Lattice-Valued Quasi-bi-Uniformizability of Lattice-Valued Quasi-bi-Topological Neighborhood Groups

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Abstract

Considering a frame $L$, we introduce the notions of stratified $L$-semi-topological neighborhood group, stratified $L$-quasi-topological neighborhood group, and stratified $L$-quasi-bi-topological neighborhood group. In so doing, we look at the notion of stratified $L$-right (left) semi-topological neighborhood group, provide some basic facts, and present a construction of a stratified $L$-right semi-topological neighborhood group. We prove that every stratified $L$-quasi-bi-topological neighborhood group is $L$-quasi-bi-uniformizable; and finally, we present conditions under which a

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stratified \(L\)-quasi-topological neighborhood group is a stratified \(L\)-topological neighborhood group.

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1. **Introduction**

Generalization of classical notion of topological groups into the framework of lattice-valued case have received much attention over the years [3, 4, 5, 7, 17], we provide a short list of reference which by no means is complete (for an almost complete list, we refer the interested reader to [3]). However, we do not see much work on the generalization of classical notion of quasi-topological groups (which is also widely known as paratopological groups in the sense of Bourbaki [6]) within the scope of lattice-valued mathematics. While studying the classical notion of quasi-topological group, the researchers, among many others, shown interest to find conditions under which quasi-topological group produces topological group; and also, to give bi-topological view of quasi-topological group as well as quasi-uniformities of quasi-bi-topological groups [2, 12, 16, 18].

We introduced in [3] a notion of stratified \(L\)-topological neighborhood group and considering enriched lattices we introduced and studied the notion of enriched lattice-valued topological neighborhood groups, generalizing the notion of topological groups. We also considered there \(L\)-uniformizability of \(L\)-topological neighborhood group.

Considering a few results of classical quasi-topological groups, in this article, with \(L\) as a frame, and the same stratified \(L\)-topological neighborhood system used in [3, 4, 5] as originated from [13], we introduce the notion of stratified \(L\)-quasi-topological neighborhood group, stratified semi-topological neighborhood group and stratified \(L\)-quasi-bi-topological neighborhood group.

We present a notion of quasi-bi-topological neighborhood group and show that
every stratified $L$-quasi-bi-topological group is a stratified $L$-quasi-bi-uniform space. We give conditions under which a stratified $L$-quasi-topological neighborhood group gives rise to a stratified $L$-topological neighborhood group.

2. PRELIMINARIES

Throughout the text we consider $L = (L, \leq, \wedge)$, a frame. The set of all $L$-sets, denoted by $L^X (= \{ \nu: X \rightarrow L \})$. If $\alpha \in L$ and $A \subseteq X$, then the map $\alpha_A : X \rightarrow L$ is defined by

$$
\alpha_A(x) = \begin{cases} 
\alpha, & \text{if } x \in A; \\
\bot, & \text{otherwise.}
\end{cases}
$$

In particular, $\top_X(x) = \top$, the characteristic function of $X$ and $\bot_X(x) = \bot$, the zero function. For singleton set $\{x\}$, we simply write $\top_{\{x\}} := x$, if there is no danger of confusion.

**Definition 2.1.** [13] A map $F: L^X \rightarrow L$ is called an $L$-filter on $X$ if and only if the conditions below are satisfied:

- (LF1) $F(\top_X) = \top, F(\bot_X) = \bot$;
- (LF2) if $\nu_1, \nu_2 \in L^X$ with $\nu_1 \leq \nu_2$, then $F(\nu_1) \leq F(\nu_2)$;
- (LF3) $F(\nu_1) \land F(\nu_2) \leq F(\nu_1 \land \nu_2)$, $\forall \nu_1, \nu_2 \in L^X$.
- (SL) An $L$-filter $F$ is called a stratified $L$-filter if $\forall \alpha \in L, \forall \mu \in L^X$, $\alpha \land F(\mu) \leq F(\alpha \land \mu)$.

The set of all stratified $L$-filters on $X$ is denoted by $F^s_L(X)$. On $F^s_L(X)$, partial ordering $\leq$ is defined by: if $F, G \in F^s_L(X)$, then $F \leq G \iff F(\nu) \leq G(\nu)$, $\forall \nu \in L^X$. If $x \in X$, then $[x] \in F^s_L(X)$, called point stratified $L$-filter on $X$, and is defined as $[x](\nu) = \nu(x)$, for all $\nu \in L^X$.

If $f: X \rightarrow Y$ is a function, then $f^+: L^Y \rightarrow L^X$ is defined for any $\mu \in L^Y$ by $f^+(\mu) = \mu \circ f$; and $f^-: L^X \rightarrow L^Y$ is defined by: $f^-(\nu)(y) = \bigvee \{\nu(x)|f(x) = y\}$, $\forall \nu \in L^X, y \in Y$. Moreover, if $F \in F^s_L(X)$, then the stratified $L$-filter $f^s(F): L^Y \rightarrow L$ on $Y$ is defined for any $\mu \in L^Y$ by: $[f^s(F)](\mu) = F(f^+(\mu)) = F(\mu \circ f)$.

If $F \in F^s_L(Y)$, then $f^s(F): L^X \rightarrow L$ is defined by: $[f^s(F)](\nu) = \bigvee \{F(\mu)|\mu \in L^Y, f^-(\mu) \leq \nu\}$, for all $\nu \in L^X$, is a stratified $L$-filter on $X$ if and only if for
all \( \mu \in L^Y, f^\leftarrow(\mu) = \perp_X \Rightarrow \mathcal{F}(\mu) = \perp \).

If \( \nu \in L^X \) and \( \mu \in L^Y \), then the product \( \nu \times \mu \in L^{X \times Y} \) is defined by

\[ \nu \times \mu = \nu \circ pr_1 \land \mu \circ pr_2, \]

where \( pr_1 : X \times Y \rightarrow X, (x, y) \mapsto x \) and \( pr_2 : X \times Y \rightarrow Y, (x, y) \mapsto y \). If \( (X, \cdot) \) is a group with neutral element \( e \), and \( \mathcal{F}, \mathcal{G} \in \mathcal{F}_L^X(X) \),

then the map \( \mathcal{F} \circ \mathcal{G} : L^X \rightarrow L \) is defined for any \( \nu \in L^X \) by:

\[ \mathcal{F} \circ \mathcal{G}(\nu) = \bigvee \{ \mathcal{F}(\nu_1) \land \mathcal{G}(\nu_2) \mid \nu_1, \nu_2 \in L^X, \nu_1 \land \nu_2 \leq \nu \} \]

If \( (X, \cdot) \) is a group and \( \mathcal{F} \in \mathcal{F}_L^X(X) \), then \( \mathcal{F}^{-1} \) is defined by \( \mathcal{F}^{-1}(\nu) = \mathcal{F}(\nu^{-1}) \), where \( \nu^{-1} : X \rightarrow L, x \mapsto \nu(x)^{-1} \). Clearly, \( \mathcal{F}^{-1} \in \mathcal{F}_L^X(X) \), since for any \( \nu \in L^X \),

\[ j^\circ(\mathcal{F})(\nu) = \mathcal{F}(j^\leftarrow(\nu)) = \mathcal{F}(\nu^{-1}) = \mathcal{F}^{-1}(\nu), \]

where \( j : X \rightarrow X, x \mapsto x^{-1} \).

Also, if \( m : X \times X \rightarrow X, (x, y) \mapsto xy \), then for any \( \nu_1, \nu_2 \in L^X \) and \( z \in X \),

\[ m^\rightarrow (\nu_1 \times \nu_2)(z) = \bigvee_{m(x,y) = z} (\nu_1 \times \nu_2)(x, y) = \bigvee_{x=y=z} (\nu_1 \circ pr_1 \land \nu_2 \circ pr_2)(x, y) \]

\[ = \bigvee_{x=y=z} \nu_1 \circ pr_1(x, y) \land \nu_2 \circ pr_2(x, y) = \bigvee_{x=y=z} \nu_1(x) \land \nu_2(y) = \nu_1 \land \nu_2(z) \]

**Definition 2.2.** [13] A subset \( \Delta \subset L^X \) is called a stratified \( L \)-topology on a set \( X \) if it fulfills the following:

\[ \text{(LT1)} \quad \top_X, \perp_X \in \Delta; \]
\[ \text{(LT2)} \quad \nu_1, \nu_2 \in \Delta \text{ implies } \nu_1 \land \nu_2 \in \Delta; \]
\[ \text{(LT3)} \quad \forall j \in J, \nu_j \in \Delta \text{ implies } \bigvee_{j \in J} \nu_j \in \Delta; \]
\[ \text{(SL)} \quad \forall \alpha \in L, \nu \in \Delta \text{ implies } \alpha_X \land \nu \in \Delta. \]

The pair \( (X, \Delta) \) is called a stratified \( L \)-topological space.

A map \( f : (X, \Delta) \rightarrow (Y, \Gamma) \) between two stratified \( L \)-topological spaces is called continuous if and only if for any \( \gamma \in \Gamma \) implies \( f^\leftarrow(\gamma) \in \Delta \).

The category of all stratified \( L \)-topological spaces and continuous maps is denoted by \( \text{SL-\text{TOP}} \).

**Definition 2.3.** [13] A family of stratified \( L \)-filters \( \Psi = (\Psi_x)_{x \in X} \) is called a stratified \( L \)-neighborhood system on a set \( X \) if and only if the following conditions are satisfied:

\[ \text{(LN1)} \quad \forall x \in X, \Psi_x \leq [x]; \]
\[ \text{(LN2)} \quad \forall x \in X, \forall \nu \in L^X : \Psi_x(\nu) \leq \bigvee \{ \Psi(\sigma) : \sigma \in L^X, \sigma(y) \leq \Psi_y(\nu), \forall y \in X \}. \]

The pair \( (X, \Psi) \) is called a stratified \( L \)-neighborhood space.

A map \( f : (X, \Psi) \rightarrow (Y, \Psi') \) between stratified \( L \)-neighborhood spaces is called continuous at \( x \in X \) if and only if \( \Psi'_{f(x)}(\varphi) \leq \Psi_x(f^\leftarrow(\varphi)), \forall \varphi \in L^Y \).
The mapping \( f \) is continuous if it is continuous at each point \( x \in X \).

The category of all stratified \( L \)-neighborhood spaces and continuous maps is denoted by \( \text{SL-NS} \).

Note that given \((X, \Delta) \in |\text{SL - TOP}|\), one obtains: \((X, \Psi^\Delta) \in |\text{SL - NS}|\), where for any \( \nu \in L^X \) and \( x \in X \), \( \Psi^\Delta_x(\nu) = \bigvee\{\varphi(x) : \varphi \in \Delta, \varphi \leq \nu\} \). Conversely, given \((X, \Psi) \in |\text{SL - NS}|\), one obtains: \((X, \Delta^\Psi) \in |\text{SL - TOP}|\), where for any \( \nu \in L^X: \nu \in \Delta^\Psi \iff \forall x \in X, \nu(x) \leq \Psi_x(\nu) \).

Due to the preceding connections between stratified \( L \)-topology and stratified \( L \)-neighborhood system, and the proposition that follow, one can easily prove that the categories \( \text{SL-TOP} \) and \( \text{SL-NS} \) are isomorphic.

**Proposition 2.4.** [14] Let \((X, \Delta), (X', \Delta') \in |\text{SL - TOP}|\) with corresponding stratified \( L \)-neighborhood system \( \Psi^\Delta \) and \( \Psi'^{\Delta'} \) on \( X \) and \( X' \) respectively. If \( f : X \to X' \), then the following are equivalent:

(a) \( \Psi'_{f(x)}(\sigma) \leq \Psi^\Delta_x(f^\sim(\sigma)), \forall \sigma \in L^{X'}, x \in X; \)
(b) \( \sigma \in \Delta' \) implies \( f^\sim(\sigma) \in \Delta \).

**Definition 2.5.** [9] Let \( \Psi = (\Psi_x)_{x \in X} \) be a stratified \( L \)-neighborhood system on \( X \). Then a map \( B : X \to L^X, x \mapsto B_x \) is called a stratified \( L \)-neighborhood base of \( \Psi \) if it satisfies \( \forall x \in X \) and \( \nu \in L^X \), the following

(i) \( B_x(\top_x) = \top \);
(ii) \( B_x \leq \Psi_x \);
(iii) \( \Psi_x(\nu) = \bigvee\{B_x(\beta) : \beta \in L^X, \beta \leq \nu\} \);
(iv) \( \alpha \in L, \nu \in L^X: \alpha \wedge B_x(\nu) \leq B(\alpha_x \wedge \nu) \).

**Theorem 2.6.** [9] Let \((X, \Psi = (\Psi_x)_{x \in X}) \in |\text{SL-NS}|\). Then an \( L \)-neighborhood base \( B = (B_x)_{x \in X} \) of \( \Psi \) satisfies the following conditions:

(LNB1) \( B_x(\top) = \top, \forall x \in X; \)
(LNB2) \( B_x(\nu_1) \wedge B_x(\nu_2) \leq \bigvee\{B_x(\beta) : \beta \in L^X, \beta \leq \nu_1 \wedge \nu_2\}, \forall \nu_1, \nu_2 \in L^X \) and \( x \in X; \)
(LNB3) \( B_x(\beta) \leq \beta(x), \forall \beta \in L^X \) and \( x \in X; \)
(LNB4) \( B_x(\beta) \leq \bigvee\{B_x(\xi) : \xi \in L^X, \xi(y) \leq \bigvee_{\eta \leq \beta} B_y(\eta), \forall y \in X\}, \forall x \in X, \beta \in L^X. \)
(LNB5) \( \forall \alpha \in L, \beta \in L^X: \alpha \wedge B_x(\beta) \leq B_x(\alpha_x \wedge \beta) \).
Conversely, for a given map \( \mathbb{B} : X \rightarrow L^X \) satisfying (LNB1)-(LNB5), there exists a stratified \( L \)-neighborhood system \( \Psi \) on \( X \) such that \( \mathbb{B} \) is exactly the stratified \( L \)-neighborhood base of \( \Psi \).

**Lemma 2.7.** Let \((X, \cdot)\) be a group, \( F \in F^s_L(X) \) and \( x_0 \in X \). Then for any \( \nu \in L^X \), \( ([x_0] \odot F)(\nu) = \mathcal{F}(x_0^{-1} \odot \nu) \).

**Proof.** Let \( F \in F^s_L(X) \) and \( x_0 \in X \). Then for any \( \nu \in L^X \), we have

\[
([x_0] \odot F)(\nu) = \bigvee \{ [x_0](\nu_1) \land \mathcal{F}(\nu_2) \mid \nu_1, \nu_2 \in L^X, \nu_1 \odot \nu_2 \leq \nu \} 
\]

\[
= \bigvee \{ \mathcal{F}(\nu_1(x_0)) \land \mathcal{F}(\nu_2) \mid \nu_1, \nu_2 \in L^X, \nu_1(x_0) \land \nu_2(y) \leq x_0^{-1} \land \nu(y), \forall y \in X \} 
\]

(by stratification of \( \mathcal{F} \))

\[
= \bigvee \{ \mathcal{F}(\nu_1(x_0)) \land \nu_2 \mid \nu_1, \nu_2 \in L^X, \nu_1(x_0) \land \nu_2 \leq x_0^{-1} \land \nu \} = \mathcal{F}(x_0^{-1} \odot \nu).
\]

On the other hand, \( ([x_0] \odot F)(\nu) = ([x_0] \odot F)(x_0 \odot (x_0^{-1} \odot \nu)) \)

\[
\geq [x_0](x_0) \land \mathcal{F}(x_0^{-1} \odot \nu) = \top \land \mathcal{F}(x_0^{-1} \odot \nu) = \top \land \mathcal{F}(x_0^{-1} \odot \nu) = \mathcal{F}(x_0^{-1} \odot \nu). \]

\( \square \)

**Proposition 2.8.** [5] Let \((X, \Delta)\), and \((Y, \Delta')\) be stratified \( L \)-topological spaces with corresponding stratified \( L \)-neighborhood systems \( \Psi \) and \( \Psi' \) on \( X \) and \( Y \), respectively. Then a function \( f : X \rightarrow Y \) is \( L \)-open if and only if it is \( L \)-neighborhood open, i.e. for all \( x \in X \), and and for all \( \mu \in L^X \), \( \Psi_x(\mu) \leq \Psi'_{f(x)}(f^{-1}(\nu)) \).

3. **Stratified \( L \)-semi-topological neighborhood groups and stratified \( L \)-quasi-topological neighborhood groups**

From now on for some technical reasons, we call a stratified \( L \)-neighborhood space, a stratified \( L \)-topological neighborhood space.

**Definition 3.1.** Let \((X, \cdot)\) be a group and \( \Psi = (\Psi_x)_{x \in X} \) a stratified \( L \)-topological neighborhood system on \( X \). Then

(a) the triple \((X, \cdot, \Psi = (\Psi_x)_{x \in X})\) is called a **stratified \( L \)-left topological neighborhood group** if and only if for each fixed \( x_0 \in X \): \( \Psi_{x_0y} \leq \mathcal{L}^{x_0}_{x_0}(\Psi_y) = [x_0] \odot \Psi_y, \forall y \in X \);
(b) the triple \((X, \cdot, \Psi = (\Psi_x)_{x \in X})\) is called a *stratified L-right topological neighborhood group* if and only if for each fixed \(y_0 \in X\): \(\Psi_{xy_0} \leq R_{y_0}^\Rightarrow (\Psi_x) = \Psi_x \circ [y_0], \forall x \in X;\)

(c) the triple \((X, \cdot, \Psi = (\Psi_x)_{x \in X})\) is called a *stratified L-semi-topological neighborhood group* if and only if it is both stratified L-left and right topological neighborhood groups, i.e., the map \(m : (x, y) \mapsto xy\) is separately continuous;

(d) the triple \((X, \cdot, \Psi = (\Psi_x)_{x \in X})\) is called a *stratified L-quasi-topological neighborhood group* if and only if the mapping \(m : (X \times X, \Psi \times \Psi) \rightarrow (X, \Psi), (x, y) \mapsto xy\) is jointly continuous: \(\forall x, y \in X, \Psi_{xy} \leq \Psi_x \circ \Psi_y;\)

(e) the triple \((X, \cdot, \Psi = (\Psi_x)_{x \in X})\) is called a *stratified L-topological neighborhood group* if and only if it is a stratified L-quasi-topological neighborhood group, that is, the map \(m : (X \times X, \Psi \times \Psi) \rightarrow (X, \Psi), (x, y) \mapsto xy\) is jointly continuous, and the inversion map \(j : (X, \Psi) \rightarrow (X, \Psi), x \mapsto x^{-1}\) is continuous: \(\forall x \in X, \Psi_{x^{-1}} \leq \Psi_x^{-1}.\)

**Lemma 3.2.** Let \((X, \cdot, \Psi = (\Psi_x)_{x \in X})\) be a stratified L-semi-topological neighborhood group and \(a\) be a fixed element of \(X\). Then the mappings (left and right translations)

1. \(L_a : X \rightarrow X, x \mapsto ax;\)
2. \(R_a : X \rightarrow X, x \mapsto xa,\)

are homeomorphisms.

**Proof.** Clearly \(R_a\) is bijective. Now if we define \(\psi : X \rightarrow X \times X\) by \(\psi(x) = (x, a)\), then \(\psi\) is continuous, and since \(R_a(x) = m \circ \psi(x) = m(x, a) = xa\), being composition of continuous functions, is continuous and \(R_a^{-1} = R_{a^{-1}}, R_a\) is a homeomorphism. Similarly, one can show that \(L_a\) is a homeomorphism. \(\square\)

**Definition 3.3.** A stratified L-topological neighborhood space \((X, \Psi = (\Psi_x)_{x \in X})\) is called *homogeneous* if and only if for any \(x, y \in X\), there exists a homeomorphism \(f : X \rightarrow X\) such that \(f(x) = y.\)

**Corollary 3.4.** Every stratified L-semi-topological neighborhood group is a homogeneous space.
Proof. This follows from the fact that for \(x, y \in X\), \(R_{x^{-1}y}\) is a homeomorphism. \(\Box\)

Below is a construction of stratified \(L\)-right topology on a group.

**Proposition 3.5.** Let \((X, \cdot)\) a group and \(\mathfrak{P} : X \to L^X\) be a map fulfilling the following conditions:

1. \(\mathfrak{P}_e(\top_X) = \top;\)
2. \(\mathfrak{P}_e(\nu) \leq \nu(e), \forall \nu \in L^X;\)
3. \(\forall \nu_1, \nu_2 \in L^X, \mathfrak{P}_e(\nu_1) \land \mathfrak{P}_e(\nu_2) \leq \bigvee_{\eta \in L^X, \eta \leq \nu_1 \land \nu_2} \mathfrak{P}_e(\eta);\)
4. \(\forall \alpha \in L, \forall \nu \in L^X, \alpha \land \mathfrak{P}_e(\nu) \leq \mathfrak{P}_e(\alpha \land \nu).\)

If we put \(\mathfrak{P}_x(\nu) = \bigvee\{\mathfrak{P}_e(\xi)|\xi \in L^X, \xi \cdot x \leq \nu\}\), then for each \(x \in X\), \(\mathfrak{P}_x\) satisfies (LNB1)-(LNB3) and (LNB5). Define \(\Delta^\mathfrak{P}\) on \(X\) by

\[\nu \in \Delta^\mathfrak{P} \iff \forall x \in X, \nu(x) \leq \bigvee\{\mathfrak{P}_x(\xi)|\xi \in L^X, x \leq \nu\}.\]

Then \(\Delta^\mathfrak{P}\) is a stratified \(L\)-topology and the triple \((X, \cdot, \Delta^\mathfrak{P})\) is a stratified \(L\)-right topological neighborhood group.

**Proof.** First of all we check that for all \(x \in X\), \(\mathfrak{P}_x\) satisfies conditions (LNB1)-(LNB4).

Clearly (LNB1) is true.

(LNB2) We take \(\nu_1, \nu_2 \in L^X\) and \(x \in X\). Then

\[
\mathfrak{P}_x(\nu_1) \land \mathfrak{P}_x(\nu_2) = \bigvee\{\mathfrak{P}_e(\eta_1)|\eta_1 \in L^X, \eta_1 \cdot x \leq \nu_1\} \land \bigvee\{\mathfrak{P}_e(\eta_2)|\eta_2 \in L^X, \eta_2 \cdot x \leq \nu_2\} \\
= \bigvee\{\mathfrak{P}_e(\eta_1) \land \mathfrak{P}_e(\eta_2)|\eta_1, \eta_2 \in L^X, \eta_1 \cdot x \leq \nu_1, \eta_2 \cdot x \leq \nu_2\} \\
\leq \bigvee\{\mathfrak{P}_e(\eta)|\eta \in L^X, (\eta_1 \land \eta_2) \cdot x \leq \nu_1 \land \nu_2\} \\
= \mathfrak{P}_x(\nu_1 \land \nu_2) \\
\leq \bigvee\{\mathfrak{P}_x(\xi)|\xi \in L^X, \xi \leq \nu_1 \land \nu_2\}.\]

(LNB3) Let \(x \in X\) and \(\nu \in L^X\). Then

\[
\mathfrak{P}_x(\nu) = \bigvee\{\mathfrak{P}_e(\eta)|\eta \in L^X, \eta \cdot x \leq \nu\} \\
\leq \bigvee\{\eta(e)|\eta \in L^X, \eta \cdot x \leq \nu\} \\
= \nu \circ x^{-1}(e) = \nu(x).\]

(LNB5) We take \(\alpha \in L, \nu \in L^X\) and \(x \in X\). Then
Now we show that $\Delta$ is a stratified $L$-topology on $X$.

(LT1) Clearly $T_X, \bot_X \in \Delta$.

(LT2) Let $\nu_1, \nu_2 \in \Delta$ and $x \in X$. Then

$$(\nu_1 \wedge \nu_2)(x) = \nu_1(x) \wedge \nu_2(x)$$

$$\leq \bigvee \{\mathfrak{P}_x(\xi_1) | \xi_1 \in L^X, \xi_1 \leq \nu_1 \} \wedge \bigvee \{\mathfrak{P}_x(\xi_2) | \xi_2 \in L^X, \xi_2 \leq \nu_2 \}$$

$$= \bigvee \{\mathfrak{P}_x(\xi_1) \wedge \mathfrak{P}_x(\xi_2) | \xi_1, \xi_2 \in L^X, \xi_1 \leq \nu_1, \xi_2 \leq \nu_2 \}$$

$$\leq \bigvee \{\mathfrak{P}_x(\xi) : \xi \in L^X, \xi \leq \xi_1 \wedge \xi_2 \} | \xi_1, \xi_2 \in L^X, \xi_1 \leq \nu_1, \xi_2 \leq \nu_2 \}$$

$$\leq \bigvee \{\mathfrak{P}_x(\xi) | \xi \in L^X, \xi \leq \nu_1 \wedge \nu_2 \}$$,

implying that $\nu_1 \wedge \nu_2 \in \Delta$.

(LT3) Let for all $j \in J$, $\nu_j \in \Delta$. Then for each $\nu_j$, and $x \in X$, $\nu_j(x) \leq \bigvee \{\mathfrak{P}_x(\xi) | \xi \in L^X, \xi \leq \nu_j \}$. Thus $\bigvee_{j \in J} \nu_j(x) \leq \bigvee \{\mathfrak{P}_x(\xi) | \xi \in L^X, \xi \leq \bigvee_{j \in J} \nu_j \}$, which yields that $\bigvee_{j \in J} \nu_j \in \Delta$.

(STL) Let $\alpha \in L$, $\nu \in \Delta$ and $x \in X$. Then

$$(\alpha_X \wedge \nu)(x) = \alpha \wedge \nu(x) \leq \alpha \wedge \bigvee \{\mathfrak{P}_x(\xi) | \xi \in L^X, \xi \leq \nu \}$$

$$\leq \alpha \wedge \bigvee_{\xi \in L^X, \xi \leq \nu} \{\bigvee \{\mathfrak{P}_x(\xi) | \xi_1 \in L^X, \xi_1 \ominus x \leq \xi \}\}$$

$$\leq \alpha \wedge \bigvee \{\mathfrak{P}_x(\xi) | \xi_1 \in L^X, \xi_1 \ominus x \leq \nu \}$$

$$\leq \bigvee \{\alpha \wedge \mathfrak{P}_x(\xi) | \xi_1 \in L^X, \xi_1 \ominus x \leq \nu \}$$

$$\leq \bigvee \{\mathfrak{P}_x(\alpha_X \wedge \xi_1) | \alpha_X \wedge \xi_1 \in L^X, \alpha_X \wedge (\xi_1 \ominus x) \leq \alpha_X \wedge \nu \}$$

$$\leq \bigvee \{\mathfrak{P}_x(\alpha_X \wedge \xi_1) | \alpha_X \wedge \xi_1 \in L^X, (\alpha_X \wedge \xi_1) \ominus x \leq \alpha_X \wedge \nu \}$$

$$= \mathfrak{P}_x(\alpha_X \wedge \nu) \leq \bigvee \{\mathfrak{P}_x(\xi) | \xi \in L^X, \xi \leq \alpha_X \wedge \nu \}$$

implying that $\alpha_X \wedge \nu \in \Delta$.

It remains to be shown that $R_{x_0} : X \rightarrow X, x \mapsto xx_0$ is continuous. To this end, consider $\nu \in \Delta$ and $x \in X$. Then

$$(R_{x_0}(\nu))(x) = \nu(xx_0) \leq \bigvee_{\xi \in L^X, \xi \leq \nu} \mathfrak{P}_{xx_0}(\xi)$$

$$= \bigvee_{\xi \in L^X, \xi \leq \nu} \bigvee \{\mathfrak{P}_x(\xi_1) | \xi_1 \in L^X, \xi_1 \ominus xx_0 \leq \xi \}$$

$$= \bigvee_{\xi \in L^X, \xi \leq \nu} \bigvee \{\mathfrak{P}_x(\xi_1) | \xi_1 \in L^X, \xi_1 \ominus x \leq \xi \ominus xx_0^{-1} \}$$

$$= \bigvee_{\xi \in L^X, \xi \leq \nu} \bigvee \{\mathfrak{P}_x(\xi_1) | \xi_1 \in L^X, \xi_1 \ominus x \leq R_{x_0}(\xi) \}$$
Proposition 3.6. Let \( (X, \cdot) \) be an Abelian group with neutral element \( e \) and \( \Psi_e \) fulfills conditions (1)-(4) of the Proposition 3.5 such that \( \forall x \in X, \forall \nu \in L^X, \Psi_x(\nu) \leq \bigvee \{ \Psi_e(\xi) | \xi \in L^X, \xi \cdot x \leq \nu \} \). Then \( \nu \cdot x_0 \in \Delta \) according to the preceding construction of stratified \( L \)-right topological neighborhood group. In fact, every \( \mathcal{R}_{x_0} : X \rightarrow X \) is \( L \)-open, equivalently, \( L \)-topological neighborhood open.

Proof. Let \( \nu \in \Delta \) and fix \( x_0 \in X \). Then for any \( x \in X \), we have
\[
\nu \cdot x_0(x) = \nu(xx_0^{-1}) \leq \bigvee \{ \Psi_{x_0}^{-1}(e) | \xi \in L^X, \xi \leq \nu \}
\]
\[
\leq \bigvee \{ \Psi_e(\xi_1) | \xi_1 \in L^X, \xi_1 \odot xx_0^{-1} \leq \xi, \xi \leq \nu \}
\]
\[
\leq \bigvee \{ \Psi_e(\xi_1) : \xi_1 \in L^X, \xi_1 \odot x \leq \nu \}
\]
\[
= \bigvee \{ \Psi_e(\xi_1) : \xi_1 \in L^X, \xi_1 \odot x \leq \mathcal{R}_x(x_0) \nu \}
\]
This implies that \( \mathcal{R}_x(x_0) = \nu \cdot x_0 \in \Delta \), and hence \( \mathcal{R}_{x_0} \) is an \( L \)-open function, and therefore, in view of Proposition 2.8, it is \( L \)-topological neighborhood open. \( \Box \)

Proposition 3.7. Let \( (X, \cdot, \Psi = (\Psi_x)_{x \in X}) \) be a stratified \( L \)-left (resp. right) topological neighborhood group, and \( \Psi_e \) be a stratified \( L \)-neighborhood system in \( e \). Then for each \( x \in X \) and \( \nu \in L^X \), \( \mathcal{B}_x(\nu) = \bigvee \{ \Psi_e(\eta) | \eta \in L^X, x \odot \eta \leq \nu \} \) (resp. for each \( x \in X \) and \( \nu \in L^X \), \( \mathcal{B}_x(\nu) = \bigvee \{ \Psi_e(\eta) | \eta \in L^X, \eta \odot x \leq \nu \} \)) is an \( L \)-neighborhood base for \( \Psi \).

Proof. (LBN1) \( \forall \nu \in L^X : \mathcal{B}_x(T_X) = \bigvee \{ \Psi_e(\eta) | \eta \in L^X, x \odot \eta \leq T_X \} = \Psi_x(T_X) = T. \)

(LBN2) \( \forall \nu \in L^X \) and \( \forall x \in X : \mathcal{B}_x(\nu) = \bigvee \{ \Psi_e(\eta) | \eta \in L^X, x \odot \eta \leq \nu \} \)
\[
\leq \bigvee \{ \Psi_e(\eta) | \eta \in L^X, \eta \leq x^{-1} \odot \nu \} = \Psi_e(x^{-1} \odot \nu) = \Psi_x(\nu).
\]

(NB3) Need to show that \( \forall \nu \in L^X, \forall x \in X : \Psi_x(\nu) = \bigvee_{\sigma \leq \nu} \mathcal{B}_x(\sigma) \). For this, let \( \nu \in L^X \) and \( x \in X \). Then we have
\[
\bigvee_{\sigma \leq \nu} \mathcal{B}_x(\sigma) = \bigvee_{\sigma \leq \nu} \bigvee \{ \Psi_e(\eta) | \eta \in L^X, x \odot \eta \leq \sigma \}. 
\]
≤ \bigvee_{\sigma \leq \nu} \left( V \{ \Psi_x(\eta) \mid \eta \in L^X, \eta \leq x^{-1} \circ \sigma \} \right)
= \bigvee_{\sigma \leq \nu} \Psi_x(x^{-1} \circ \sigma) = \bigvee_{\sigma \leq \nu} \Psi_x(\sigma) = \Psi_x(\nu).

Next, \Psi_x(\nu) \leq \bigvee_{\sigma \leq \nu} \Psi_x(\sigma) = \bigvee_{\sigma \leq \nu} \Psi_e(x^{-1} \circ \sigma)
\leq \bigvee_{\sigma \leq \nu} \left( V \{ \Psi_e(\nu_1) \mid \nu_1 \in L^X, \nu_1 \leq x^{-1} \circ \sigma \} \right)
\leq \bigvee_{\sigma \leq \nu} \left( V \{ \Psi_e(\nu_1) \mid \nu_1 \in L^X, x \circ \nu_1 \leq \sigma \} \right)
= \bigvee_{\sigma \leq \nu} B_x(\sigma). \quad \square

**Proposition 3.8.** Let \((X, \cdot, \Psi = (\Psi_x)_{x \in X})\) and \((Y, \cdot, \Psi' = (\Psi_y)_{y \in Y})\) be stratified \(L\)-semi-topological neighborhood groups with corresponding stratified \(L\)-topological neighborhood systems \(\Psi_e\) and \(\Psi_{e'}\) on \(X\) and \(Y\), respectively. If \(f : X \to Y\) is a group homomorphism, then the following are fulfilled.

1. \(f\) is continuous if and only if \(\forall \nu \in L^Y : \Psi_{e'}(\nu) \leq \Psi_e(f^{-} (\nu))\);
2. \(f\) is stratified \(L\)-topological neighborhood open if and only if for all \(\xi \in L^X, \Psi_e(\xi) \leq \Psi_{e'}(f^{-}(\xi))\).

**Proof.** (i) We only prove that \(f\) is continuous. Let \(x \in X\) and \(\nu \in L^Y\). Then
\[
\Psi_{f(x)}(\nu) = \bigvee_{\beta \in L^Y, \beta \leq \nu} B_{f(x)}(\beta)
\leq \bigvee_{\beta \leq \nu} V \{ \Psi_{e'}(\xi) : \xi \in L^Y, \xi \circ f(x) \leq \beta \}
\leq \bigvee \{ \Psi_e(f^{-}\xi) : \xi \in L^Y, f^{-}\xi \circ f(x) \leq f^{-}\nu \}
\leq \bigvee \{ \Psi_e(f^{-}\xi) : \xi \in L^Y, f^{-}\xi \circ x \leq f^{-}\nu \} \quad \text{(as} \ f\ \text{is a group homomorphism)}
= B_x(f^{-}\nu) \leq \bigvee \{ B_x(\eta) : \eta \in L^X, \eta \leq f^{-}\nu \}
= \Psi_x(f^{-}\nu).

(ii) Upon using Proposition 2.8 and following similar rout as in (i) above, we get that \(f\) is \(L\)-topological neighborhood open. \(\square\)

**Proposition 3.9.** Let \((X, \cdot)\) be an Abelian group with neutral element \(e\) and \(\Psi_e\) fulfills conditions (1)-(4) of the Proposition 3.5 such that \(\forall x \in X, \forall \nu \in L^X, \Psi_x(\nu) \leq V \{ \Psi_e(\xi) \mid \xi \in L^X, \xi \circ x \leq \nu \}\) and \(\forall \nu \in L^X, \Psi_e(\nu) \leq V \{ \Psi_e(\nu_1) \wedge \Psi_e(\nu_2) : \nu_1, \nu_2 \in L^X, \nu_1 \circ \nu_2 \leq \nu \}\) are fulfilled. Then the stratified \(L\)-topology generated by \(\Psi_e\) in the preceding construction gives rise to a stratified \(L\)-quasi-topological neighborhood group \((X, \cdot, \Psi = (\Psi_x)_{x \in X})\).

**Proof.** Let \(x, y \in X\) and \(\nu \in L^X\). Then
\[
\Psi_{xy}(\nu) \leq \bigvee_{\xi \leq \nu} \Psi_{xy}(\xi)
\]
\[
\begin{align*}
&\leq \bigvee_{\xi \leq \nu} \left[ \left\{ \Psi_e(\xi_1) \mid \xi_1 \in L^X, \xi_1 \odot xy \leq \xi \right\} \right] \\
&\leq \bigvee_{\xi_1 \odot xy \leq \nu} \left[ \left\{ \Psi_e(\nu_1) \land \Psi_e(\nu_2) \mid \nu_1, \nu_2 \in L^X, \nu_1 \odot \nu_2 \leq \xi_1 \right\} \right] \\
&\leq \bigvee \left[ \left\{ \Psi_e(\nu_1) \land \Psi_e(\nu_2) \mid \nu_1, \nu_2 \in L^X, (\nu_1 \odot \nu_2) \odot xy \leq \nu \right\} \right] \\
&= \bigvee \left[ \left\{ \Psi_x(\nu_1 \odot x) \land \Psi_y(\nu_2 \odot y) \mid \nu_1, \nu_2 \in L^X, (\nu_1 \odot x) \odot (\nu_2 \odot y) = (\nu_1 \odot \nu_2) \odot xy \leq \nu \right\} \right] \\
&\leq \bigvee \left[ \left\{ \Psi_x(\eta_1) \land \Psi_y(\eta_2) \mid \eta_1, \eta_2 \in L^X, \eta_1 \odot \eta_2 \leq \nu \right\} \right] \\
&= m^{-1}(\Psi_x \times \Psi_y)(\nu) = \Psi_x \odot \Psi_y(\nu), \text{ which proves that the group operation } \\
m : X \times X \to X, (x, y) \mapsto xy \text{ is jointly continuous.} \quad \square
\end{align*}
\]

**Proposition 3.10.** Every stratified $L$-left topological neighborhood group with continuous inverse is a stratified $L$-semi-topological neighborhood group.

**Proof.** Need show that $R_{x_0} : X \to X, x \mapsto xx_0$ is continuous. But it follows from the observation that $R_{x_0}(x) = j \circ L_{x_0}^{-1} \circ j(x) = xx_0$, where all the maps are continuous and hence their composition. \quad \square

**Proposition 3.11.** Every stratified $L$-topological neighborhood group is a stratified $L$-semi-topological neighborhood group.

**Proof.** Let $(X, \cdot, \Psi = (\Psi_x)_{x \in X})$ be a stratified $L$-topological neighborhood group. To prove it is a stratified $L$-semi-topological neighborhood group, we let $x_0 \in X$ and $\nu \in L^X$. Then for any $y \in X$, we have

\[
\Psi_{x_0y}(\nu) \leq m^{-1}(\Psi_{x_0} \times \Psi_y)(\nu) \\
= (\Psi_{x_0} \times \Psi_y)(m^{-1}(\nu)) \\
= \bigvee \left[ \left\{ \Psi_{x_0}(\nu_1) \land \Psi_{y}(\nu_2) \mid \nu_1, \nu_2 \in L^X, pr_1^+^{-1}(\nu_1) \times pr_2^+^{-1}(\nu_2) \leq m^{-1}(\nu) \right\} \right] \\
\leq \bigvee \left[ \left\{ \nu_1(x_0) \land \Psi_{y}(\nu_2) \mid \nu_1, \nu_2 \in L^X, \nu_1(x_0) \land \nu_2(x_0) \leq \nu(0) \right\} \right] \\
\leq \bigvee \left[ \left\{ \Psi_{x}(\nu_1(x_0) \land \nu_2) \mid \nu_1, \nu_2 \in L^X, \nu_1(x_0) \land \nu_2 \leq x_0^{-1} \odot \nu \right\} \right] \\
= \Psi_{y}(x_0^{-1} \odot \nu) = [x_0] \odot \Psi_y(\nu)
\]

Similarly, the right part can be proved. \quad \square

4. **Stratified $L$-quasi-bi-topological neighborhood groups and their $L$-quasi-bi-uniformizability**

**Definition 4.1.** A stratified $L$-quasi-bi-topological neighborhood group is a quadruple $(X, \cdot, \Psi = (\Psi_x)_{x \in X}, \Psi^{-1} = (\Psi^{-1}_x)_{x \in X})$, where $(X, \cdot, \Psi = (\Psi_x)_{x \in X})$ is a stratified $L$-quasi-topological neighborhood group, and $\Psi^{-1}$ is the conjugate
stratified $L$-neighborhood system of $\Psi$, where $\Psi_{x}^{-1}(\nu) = \psi_{x}(\nu^{-1})$, $\forall \nu \in L^{X}$, and $x \in X$.

Lemma 4.2. Let $(X, \cdot, \Psi, \Psi^{-1})$ be a stratified $L$-quasi-bi-topological neighborhood group, $x \in X$ and $\nu \in L^{X}$. Then the following assertions are satisfied:

(i) $\Psi_{x}(\nu) = \bigvee \{ \psi_{x}(\lambda) : \lambda \in L^{X}, x \odot \lambda \leq \nu \} = \bigvee \{ \psi_{x}(\zeta) : \zeta \in L^{X}, x^{-1} \odot \zeta \leq \nu \}$.

(ii) $\Psi_{x}^{-1}(\nu) = \bigvee \{ \psi_{x}(\lambda) : \lambda \in L^{X}, x \odot \lambda^{-1} \leq \nu \} = \bigvee \{ \psi_{x}(\zeta) : \zeta \in L^{X}, \zeta^{-1} \odot x \leq \nu \}$.

Proof. This is similar to the proof of Proposition 2.11 [4].

Definition 4.3. [8] Let $X$ be a nonempty set. Then a stratified $L$-filter $U$ on $X \times X$ is called a stratified $L$-quasi-uniformity on $X$ if and only if the following are fulfilled:

(LQU1) $U(d) \leq \bigwedge_{x \in X} d(x, x), \forall d \in L^{X \times X}$;

(LQU2) $U(d) \leq \bigvee \{ U(d_{1}) \wedge U(d_{2}) : d_{1}, d_{2} \in L^{X}, d_{2} \circ d_{1} \leq d \}$.

The pair $(X, U)$ is called a stratified $L$-quasi-uniform space.

Theorem 4.4. [8] Let $(X, U)$ be a stratified $L$-uniform space. Then $(X, (\Psi_{x}^{U})_{x \in X})$ is a stratified $L$-topological neighborhood space, where the associated stratified $L$-topological neighborhood system is given for any $\nu \in L^{X}$ and $x \in X$ by

$\Psi_{x}^{U}(\nu) = \bigvee \{ U(d) : d \in L^{X \times X}, d(x, -) \leq \nu \}$,

where $[d(x, -)](y) = d(x, y), \forall y \in X$.

Definition 4.5. A stratified $L$-quasi-uniform space $(X, U)$ is called weakly locally symmetric if and only if for all $x \in X$ and for all $d \in L^{X \times X}$: $U(d) \leq \bigvee \{ U(d_{1}) : d_{1} \in L^{X \times X}, d_{1} \text{ symmetric and } d_{1}(x, -) \leq d(x, -) \}$.

Definition 4.6. A stratified $L$-quasi-bi-topological neighborhood space is called $L$-quasi-bi-uniformizable if and only if there are $L$-quasi uniformities $U$ and $U^{-1}$ such that $\Psi^{U} = \Psi$ and $\Psi^{-1}U^{-1} = \Psi^{-1}$.

Let $(X, \psi, \psi^{-1})$ be a stratified $L$-quasi-bi-topological neighborhood group. Then it has always three $L$-quasi-uniformities: $U_{L}$ (the left $L$-quasi-uniformity), $U_{R}$ (the right $L$-quasi-uniformity) and $U_{B}$ (the two-sided $L$-quasi-uniformity, Proposition 3.5 [4]). Having the same notations as in Section 3 [4] (see also
Theorem 4.7. Each stratified $L$-quasi-bi-topological neighborhood group $(X, \Psi = (\Psi_x)_{x \in X}, \Psi^{-1} = (\Psi_x^{-1})_{x \in X})$ is $L$-quasi-uniformizable.

Proof. It follows from Theorem 4.2 [3] that $\Psi^{UL} = \Psi$. We only prove the other part. For any $x \in X$ and $\mu \in L^X$, we have
\[
\Psi^{-1}_x(\mu) = \bigvee \{ U^{-1}_L(d) : d \in L^{X \times X}, d(x, -) \leq \mu \}
\]
\[
= \bigvee \{ U_L(d^{-1}) : d \in L^{X \times X}, d(x, -) \leq \mu \}
\]
\[
= \bigvee_{d \in L^{X \times X}, d(x, -) \leq \mu} \bigvee \{ \Psi_e(\nu) : \nu \in L^X, \nu_L \leq d^{-1} \}
\]
\[
= \bigvee_{d \in L^{X \times X}, d(x, -) \leq \mu} \bigvee \{ \Psi_e(\nu) : \nu \in L^X, (\nu_L)^{-1} \leq d \}
\]
\[
= \bigvee \{ \Psi_e(\nu) : \mu \in L^X, x \odot \nu^{-1} \leq \mu \}
\]
\[
= \Psi^{-1}_x(\mu) \text{ (by Lemma 4.2(ii)).}
\]
Thus we have shown that $\Psi^{UL} = \Psi$ and $\Psi^{-1}U^{-1}_L = \Psi^{-1}$, proving that
\[
(X, \Psi = (\Psi_x)_{x \in X}, \Psi^{-1} = (\Psi_x^{-1})_{x \in X})
\]
is $L$-quasi-uniformizable. \hfill \Box

Theorem 4.8. Let $(X, \cdot, \Psi)$ be a stratified $L$-quasi-topological neighborhood group. Then $(X, \cdot, \Psi)$ is a stratified $L$-topological neighborhood group if and only if any one of $U_L$, $U_R$ or $U_B = U_L \land U_R$ is weakly locally symmetric.

Proof. If $(X, \cdot, \Psi)$ is a stratified $L$-topological neighborhood group, then it follows from Theorem 4.2 [3] that $U_L$, $U_R$ and $U_B$ always exist. To prove the converse, we consider the case for $U_L$, assuming that $(X, U_L)$ is weakly locally symmetric. We need to show that the inversion map $j : X \to X, x \mapsto x^{-1}$ is continuous at $e \in X$. Let $\nu \in L^X$. Then in view of Theorem 4.4, we deduce that
\[
\Psi^e_L(\nu) = \bigvee \{ U_L(d) : d \in L^{X \times X}, d(e, -) \leq \nu \}
\]
\[
= \bigvee_{d(e, -) \leq \nu} \bigvee \{ U_L(d_1) : d_1 \in L^{X \times X}, d_1 \text{ symmetric and } d_1(e, -) \leq d(e, -) \}
\]
\[
= \bigvee \{ U_L(d_1) : d_1 \in L^{X \times X}, d_1 \text{ symmetric } d_1(e, -) \leq \nu \}
\]
\[
= \bigvee_{d_1(e, -) \leq \nu} \bigvee \{ \Psi_e(\mu) : \mu \in L^X, \mu_L \leq d_1 \}.
\]
We have for any $y \in X$, $\mu(y) = \mu_L(y^{-1}, e) \leq d_1(e, y^{-1}) \leq \nu(y^{-1}) = j^{-1}(\nu)(y)$, i.e. $\mu \leq j^{-1}(\nu)$.
Thus we get $\Psi^e_L(\nu) \leq \bigvee \{ \Psi_e(\mu) : \mu \leq j^{-1}(\nu) \} = \Psi_e(j^{-1}(\nu))$, that is, $\Psi^e_L(\nu) \leq \Psi_e(j^{-1}(\nu))$, showing that $j : X \to X, x \mapsto x^{-1}$ is continuous at $e \in X$. Hence
the inversion is continuous. This ends the proof that the triple \((X, \cdot, \Psi)\) is a stratified \(L\)-topological neighborhood group. □

**References**


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