000298</td>
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</table>

Adomian decomposition method performed well as the exact method at \(h=0.1\)

(ii). **Overdamped Motion:** - Considering the mass/string system with an initial displacement and velocity \(y(0) = y_1\) and \(y'(0) = y_2\) respectively and a damping force, then by equation (3.2) the model equation is

\[
\frac{md^2y}{dt^2} + w^2 y + \beta \frac{dy}{dt} = 0,
\]

where \(\beta\) is a positive damping constant.

Example: \(y''(t) = -5y' - 4y\), \(y(0) = 1\) and \(y'(0) = 1\), has analytical solution
The numerical solution by the Adomian decomposition method of (2.4 – 2.5) for the first 9th term are;

\[
y_0(t) = 1 + t
\]
\[
y_1(t) = -\frac{1}{6} t^2(27 + 4t)
\]
\[
y_2(t) = \frac{1}{30} t^3(225 + 70 + 4t^2)
\]
\[
y_3(t) = -\frac{1}{2520} t^4(23625 + 9660t + 1064r^2 + 32t^3)
\]
\[
y_4(t) = \frac{75}{8} t^5 + \frac{40}{9} t^6 + \frac{2}{3} t^7 + \frac{4}{105} t^8 + \frac{2}{2835} t^9
\]
\[
y_5(t) = -\frac{1}{2494800} t^6(19490625 + 10147500r + 1831500r^2 + 145200r^3 + 5104r^4 + 64r^5)
\]
\[
y_6(t) = \frac{1}{97297200} t^7(542953125 + 301640625r + 61668750r^2 + 600600r^3 + 296400r^4 + 7072r^5 + 64r^6)
\]
\[
y_7(t) = -\frac{1}{8172964800} t^8(285050390625 + 166103437500r + 3716212500r^2 + 4176900000r^3
\]
\[+ 256620000r^4 + 8668800r^5 + 149760r^6 + 1024r^7)
\]
\[
y_8(t) = \frac{15625}{8064} t^9 + \frac{10625}{9072} t^{10} + \frac{250}{891} t^{11} + \frac{125}{3564} t^{12} + \frac{175}{69498} t^{13} + \frac{2}{18711} t^{14} + \frac{16}{6081075} t^{15}
\]
\[+ \frac{2}{58046625} t^{16} + \frac{2}{10854718875} t^{17}
\]

By equation (2.3),

\[
\sum_{n=0}^{10} y_n(t) = y_0 + y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + \ldots + O(h),
\]

Hence
Solving boundary value problems

\[ \sum_{n=0}^{\infty} y_n(t) = 1 + t - \frac{1}{6} t^2 (27 + 4t) + \frac{1}{30} t^3 (225 + 70 + 4t^3) - \frac{1}{2520} t^4 (23625 + 9660r + 1064r^2 + 32r^3) + \frac{75}{8} t^5 + \frac{40}{9} t^6 + \frac{2}{3} t^7 + \frac{4}{105} t^8 + \frac{2}{2835} t^9 - \left( \frac{1}{2494800} t^6 (19490625 + 10147500r) + 1831500r^2 + 145200r^3 + 5104r^4 + 64r^5 \right) + \left( \frac{1}{97297200} t^7 (542953125 + 301640625t) + 61668750r^2 + 600600r^3 + 296400r^4 + 7072r^5 + 64r^6 \right) - \left( \frac{1}{8172964800} t^8 (285050390625 + 166103437500r + 3716212500r^2 + 4176900000r^3 + 2566200000r^4 + 8668800r^5 + 149760r^6 + 1024r^7) \right) + \frac{15625}{8064} t^9 + \frac{10625}{9072} t^{10} + \frac{250}{891} t^{11} + \frac{125}{3564} t^{12} + \frac{175}{6948} t^{13} + \frac{2}{18711} t^{14} + \frac{16}{6081075} t^{15} + \frac{2}{58046625} t^{16} + \frac{2}{10854718875} t^{17} \]

### Table 3.2: Comparison Results \((h = 0.01)\)

<table>
<thead>
<tr>
<th>t</th>
<th>Solution by Adomian</th>
<th>Analytical solution</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.000</td>
<td>1.00000000000</td>
<td>1.00000000000</td>
<td>0.00000000000</td>
</tr>
<tr>
<td>0.010</td>
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<td>1.0095567703</td>
<td>0.00000000000</td>
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<tr>
<td>0.020</td>
<td>1.0182535648</td>
<td>1.0182535648</td>
<td>0.00000000000</td>
</tr>
<tr>
<td>0.030</td>
<td>1.0261288881</td>
<td>1.0261288881</td>
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</tr>
<tr>
<td>0.040</td>
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<tr>
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</table>

Adomian decomposition method performed well as the exact method at \(h=0.01\)
iii. **Critically damped**: Considering the mass/string system with an initial displacement and velocity \( y(0) = y_1 \) and \( y'(0) = y_2 \) respectively and a damping force \( \beta \frac{dy}{dt} \) at a critical state where \( \lambda^2 = w^2 \), then by equation (3.2) the model equation is

\[
\frac{md^2 y}{dt^2} + w^2 y + \beta \frac{dy}{dt} = 0. \tag{3.11}
\]

Example;

\[
y^\prime\prime(t) = -8y^\prime - 16y, \quad y(0) = 0 \text{ and } y'(0) = -1,
\]

has analytical solution

\[
y(t) = -3te^{-4t}. \tag{3.12}
\]

The numerical solution by the Adomian decomposition method of (2.4 – 2.5) for

The first 7th term are;

\[
\begin{align*}
y_0(t) &= -3t \\
y_1(t) &= 12t^2 \\
y_2(t) &= -16t^3(2 + t) \\
y_3(t) &= \frac{64}{15}t^4(15 + 12t + 2t^2) \\
y_4(t) &= -\frac{256}{105}t^5(42 + 42t + 12t^2 + t^3) \\
y_5(t) &= \frac{2048}{4725}t^6(315 + 360t + 135t^2 + 20t^3 + t^4) \\
y_6(t) &= -\frac{4096}{155925}t^7(5940 + 7425t + 3300t^2 + 660t^3 + 60t^4 + 2t^5) + \ldots + O(h)
\end{align*}
\]

By equation (2.3),

\[
\sum_{n=0}^{7} y_n(t) = y_0 + y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + \ldots + O(h), \tag{3.13}
\]

Hence

\[
\begin{align*}
\sum_{n=0}^{6} y_n(t) &= -3t + 12t^2 - 16t^3(2 + t) + \frac{64}{15}t^4(15 + 12t + 2t^2) - \frac{256}{105}t^5(42 + 42t + 12t^2 + t^3) \\
&\quad + \frac{2048}{4725}t^6(315 + 360t + 135t^2 + 20t^3 + t^4) \\
&\quad - \frac{4096}{155925}t^7(5940 + 7425t + 3300t^2 + 660t^3 + 60t^4 + 2t^5) + \ldots + O(h)
\end{align*}
\]
### Table 4.3: Comparison Results (h=0.01)

<table>
<thead>
<tr>
<th>t</th>
<th>Solution by Adomian</th>
<th>Analytical solution</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>.000</td>
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</tbody>
</table>

Adomian decomposition method performed well as the exact method at $h=0.01$

(iv). **Underdamped Motion**; considering the mass/string system with an initial displacement and velocity $y(0) = y_1$ and $y'(0) = y_2$ respectively and a damping force $\beta \frac{dy}{dt}$ at a critical state where $\lambda^2 - w^2 < 0$, then by equation (3.2) the model equation is

$$\frac{md^2y}{dt^2} + w^2 y + \beta \frac{dy}{dt} = 0.$$  \hspace{1cm} (3.14)

with a general solution

$$y(t) = e^{-\lambda t} (c_1 \cos \sqrt{w^2 - \lambda^2} t + c_2 \sin \sqrt{w^2 - \lambda^2} t).$$  \hspace{1cm} (3.15)

Example; $y''(t) = -2y' - 10y, \ y(0) = -2$ and $y'(0) = 0$, has analytical solution

$$y(t) = e^{-t} (-2 \cos 3t - \frac{2}{3} \sin 3t).$$  \hspace{1cm} (3.16)

The numerical solution by the Adomian decomposition method of (2.3–2.5) for the first $9^{th}$ term are;

$$\sum_{n=0}^{7} y_n(t) = y_0 + y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + \ldots + O(h),$$  \hspace{1cm} (3.17)
\[
\sum_{n=0}^{8} y_n(t) = -2 + 10t^2 - \frac{5}{3} t^2 (4 + 5t) + \frac{5}{9} t^4 (6 + 12t + 5t^2) - \frac{1}{252} t^8 (336 + 840r + 600r^2 + 125r^3)
\]
\[+ \frac{1}{2268} t^6 (1008 + 2880r + 2700r^2 + 100r^3 + 125r^4)
\]
\[- \frac{1}{14968} t^7 (19008 + 59400r + 66000r^2 + 33000r^3 + 7500r^4 + 625r^5)
\]
\[+ \frac{1}{13621608} t^8 (432432 + 1441440r + 1801800r^2 + 1092000r^3 + 341250r^4 + 52500r^5 + 3125r^6)
\]
\[- \frac{1}{326918592} t^9 (2306304 + 8072064r + 11007360r^2 + 7644000r^3 + 2940000r^4
\]
\[+ 630000r^5 + 70000r^6 + 3125r^7) + O(h)
\]

Table 4.4: Comparison Results (h=0.01)

<table>
<thead>
<tr>
<th>t</th>
<th>Solution by Adomian</th>
<th>Analytical solution</th>
<th>Error</th>
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</table>

Adomian decomposition method performed well as the exact method at h=0.01

4 Conclusion

In all the various stages of the damped motion illustrated in section (3), the numerical solutions by the Adomian decomposition method approximated the analytical solutions with very minimal errors. Therefore it is a good mathematical tool for evaluating higher order boundary value problems.

References


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