A New Method on the Fourier Chebyshev Expansion of the Product of Special Chebyshev Sums with Combinatorial Coefficients

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Abstract

In this work we consider the Chebyshev Fourier finite series expansions of the different products that can be formed of the following two Chebyshev sums:

\[ S_1(n, x) = \frac{1}{2} \binom{2n}{n} + \sum_{r=1}^{n} \binom{2n}{n-r} T_r(x) \]

and

\[ S_2(n, x) = \frac{1}{2} \binom{2n}{n} + \sum_{r=1}^{n} (-1)^r \binom{2n}{n-r} T_r(x) \quad \text{with} \quad x = \cos(\theta). \]

First we like to indicate that \( T_r(x) \) represent the Chebyshev polynomials of the first kind, and of degree \( r \). We further like to note that the cause, and the origin of the above two Chebyshev sums was generated in the Key lemma by Ziad S. Ali In [2]; Our new method uses the Key lemma providing us with a simple, and new method for finding the Chebyshev expansion of the different products of the above Chebyshev sums. In fact for any integer \( s > 0 \) we give the Fourier Chebyshev expansion of \( S_1^{s}(n, x) \), and \( S_2^{s}(n, x) \) We also give in addition a different method of our presentation. The other method we use depend on already known expressions for computing the coefficients of the product expansion. As a result the new method shows us that it’s a much easier method, more practical, and with it more general cases are easily obtained; furthermore by using the new method we are avoiding what could be
thought of as an undesired or unnecessary computations of the coefficients. The method of computing the coefficients which uses already known formulas for computing the coefficients of the expansion of the product is challenging in our case as in it one is confronted in proving combinatorial identities.

we like to indicate further that use of the Key lemma, and the Chebyshev Fourier expansion of \((1 - x^2)^n\) was very handy in obtaining the Chebyshev Fourier expansion of \(S_1(n, x).S_2(n, x)\). In our work we also give general Theorems related to the coefficients in a Fourier Chebyshev expansion, and hence adding to what was given by Ziad S.Ali in [1] on the topic of Fourier Chebyshev series expansion.

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1 Introduction

The Chebyshev’s polynomials of the first kind \(T_n(x)\), and of the second kind \(U_n(x)\) are respectively defined by:

\[
T_n(x) = \cos n\theta, \quad U_n(x) = \frac{\sin(n + 1)\theta}{\sin \theta}, \quad x = \cos \theta.
\]

A continuous function \(f(x)\) in \(|x| \leq 1\) can have a Generalized Fourier Series expansion or a Fourier Chebyshev Series expansion of the form

\[
f(x) = \sum_{r=1}^{\infty} a_r T_r(x), \quad \text{where} \quad a_r = \frac{2}{\pi} \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f(x) T_r(x) dx.
\]

The orthogonality conditions are given by

\[
\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_s(x) T_r(x) dx = \begin{cases} 
0 & \text{if } s \neq r \\
\pi & \text{if } s = r = 0 \\
\frac{\pi}{2} & s = r = 1, 2, \ldots
\end{cases}
\]

Similarly,

\[
g(x) = \sum_{r=1}^{\infty} b_r U_r(x), \quad \text{where} \quad b_r = \frac{2}{\pi} \int_{-1}^{1} \sqrt{1-x^2} g(x) U_r(x) dx
\]

with the orthogonality conditions

\[
\int_{-1}^{1} U_r(x) U_s(x) \sqrt{1-x^2} dx = \begin{cases} 
0 & \text{if } r \neq s \\
\frac{\pi}{2} & r = s
\end{cases}
\]
2 Known key lemma :

In [3] Ziad S.Ali has the following key lemma:

Lemma 2.1: For $1 \leq r \leq n$, and $\theta$ real we have:

(i) $\sum_{r=1}^{n} \binom{2n}{n-r} \cos r\theta + \frac{1}{2} \binom{2n}{n} = 2^{n-1}(1 + \cos \theta)^n$.

(ii) $\sum_{r=1}^{n} (-1)^r \binom{2n}{n-r} \cos r\theta + \frac{1}{2} \binom{2n}{n} = 2^{n-1}(1 - \cos \theta)^n$.

We like to note that each of the above two sums as given is very similar in their format to the truncated Fourier series expansion of an even function, which as we know is given by:

$$\frac{1}{2} a_0 + \sum_{r=1}^{n} a_r \cos(r\theta)$$

3 Main Theorems:

In this section we have the following main Theorems:

Theorem 3.1. The Fourier Chebyshev finite series expansion of the product:

$$S_1(n, x) \cdot S_1(n, x)$$

is given by:

$$\frac{1}{4} \binom{4n}{2n} + \sum_{r=1}^{2n} \frac{1}{2} \binom{4n}{2n-r} T_r(x)$$

Proof of Theorem 3.1: The New Method in the proof of Theorem 3.1:
By using the Key lemma stated above the proof of Theorem 3.1 is immediate. Knowing the right hand side of the key lemma has helped, and in general knowing how to find the closed form of a Chebyshev sum can always be helpful. For the proof of the Key lemma see [2] by Ziad S.Ali.

Proof of Theorem 3.1: The method of computing the coefficients in the proof of Theorem 3.1:
In considering the product

$$S_1(n, x) \cdot S_1(n, x)$$

one can easily see that by using the Chebyshev identity

$$2T_i T_j = T_{i+j} + T_{i-j}$$
that the constant term independant of $T_r$ is given by:

$$\frac{1}{4} \binom{2n}{n}^2 + \frac{1}{2} \sum_{r=1}^{n} \binom{2n}{n-r}^2 = \frac{1}{4} \binom{4n}{2n}$$

So if

$$p_0, p_1, p_2, p_3, \ldots, p_{n-1}, p_n, \ldots, p_{2n-1}, p_{2n}$$

are the coefficients of the expansion of $S_1^2(n, x)$, then $p_0 = \frac{1}{4} \binom{4n}{2n}$.

Now by using the formulas given by Baszenki, and Tasche [3] iwe have:

$$p_r = \frac{1}{2} \sum_{j=0}^{r} \binom{2n}{n-j} \binom{2n}{n-r+j} + \sum_{j=1}^{n-r} \binom{2n}{n-j} \binom{2n}{n-r-j} \quad \text{with} \quad r = 1, 2, 3, \ldots, n-1.$$

$$p_r = \frac{1}{2} \sum_{j=r-n}^{n} \binom{2n}{n-j} \binom{2n}{n-r+j} \quad \text{with} \quad r = n, n+1, \ldots, 2n.$$

What we like to do now is to find a closed form expression for each of the above two sums each representing $p_r$ for different values of $r$.

For $r=1, 2, \ldots, n-1$, we can show that:

$$p_r = \frac{1}{2} \sum_{j=0}^{r} \binom{2n}{n-j} \binom{2n}{n-r+j} + \sum_{j=1}^{n-r} \binom{2n}{n-j} \binom{2n}{n-r-j} =$$

$$\frac{1}{2} \sum_{j=-n}^{n-r} \binom{2n}{n+j} \binom{2n}{(n-r)-j} = \frac{1}{2} \binom{4n}{2n-r}.$$

We like to remark that the just above equations are possible by using the Vandermonde combinatorial identity as well as using some of the properties of finite sums.

For $r = n, n+1, n+2, \ldots, 2n$, we have by the symmetric Vandermonde combinatorial identity the following:

$$p_r = \frac{1}{2} \sum_{j=r-n}^{n} \binom{2n}{(n)-j} \binom{2n}{(n-r)+j} = \frac{1}{2} \binom{4n}{2n-r}$$

Accordingly the the Fourier Chebyshev finite series expansion of the product:

$$S_1(n, x) \cdot S_1(n, x)$$

is given by:

$$\frac{1}{4} \binom{4n}{2n} + \sum_{r=1}^{2n} \frac{1}{2} \binom{4n}{2n-r} T_r(x)$$
Fourier Chebyshev expansion

We like to add that using the key lemma has saved from lengthy computations, at the same time knowing a different method is always better.
Accordingly in appreciation of the Key lemma given above we state the following general Theorem which gives the Fourier Chebyshev finite series expansion of $S_1^s(n, x)$, the product of $S_1(n, x)$ by itself $s$ number of times.

**Theorem 3.2.** Let $s > 0$ be an integer, then the Fourier Chebychev finite series expansion of the product:

$$S_1^s(n, x)$$

is given by:

$$\frac{1}{2^s} \left( \frac{2ns}{ns} \right) + \frac{1}{2^{s-1}} \sum_{r=1}^{ns} \left( \frac{2ns}{ns - r} \right) T_r(x)$$

Proof of Theorem 3.2: The proof is immediate by the Key lemma above.

The following Theorem is related to the coefficients of the above Chebyshev Fourier series expansion, and for more detailed view see Generalized Theorems on the finite Chebyshev Fourier series expansion by Ziad S.Ali i In [3]

**Theorem 3.3.** Let $n$, and $s$ be a positive fixed integer, let $r \leq n.s$, and let $|x| \leq 1$ then we have:

$$\left( \frac{2ns}{ns - r} \right) = \frac{2}{\pi} \int_{-1}^{1} \frac{(2n^{s-1}(x + 1)^{ns} - \frac{1}{2}(2^{ns}))}{\sqrt{1 - x^2}} T_r(x) dx.$$ 

Proof of Theorem 3.3: The proof is immediate, and is omitted.

**Theorem 3.4.** The Fourier Chebychev finite series expansion of the product:

$$S_2(n, x).S_2(n, x)$$

is given by:

$$\frac{1}{4} \left( \frac{4n}{2n} \right) + \frac{1}{2} \sum_{r=1}^{2n} (-1)^r \left( \frac{4n}{2n - r} \right) T_r(x)$$

Proof of Theorem 3.4: The New Method in the proof of Theorem 3.4: By using the Key lemma stated above the proof of Theorem 3.4 is immediate.
Proof of Theorem 3.4: The method of computing the coefficients in the proof of Theorem 3.4: In considering the product

$$S_2(n, x).S_2(n, x)$$
one can easily see that by using the Chebyshev identity
\[ 2T_iT_j = T_{i+j} + T_{i-j} \]
that the constant term independant of \( T_r \) is given by :
\[ \frac{1}{4} \binom{2n}{n}^2 + \frac{1}{2} \sum_{r=1}^{n} (-1)^{2r} \binom{2n}{n-r}^2 = \frac{1}{4} \binom{4n}{2n} \]
We add by remarking that a constant term for example is generated by considering the product
\[ (-1)^r \binom{2n}{n-r} (-1)^r \binom{2n}{n-r} T_1T_1 \]
Now related to the coefficients
\[ p_1, p_2, p_3, \ldots, p_{n-1}, p_n, \ldots, p_{2n-1}, p_{2n} \]
of \( S_2^2(n, x) \) we like to indicate that method 2 in this case is the same as method 2 indicated in the proof of Theorem 3.1 above. Accordingly we can easily see that The Fourier Chebychev finite series expansion of the product :
\[ S_2(n, x).S_2(n, x) \]
is given by :
\[ \frac{1}{4} \binom{4n}{2n} + \frac{1}{2} \sum_{r=1}^{2n} (-1)^r \binom{4n}{2n-r} T_r(x) \]
Again we state the following general Theorem which gives the Fourier Chebychev finite series expansion of \( S_2^s(n, x) \), the product of \( S_2(n, x) \) by itself \( s \) number of times.

**Theorem 3.5.** Let \( s > 0 \) be an integer, then the Fourier Chebychev finite series expansion of the product :
\[ S_2^s(n, x) \]
is given by :
\[ \frac{1}{2^s} \binom{2ns}{ns} + \frac{1}{2^{s-1}} \sum_{r=1}^{ns} (-1)^r \binom{2ns}{ns-r} T_r(x) \]

Proof of Theorem 3.5 : The proof is immediate by the Key lemma above.

The following Theorem is related to the coefficients of the above Chebyshev Fourier series expansion, and for more detailed view see Generalized Theorems on the finite Chebyshev Fourier series expansion by Ziad S.Ali i In [3]
Theorem 3.6. Let $n$, and $s$ be a positive fixed integer, let $r \leq n.s$, and let $|x| \leq 1$ then we have:

$$(-1)^r \left( \frac{2ns}{ns-r} \right) = \frac{2}{\pi} \int_{-1}^{1} \frac{(2^{ns-1} (1-x)^{ns} - \frac{1}{2}(2^{ns})}{\sqrt{1-x^2}} T_r(x) dx.$$ 

Proof of Theorem 3.6 : The proof is immediate, and is omitted.

Now the basic idea of the following Theorem is to consider the Fourier Chebychev finite series expansion of the product : $S_1(n,x).S_2(n,x)$, and hence we would have considered in this paper the different products which arise from $S_1(n,x)$, and $S_2(n,x)$.

Theorem 3.7. The Fourier Chebychev finite series expansion of the product :

$$S_1(n,x).S_2(n,x)$$

is given by :

$$\frac{1}{4} \left( \frac{2n}{n} \right) + \frac{1}{2} \sum_{r=1}^{n} (-1)^r \left( \frac{2n}{n-r} \right) T_{2r}(x)$$

Proof of Theorem 3.7 : The New Method in the proof of Theorem 3.7 : By using the Key lemma given above we can easily see that :

$$S_1(n,x).S_2(n,x) = 2^{(2n-2)}(1 - x^2)^n$$

Now by noting that

$$2^{(2n-2)}(1 - x^2)^n = 2^{(2n-2)}\left( \frac{2n}{2^{2n}} \right) + 2^{1-2n} \sum_{r=0}^{n} (-1)^r \left( \frac{2n}{n-r} \right) T_{2r}(x)$$

Accordingly :

$$S_1(n,x).S_2(n,x) = \frac{2n}{4} + \frac{1}{2} \sum_{r=1}^{n} (-1)^r \left( \frac{2n}{n-r} \right) T_{2r}(x)$$

which is the desired expansion by using the Key lemma given above, and by using the Chebyshev expansion of $(1 - x^2)^n$.

Proof of Theorem 3.7 : The method of computing the coefficients proof of Theorem 3.7: For any $n$ the coefficients $C_1, C_2, \ldots, C_{n-2}, C_{n-1}$ are given by :

$$C_r(1 \leq r < n) = \frac{1}{2} \sum_{j=0}^{r} (-1)^j \left( \frac{2n}{n-r+j} \right) \left( \frac{2n}{n-j} \right) +$$
\[
\frac{1}{2}(-1)^r \sum_{j=1}^{n-r} (-1)^j \binom{2n}{n-r-j} \binom{2n}{n-j} +
\frac{1}{2} \sum_{j=1}^{n-r} (-1)^j \binom{2n}{n-r-j} \binom{2n}{n-j}
\]

We need to note now that for \( r \) odd with \( 1 \leq r < n \) \( C_r = 0 \).

Note for example that the first sum just given above is one where the zero term, and the \( k \)th term are both equal, but of different signs. Similarly the first term, and the \((-1)\) term are both equal, but of different signs. Accordingly we continue until we exhaust all the terms.

For any \( n \) the coefficients \( C_n, C_{n+1}, \ldots, C_{2n-3}, C_{2n-1} \) are given by:

\[
C_r(n \leq r < 2n) = \frac{1}{2} \sum_{j=r-n}^{n} (-1)^j \binom{2n}{n-r+j} \binom{2n}{n-j}
\]

To show that for any \( r \) odd with \( n \leq r < 2n \) \( C_r = 0 \), we use an argument very similar to the just above argument. Let us say that \( n \) is even, then clearly \( C_{n+1} = 0 \), for the first term in the sum equals the last term, and both have different signs. Similarly the second term, and the term before the last add up to zero. So we continue in this manner until we exhaust all the terms.

So we have thus shown that in the expansion of \( S_1(n, x)S_2(n, x) \) all the odd coefficients are zero.

Accordingly our expansion of

\[
S_1(n, x)S_2(n, x)
\]

is now generally given by:

\[
\frac{1}{2}C_0 + \sum_{k=1}^{n} C_{2k}T_{2k}(x)
\]

Now with

\[
a_0, a_1, a_2, \ldots, a_n,
\]

being the coefficients \( S_1(n, x) \), and

\[
b_0, b_1, b_2, \ldots, b_n,
\]

being the coefficients of \( S_2(n, x) \) it follows easily that
\[ C_0 = \frac{1}{2} a_0 b_0 + \sum_{j=1}^{n} a_j b_j \]

Now since
\[ \sum_{j=1}^{n} (-1)^j \binom{2n}{n-j} = \frac{1}{2} \binom{2n}{n} - \frac{1}{2} \binom{2n}{n}^2 \]

It follows easily that
\[ C_0 = \frac{1}{2} \binom{2n}{n}^2 + \sum_{j=1}^{n} (-1)^j \binom{2n}{n-j}^2 = \frac{1}{2} \binom{2n}{n} \]

Hence our expansion as it stands now is of the form:
\[ \frac{1}{4} \binom{2n}{n} + \sum_{k=1}^{n} C_{2k} T_{2k}(x) \]

Now for the coefficients \( C_2, C_4, C_6, \ldots, C_{n-2} \)
we have with \( k = 1, 2, 3, \ldots, \frac{n-2}{2} \):

\[ C_{2k} = \frac{1}{2} \sum_{j=0}^{2k} (-1)^j \binom{2n}{n-j} \binom{2n}{n-2k+j} + \sum_{j=1}^{n-2k} (-1)^j \binom{2n}{n-j} \binom{2n}{n-2k-j} \]

Now related to the above combinatorial identity we can say that by a change in the variable index \( j \), and by noting that
\[ \sum_{j=n-2k}^{n} (-1)^j \binom{2n}{j} \binom{2n}{2n-2k-j} + \sum_{j=n+1}^{2n-2k} (-1)^j \binom{2n}{j} \binom{2n}{2n-2k-j} = \]
\[ \sum_{j=0}^{n} (-1)^n \binom{2n}{j} \binom{2n}{2n-2k-j} \]

one can easily see that
\[ C_{2k} = \frac{1}{2} (-1)^k \binom{2n}{n-k}, k = 1, 2, 3, \ldots, \frac{n-2}{2}. \]

Now for the coefficients
\[ C_n, C_{n+2}, C_{n+4}, \ldots, C_{2n} \]
we have with \( k = \frac{n}{2}, \frac{n+2}{2}, \ldots, n \):

\[
C_{2k} = \frac{1}{2} \sum_{j=2k-n}^{n} (-1)^j \binom{2n}{n-j} \binom{2n}{n+j-2k}
\]

Accordingly for example with \( k = \frac{n}{2} \) we have:

\[
C_n = \frac{1}{2} \sum_{j=0}^{n} (-1)^j \binom{2n}{j} \binom{2n}{n-j} = \frac{1}{2} (-1)^{\frac{n}{2}} \binom{2n}{n} = \frac{1}{2} (-1)^k \binom{2n}{n-k}
\]

Similarly for \( k = \frac{n+2}{2} \) we have:

\[
C_{n+2} = \frac{1}{2} \sum_{j=0}^{n-2} (-1)^j \binom{2n}{j} \binom{2n}{n-2-j} = \frac{1}{2} (-1)^{\frac{n-2}{2}} \binom{2n}{n-2} = \frac{1}{2} (-1)^k \binom{2n}{n-k}
\]

The other coefficients \( C_{n+4}, \ldots, C_{2n} \) are treated similarly to above. Accordingly both methods agree in computing the coefficients of the product expansion \( S_1(n, x).S_2(n, x) \).

**References**


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