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Upper Geodetic Domination Number of a Graph

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Abstract

A subset S of vertices in a graph G is called a geodetic set if every vertex not in S lies on a shortest path between two vertices from S. A subset D of vertices in G is called a dominating set if every vertex not in D has at least one neighbor in D. A set $S \subseteq V(G)$ is called a geodetic dominating set of G if S is both a geodetic and a dominating set of G. The geodetic domination number $\gamma_g(G)$ of G is the minimum cardinality of a geodetic dominating set in G. A geodetic dominating set S in G is called a minimal geodetic dominating set if no proper subset of S is a geodetic dominating set of G. Upper geodetic domination number $\gamma_g^+(G)$ of G is the maximum cardinality of a minimal geodetic dominating set of G. The upper geodetic domination number of certain classes of graphs are determined and some of its general properties are studied. It is shown that for any three integers a, b and p, where $2 \leq a \leq b \leq p$, there exists a connected graph G of order p with $\gamma_g(G) = a$ and $\gamma_g^+(G) = b$.

Mathematics Subject Classification: 05C12, 05C69

Keywords: Geodesic, geodetic domination number, upper geodetic number, upper domination number, upper geodetic domination number

1 Introduction

We consider only finite simple connected graphs with at least two vertices. For any graph G, the vertex set is denoted by $V(G)$ and the edge set by $E(G)$. The *order* and *size* of G are denoted by p and q, respectively. For a vertex $v \in V(G)$, the *open neighborhood* $N(v)$ is the set of all vertices adjacent to v, and $N[v] = N(v) \cup \{v\}$ is the *closed neighborhood* of v. The *degree* $deg(v)$ of a vertex v is defined by $deg(v) = |N(v)|$. The *minimum* and *maximum degrees* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. For $X \in V(G)$, let $G[X]$ be the subgraph of G induced by X, $N(X) = \bigcup_{x \in X} N(x)$ and $N[X] = \bigcup_{x \in X} N[x]$. If G is a connected graph, then the *distance* $d(x, y)$ is the length of a shortest $x - y$ path in G. The *diameter* of a connected graph G is defined by $diam(G) = max_{x,y \in V(G)} d(x, y)$. If $e = \{u, v\}$ is an edge of a graph G with $d(u) = 1$ and $d(v) > 1$, then e is a *pendant edge*, u a *leaf* and v a *support vertex*. The *corona* of a graph G is the graph $Cor(G)$ constructed from a copy of G, where for each vertex $v \in V(G)$, a new vertex v' and a pendant edge vv' are added. A vertex of G is an extreme if the subgraph induced by its neighborhood is complete. An $x - y$ path of length $d(x, y)$ is called an *$x - y$ geodesic*. A vertex v is said to *lie* on an $x - y$ geodesic P if v is an internal vertex of P. The closed interval $I[x, y]$ consists of x, y and all vertices lying on some $x - y$ geodesic of G, and for a nonempty set $S \subseteq V(G)$, $I[S] = \bigcup_{x,y \in S} I[x, y]$.

The concept of geodetic number and upper geodetic number of a graph was introduced in [1, 3]. A set $S \subseteq V(G)$ is a *geodetic set* of G if $I[S] = V(G)$. The minimum cardinality of a geodetic set of G is the *geodetic number* $g(G)$ of G. A geodetic set S in G is called a minimal geodetic set if no proper subset of S is a geodetic set of G. The maximum cardinality of a minimal geodetic set of G is the *upper geodetic number* $g^+(G)$ of G.

The concept of domination number and upper domination number of a graph was introduced in [6]. A set of vertices S in a graph G is a *dominating set* if $N[S] = V(G)$. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. A dominating set S of G is a minimal dominating set if no proper subset of S is a dominating set of G. The maximum cardinality of a minimal dominating set of G is the *upper domination number* $\Gamma(G)$ of G. A minimal dominating set with cardinality $\Gamma(G)$ is called a Γ-set of G.

The concept of geodetic domination number of a graph was introduced in [4]. A set $S \subseteq V(G)$ is a geodetic dominating set of G if S is both a geodetic and a dominating set of G. The minimum cardinality of a geodetic dominating set of a graph G is its *geodetic domination number* $\gamma_g(G)$.

2 Preliminary Notes

In this section we cite some results to be used in the sequel.

Theorem 2.1. [2, 4] *Each extreme vertex of a connected graph G belongs to every geodetic dominating set of G.*

Theorem 2.2. [2, 4] *If G is a connected graph of order $p \geq 2$, then $2 \leq max\{\gamma(G), g(G)\} \leq \gamma_g(G) \leq p$.*

Theorem 2.3. [2] *Let G be a connected graph of order $p \geq 2$. Then $\gamma_g(G) = 2$ if and only if there exists a geodetic set $S = \{u, v\}$ of G such that $d(u, v) \leq 3$.*

3 Upper Geodetic Domination Number of a Graph

Definition 3.1. *A geodetic dominating set S in a connected graph G is called a minimal geodetic dominating set if no proper subset of S is a geodetic dominating set of G. The maximum cardinality of a minimal geodetic dominating set of G is the upper geodetic domination number of G and is denoted by $\gamma_g^+(G)$. A minimal geodetic dominating set with cardinality $\gamma_g^+(G)$ is called a γ_g^+-set of G.*

Example 3.2. *For the graph G given in Figure 3.1, $S_1 = \{v_1, v_3\}$ and $S_2 = \{v_2, v_4, v_5\}$ are geodetic dominating sets of G. It is clear that no proper subset of S_1 and S_2 are geodetic dominating sets of G and so S_1 and S_2 are minimal geodetic dominating sets of G. Therefore $\gamma_g(G) = 2$ and $\gamma_g^+(G) = 3$.*

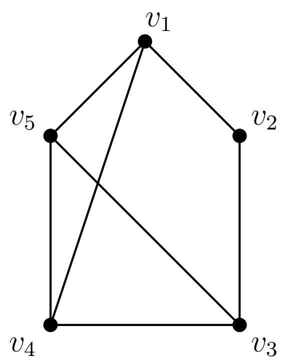

Figure 3.1: G

Observation 3.3. *Let G be a connected graph of order p. Then* (*i*). *every minimal geodetic dominating set of a graph G contains its extreme vertices.*

(*ii*). *every end vertex belongs to every minimal geodetic dominating set of G.*

(*iii*). *if G has the unique minimal geodetic dominating set, then* $\gamma_g(G) = \gamma_g^+(G)$.

Corollary 3.4. *For the complete graph* K_p, $\gamma_g^+(K_p) = \gamma_g(K_p) = p$.

Proof. This follows from Observation 3.3. □

Theorem 3.5. *Let G be a connected graph of order p. Then* $2 \leq \gamma_g(G) \leq \gamma_g^+(G) \leq p$.

Proof. As any geodetic dominating set needs at least two vertices, $\gamma_g(G) \geq 2$. Since every minimal geodetic dominating set of G is a geodetic dominating set of G, $\gamma_g(G) \leq \gamma_g^+(G)$. Since $V(G)$ is a geodetic dominating set of G, it is clear that $\gamma_g^+(G) \leq p$. Hence, $2 \leq \gamma_g(G) \leq \gamma_g^+(G) \leq p$. □

Remark 3.6. *The bounds in Theorem 3.5 are sharp. For the graph G given in Figure 3.1,* $\gamma_g(G) = 2$. *For the star* $K_{1,p-1}$, $\gamma_g(K_{1,p-1}) = \gamma_g^+(K_{1,p-1}) = p-1$. *For the complete graph* K_p, $\gamma_g(K_p) = \gamma_g^+(K_p) = p$. *Also, all the inequalities in Theorem 3.5 are strict. For the graph G given in Figure 3.2,* $\gamma_g(G) = 3$, $\gamma_g^+(G) = 4$ *and* $p = 7$. *Thus,* $2 < \gamma_g(G) < \gamma_g^+(G) < p$.

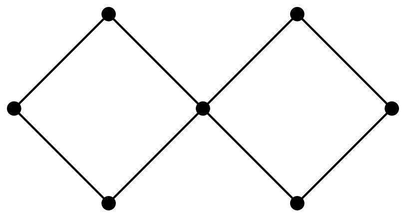

Figure 3.2: *G*

Corollary 3.7. *Let G be any connected graph of order p. If* $\gamma_g^+(G) = 2$, *then* $\gamma_g(G) = 2$.

Proof. This follows from Theorem 3.5. □

Remark 3.8. *The converse of the Corollary 3.7 is false. For the graph G given in Figure 3.1, it is clear that* $\gamma_g(G) = 2$ *and* $\gamma_g^+(G) = 3$. *Thus,* $\gamma_g^+(G) \neq 2$.

Corollary 3.9. *Let G be any connected graph of order p. If* $\gamma_g(G) = p$, *then* $\gamma_g^+(G) = p$.

Proof. This follows from Theorem 3.5. □

The following theorems characterize the graphs for which the upper geodetic domination number is p.

Theorem 3.10. *For a connected graph G of order p, $\gamma_g^+(G) = p$ if and only if $\gamma_g(G) = p$.*

Proof. Let G be a connected graph of order p and let $\gamma_g^+(G) = p$. Then $V(G)$ is a minimal geodetic dominating set of G. Since no proper subset of $V(G)$ is a geodetic dominating set of G, $V(G)$ is the minimum geodetic dominating set of G. Hence $\gamma_g(G) = |V(G)| = p$. Conversely, let $\gamma_g(G) = p$. Then by Corollary 3.9, $\gamma_g^+(G) = p$. □

Theorem 3.11. *Let G be any connected graph of order p. Then $\gamma_g(G) = p$ if and only if $G = K_p$.*

Proof. Let G be any connected graph of order p and let $\gamma_g(G) = p$. Suppose that $G \neq K_p$. Then there exist two vertices u and v such that $d(u,v) \geq 2$. Since G is connected, then there is a geodesic P from u to v. Let $x \in V(P)$ such that $x \neq u, v$ which is adjacent to either u or v. Without loss of generality, we assume that x is adjacent to u. Let $S = V(G) - \{x\}$. Then it is clear that $x \in I[u,v] \subseteq I[S]$ and $x \in N[u] \subseteq N[S]$. So that S is a geodetic dominating set of G and hence $\gamma_g(G) \leq |S| = p - 1$, which is a contradiction. Conversely, let $G = K_p$. Then it is clear that $\gamma_g(G) = p$. □

Corollary 3.12. *Let G be any connected graph of order p. Then $\gamma_g^+(G) = p$ if and only if $G = K_p$.*

Proof. This follows from Theorem 3.10 and Theorem 3.11. □

Observation 3.13. *For any connected graph G of order p, $2 \leq max\ \{\Gamma(G), g^+(G)\} \leq \gamma_g^+(G) \leq p$.*

The following theorems characterize the graphs for which the upper geodetic domination number is $p - 1$.

Theorem 3.14. *For a connected graph G of order $p \geq 3$, $\gamma_g(G) = p - 1$ if and only if $\gamma_g^+(G) = p - 1$.*

Proof. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_p\}$. Let $\gamma_g(G) = p - 1$. Then by Theorem 3.5, $\gamma_g^+(G) = p - 1$ or p. If $\gamma_g^+(G) = p$, then by Theorem 3.10 $\gamma_g(G) = p$, which is a contradiction. Hence, $\gamma_g^+(G) = p - 1$.

Conversely, let $\gamma_g^+(G) = p - 1$. Let $S = \{v_1, v_2, \ldots, v_{p-1}\}$ be a minimal geodetic dominating set of G with maximum cardinality, so that $|S| = p - 1$. We claim that every vertex in S is adjacent to v_p. Suppose, to the contrary, that some $v \in S$ is not adjacent to v_p. Since G is connected, v lies on a path joining a pair of distinct vertices $x, y \in S$ such that $d(x,y)$ is minimum. It is clear that $d(x, y) \leq 2$. Since some vertices of S are adjacent to v_p, there exists a pair of vertices $u, w \in S$ such that $u, w \neq v$ and $d(u, w) = 2$. It is clear that $v_p \in I[u, w] \subseteq I[S]$ and $v_p \in N[u] \subseteq N[S]$.

If $v \neq x, y$, then $d(x, y) = 2$, $v \in I[x, y] \subseteq I[S]$ and $v \in N[x] \subseteq N[S]$. Therefore, $S - \{v\}$ is a geodetic dominating set of G, which is a contradiction. Thus, $d(x, y) = 1$ and either $x = v$ or $y = v$. Without loss of generality, we assume that $x = v$. We consider two cases.

Case (1): Let $yv_p \in E(G)$. If every vertex $z \in S$ that is adjacent to v_p is also adjacent to y, then it is clear that $y, v_p \in I[u, w] \subseteq I[S]$ and $y, v_p \in N[u] \subseteq N[S]$. Therefore, $S - \{y\}$ is a geodetic dominating set of G, which is a contradiction. If there exists a vertex $z \in S$ such that z is adjacent to v_p but not adjacent to y, then we consider two cases.

Case (1a): Let $xz \in E(G)$. Then $x, v_p \in I[y, z] \subseteq I[S]$ and $x, v_p \in N[y] \subseteq N[S]$. Therefore, $S - \{x\}$ is a geodetic dominating set of G, which is a contradiction.

Case (1b): Let $xz \notin E(G)$. Then $y, v_p \in I[x, z] \subseteq I[S]$, $y \in N[x] \subseteq N[S]$ and $v_p \in N[z] \subseteq N[S]$. Therefore, $S - \{y\}$ is a geodetic dominating set of G, which is a contradiction.

Case (2): Let $yv_p \notin E(G)$. Since S is a minimal geodetic dominating set of G, y is not adjacent to every pair of non-adjacent vertices $u, w \in S$ such that $u, w \in N(v_p)$ and $d(u, w) = 2$. Since $yu \notin E(G)$, then let the $y - u$ geodesic P be $P : y = w_0, w_1, w_2, \ldots, w_k = u$. Since $yv_p \notin E(G)$, $w_1 \neq v_p$. Then it follows that $w_1 \in I[y, u] \subseteq I[S]$ and $w_1 \in N[y] \subseteq N[S]$. Therefore $S - \{w_1\}$ is a geodetic dominating set of G, which is a contradiction.

In both the cases, every vertex in S is adjacent to v_p. This completes the proof of the claim.

Therefore, for every pair of non-adjacent vertices u, v in S, the vertex v_p lies in the geodesic u, v_p, v. Clearly, $diam(G) = 2$. Next, we show that $G = K_1 + (K_{p_1} \cup K_{p_2} \cup \ldots K_{p_r})$ where $p_1, p_2, \ldots, p_r$ and r are positive integers with $p_1 + p_2 + \cdots + p_r = p - 1$ and $V(K_1) = \{v_p\}$. Suppose not, then there exist vertices $x, y, z \in S$ such that $d(x, y) = 2$ and $xz, zy \in E(G)$. It follows that $z, v_p \in I[x, y] \subseteq I[S]$ and $z, v_p \in N[x] \subseteq N[S]$. So $S - \{z\}$ is a geodetic dominating set of G, which is a contradiction. Hence $G = K_1 + (K_{p_1} \cup K_{p_2} \cup \ldots K_{p_r})$ and $\gamma_g(G) = p - 1$. □

Theorem 3.15. *For a connected graph G of order $p \geq 3$, $\gamma_g^+(G) = p - 1$ if and only if $G = K_1 + (K_{p_1} \cup K_{p_2} \cup \ldots K_{p_r})$ where $p_1, p_2, \ldots, p_r$ and r are positive integers with $p_1 + p_2 + \cdots + p_r = p - 1$.*

Proof. This follows from Theorem 3.14. □

Theorem 3.16. *For any two integers $p, q \geq 2$, the upper geodetic domination number of the complete bipartite graph $K_{p,q}$ is $\gamma_g^+(K_{p,q}) = max\ \{p, q\}$.*

Proof. Let $p, q \geq 2$. Let $X = \{x_1, x_2, \ldots, x_p\}$ and $Y = \{y_1,\ y_2,\ \ldots,\ y_q\}$ be the bipartite sets of $K_{p,q}$. Without loss of generality, we assume that $p \leq q$. First, we assume that $p < q$. Let $S = Y$, so that $|S| = |Y| = q$. We show that S is a minimal geodetic dominating set of $K_{p,q}$. Since $x_i \in I[y_j, y_k] \subseteq I[S]$ and $x_i \in N[y_j] \subseteq N[S]$ for every $x_i \in X$, S is a geodetic dominating set of $K_{p,q}$. Suppose there exists a geodetic dominating set S' of $K_{p,q}$ such that $S' \subset S$, then there exists a vertex $y_j \in S$ such that $y_j \notin S'$. Clearly the vertex y_j does not lie on any geodesic joining a pair of vertices in S' and $y_j \notin N[S']$. Thus S' is not a geodetic dominating set of $K_{p,q}$, which is a contradiction. Therefore S is a minimal geodetic dominating set of $K_{p,q}$. Hence $\gamma_g^+(K_{p,q}) \geq |S| = q$.

Suppose that $\gamma_g^+(K_{p,q}) > q$, then there exists a minimal geodetic dominating set S'' of $K_{p,q}$ such that $|S''| > q$. Since $y_j \in I[x_m, x_n] \subseteq I[X]$, $y_j \in N[x_m] \subseteq N[X]$ for every $y_j \in Y$ and $x_i \notin N[x_m] \subseteq N[X]$, $x_i \notin I[x_m, x_n] \subseteq I[X]$ for every $x_i \neq x_m, x_n \in X$, X is a minimal geodetic dominating set of $K_{p,q}$. Hence S'' does not contain both X and Y. Since $X \not\subseteq S''$ and $Y \not\subseteq S''$, $S'' \subset X \cup Y$. Therefore S'' contains at least two vertices of X and at least two vertices of Y. Let us assume that $x_1, x_2, y_1, y_2 \in S''$. Since $x_i \in I[y_1, y_2] \subseteq I[S'']$ and $x_i \in N[y_1] \subseteq N[S'']$ for every $x_i \in X$ and $y_j \in I[x_1, x_2] \subseteq I[S'']$ and $y_j \in N[x_1] \subseteq N[S'']$ for every $y_j \in Y$, S'' is not a minimal geodetic dominating set of $K_{p,q}$, which is a contradiction. Therefore, $\gamma_g^+(K_{p,q}) = q$. Now, if $p = q$, then we can prove by similar argument that $\gamma_g^+(K_{p,q}) = p = q$. Hence, $\gamma_g^+(K_{p,q}) = \max\ \{p, q\}$. □

Theorem 3.17. *For the cycle C_p with $p \geq 6$, $\gamma_g^+(C_p) = \left\lfloor \frac{p}{2} \right\rfloor$.*

Proof. Let C_p be the cycle of order $p \geq 6$. We consider two cases.
Case (1): Let p be even. Then $p = 2n$. Let $C_{2n} : v_1, v_2, \ldots, v_{2n}, v_1$ be the cycle of order $2n$. It is clear that the set $S = \{v_1,\ v_3,\ \ldots, v_{i-2}, v_i,\ v_{i+2}, \ldots, v_{2n-3}, v_{2n-1}\}$ is a geodetic dominating set of C_p and $|S| = n = \frac{p}{2} = \left\lfloor \frac{p}{2} \right\rfloor$. Let $S' = S - \{v_i\}$. Since $v_i \in I[v_{i-2}, v_{i+2}] \subseteq I[S']$ and $v_i \notin N[S']$ for every $v_i \in S'$, S' is not a geodetic dominating set of C_p. Therefore, S is a minimal geodetic dominating set of C_p and $\gamma_g^+(C_p) \geq |S|$. We show that $\gamma_g^+(C_p) \leq |S|$. Suppose that $\gamma_g^+(C_p) > |S|$, then there exists a minimal geodetic dominating set S'' of C_p such that $|S''| > |S| = \frac{p}{2}$. Since S'' is a minimal geodetic dominating set, $I_1 = \{v_1, v_3, \ldots, v_{2n-1}\} \not\subseteq S''$ and $I_2 = \{v_2, v_4, \ldots, v_{2n}\} \not\subseteq S''$. Hence S'' consists of vertices of both I_1 and I_2. Let $v_i, v_j \in S''$ where $v_i \in I_1$ and $v_j \in I_2$.
Case (1a): Let v_i and v_j be non-adjacent vertices. If $v_{i-1}, v_{i+1} \in S''$, then $v_i \in I[v_{i-1}, v_{i+1}] \subseteq I[S'']$ and $v_i \in N[v_{i-1}] \subseteq N[S'']$. Therefore, $S'' - \{v_i\}$ is a geodetic dominating set of C_p, which is a contradiction. If either $v_{i-1} \in S''$ and $v_{i+1} \notin S''$, or $v_{i-1} \notin S''$ and $v_{i+1} \in S''$, or $v_{i-1}, v_{i+1} \notin S''$, then it is clear that $|S''| \leq \frac{p}{2}$, which is a contradiction.
Case (1b): Let v_i and v_j be adjacent. Then let $v_j = v_{i+1}$. If $v_{i-1}, v_{i+2} \in S''$, then $v_i, v_{i+1} \in I[v_{i-1}, v_{i+2}] \subseteq I[S'']$, $v_i \in N[v_{i-1}] \subseteq N[S'']$ and $v_{i+1} \in N[v_{i+2}] \subseteq$

$N[S'']$. Therefore, $S'' - \{v_i, v_{i+1}\}$ is a geodetic dominating set of C_p, which is a contradiction. If $v_{i-1} \in S''$ and $v_{i+2} \notin S''$, then $S'' - \{v_i\}$ is a geodetic dominating set of C_p, which is a contradiction. If $v_{i-1} \notin S''$ and $v_{i+2} \in S''$, then $S'' - \{v_{i+1}\}$ is a geodetic dominating set of C_p, which is a contradiction. If $v_{i-1}, v_{i+2} \notin S''$, then it is clear that $|S''| \leq \frac{p}{2}$, which is a contradiction.
Hence, in both the cases, $\gamma_g^+(C_p) = |S| = \left\lfloor \frac{p}{2} \right\rfloor$.
Case (2): Let p be odd. Then let $p = 2n+1$. Let $C_{2n+1} : v_1, v_2, \ldots, v_{2n}, v_{2n+1}, v_1$ be the cycle of order $2n+1$. It is straightforward to verify that the set $S = \{v_1, v_3, \ldots, v_{2n-3}, v_{2n-1}\}$ is a geodetic dominating set of C_p and $|S| = n = \frac{p-1}{2} = \left\lfloor \frac{p}{2} \right\rfloor$. By the similar argument as in Case(1), we get S is a minimal geodetic dominating set of C_p with maximum cardinality. Hence, $\gamma_g^+(C_p) = |S| = \left\lfloor \frac{p}{2} \right\rfloor$. □

Observation 3.18. *The upper geodetic domination number of some standard graphs can be easily found and are given as follows:*

(*i*). *For the path P_p with $p \geq 3$, $\gamma_g^+(P_p) = \left\lceil \frac{p}{2} \right\rceil$.*

(*ii*). *For the wheel W_p with $p \geq 5$, $\gamma_g^+(W_p) = \left\lfloor \frac{2(p-1)}{3} \right\rfloor$.*

(*iii*). *For the star $K_{1,p-1}$, $\gamma_g^+(K_{1,p-1}) = p-1$.*

Theorem 3.19. *For any connected non-complete graph G of order p, $\gamma_g^+(G) \leq p - \delta(G)$.*

Proof. Let S be a γ_g^+-set of a non-complete connected graph G of order p. Then $\gamma_g^+(G) = |S|$. We show that $|S| \leq p - \delta(G)$. Let $v \in S$. Assume that v is adjacent to k distinct vertices in S. Since $deg(v) \geq \delta(G)$, v must be adjacent to at least $\delta(G) - k$ vertices in $V(G) - S$ and so $|V(G) - S| \geq \delta(G) - k$. If $k = 0$, then $|V(G) - S| \geq \delta(G)$, that is, $|S| \leq |V(G)| - \delta(G) = p - \delta(G)$. If $k > 0$, then the k distinct vertices belong to $N[S]$ and does not lie on any geodesic joining any pair of vertices of S, since S is a minimal geodetic dominating set of G. Hence $|V(G) - S| \geq (\delta(G) - k) + k = \delta(G)$, that is, $|S| \leq p - \delta(G)$. To show the sharpness of the bound, we take a graph $G = K_{1,p-1}$ of order p. It is clear that $\delta(G) = 1$, $p - \delta(G) = p - 1$ and $\gamma_g^+(G) = p - 1$. Thus, $\gamma_g^+(G) = p - \delta(G)$. □

Definition 3.20. [2] *Let G be a graph and $X \subseteq V(G)$. A set $D \subseteq V(G)$ is an X-dominating set of G if $X \subseteq N[D]$. The X-domination number $\gamma_X(G)$ is the cardinality of a minimum X-dominating set of G. An X-dominating set S of G is a minimal X-dominating set if no proper subset of S is the X-dominating set of G. The upper X-domination number $\gamma_X^+(G)$ is the maximum cardinality of a minimal X-dominating set of G.*

Theorem 3.21. *Let $L(T)$ be the set of leaves of a tree T and $X = V(T) - N[L(T)]$. Then $\gamma_g^+(T) = |L(T)| + \gamma_X^+(T)$.*

Proof. Let $L(T)$ be the set of leaves of a tree T and $X = V(T) - N[L(T)]$. Let S be a γ_g^+-set of T, so that $\gamma_g^+(T) = |S|$. Since every geodetic dominating set of T contains its extreme vertices, $L(T) \subseteq S$. Since S is a dominating set of T and $L(T)$ is the unique minimal geodetic set of T, $L(T)$ only dominates the vertices of $N[L(T)]$ and the set $S - L(T)$ is a minimal $(V(T) - N[L(T)])$-dominating set of T with maximum cardinality. Therefore, $\gamma_X^+(T) = |S - L(T)| = |S| - |L(T)|$. Hence, $\gamma_g^+(T) = |S| = \gamma_X^+(T) + |L(T)|$. □

Theorem 3.22. *If T is a tree of order $p \geq 3$, then the following conditions are equivalent:*

(i) $\gamma_g^+(T) = g^+(T) = \Gamma(T)$.

(ii) The set of leaves $L(T)$ is a Γ-set of T.

(iii) Every non-leaf of T is adjacent to at least one leaf of T.

Proof. Let T be a tree of order $p \geq 3$ and let $L(T)$ be the set of leaves of T.

First, we prove $(i) \Leftrightarrow (ii)$. Let $\gamma_g^+(T) = g^+(T) = \Gamma(T)$. Then it is clear that $L(T)$ is a minimal dominating set of T with maximum cardinality, since $L(T)$ is the unique minimal geodetic set of a tree T. Conversely, let $L(T)$ be a Γ-set of T. Then $\Gamma(T) = |L(T)|$. Since $L(T)$ is the unique minimal geodetic set of a tree T, $L(T)$ is the unique minimal geodetic dominating set of a tree T. Hence, $\gamma_g^+(T) = g^+(T) = \Gamma(T)$.

Next, we prove $(ii) \Leftrightarrow (iii)$. Assume that $L(T)$ is a Γ-set of T, so that $\Gamma(T) = |L(T)|$. Suppose there exists a non-leaf u of T such that u is not adjacent to every leaf of T. Then it follows that $u \notin N[L(T)]$. Therefore $L(T)$ is not a dominating set of T, which is a contradiction. Hence every non-leaf of T is adjacent to at least one leaf of T. Conversely, assume that every non-leaf of T is adjacent to at least one leaf of T. Then it follows that $L(T)$ is a Γ-set of T. □

Theorem 3.23. *If $T = Cor(T')$ is a tree of order $p \geq 3$, where T' is an arbitrary tree of order at least 2, then $L(T)$ is the unique minimal geodetic dominating set of T.*

Proof. Let $T = Cor(T')$ be a tree of order $p \geq 3$, where T' is an arbitrary tree of order at least 2. Then it is clear that $L(T)$ dominates $N[L(T)]$ and $\Gamma(T) = \frac{p}{2} = |L(T)|$. Since $L(T)$ is the unique minimal geodetic set of T, $L(T)$ is the unique minimal geodetic dominating set of T. □

Remark 3.24. *The converse of the Theorem 3.23 is false. For the tree T given in Figure 3.3, $L(T)$ is the unique minimal geodetic dominating set of T but $T \neq Cor(T')$, where T' is an arbitrary tree of order at least 2.*

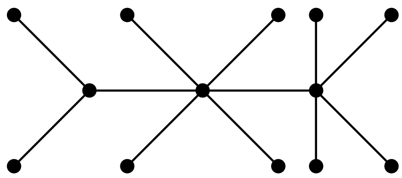

Figure 3.3: T

Corollary 3.25. *If $T = Cor(T')$ is a tree of order $p \geq 3$, where T' is an arbitrary tree of order at least 2, then $\gamma_g^+(T) = \frac{p}{2}$.*

Proof. This follows from Theorem 3.23. □

The following theorems show the effect on the upper geodetic domination number of a given graph by the removal of a vertex or adding some pendant edges.

Theorem 3.26. *Let G be a connected graph of order p and $u \in V(G)$. If $deg(u) = 1$, then $\gamma_g^+(G-u) \leq \gamma_g^+(G)$.*

Proof. Let $u \in V(G)$ and $deg(u) = 1$. Let S be a minimal geodetic dominating set of $G-u$ with maximum cardinality, so $\gamma_g^+(G-u) = |S|$. Since $deg(u) = 1$, u is an end vertex and u is adjacent to exactly one vertex, say v. By Observation 3.3, every minimal geodetic dominating set of G contains u. We consider two cases.

Case (*i*): Let $v \in S$. Since S is a geodetic dominating set of $G-u$, there exists a vertex $w \in V(G-u)$ such that $w \in I[v,x] \subseteq I[S]$, $w \in N[S]$ and $d(v,x) \leq 3$. If $d(v,x) = 3$, then consider the set $S' = (S - \{v\}) \cup \{u,w\}$. If $d(v,x) \leq 2$, then consider the set $S' = (S - \{v\}) \cup \{u\}$. It is straightforward to verify that S' is a minimal geodetic dominating set of G. So that $\gamma_g^+(G-u) = |S| \leq |S'| \leq \gamma_g^+(G)$.

Case (*ii*): Let $v \notin S$. Then consider the set $S' = S \cup \{u\}$. It is straightforward to verify that S' is a minimal geodetic dominating set of G. So that $\gamma_g^+(G-u) = |S| < |S'| \leq \gamma_g^+(G)$.

Hence, in both the cases, $\gamma_g^+(G-u) \leq \gamma_g^+(G)$. To show the sharpness of the bound, we take $G = P_4$. Let u be an end vertex of G. Then it is clear that $\gamma_g^+(G-u) = 2$ and $\gamma_g^+(G) = 2$. Hence, $\gamma_g^+(G-u) = \gamma_g^+(G)$. □

Remark 3.27. *The converse of the Theorem 3.26 is false. For the complete graph K_p, it is clear that $\gamma_g^+(K_p) = p$, $\gamma_g^+(K_p - u) = p-1$ and $deg(u) = p-1$ for every $u \in V(K_p)$. Hence, $\gamma_g^+(K_p - u) \leq \gamma_g^+(K_p)$ but $deg(u) \neq 1$.*

Remark 3.28. *Theorem 3.26 is not true if $deg(u) \neq 1$. For the graph $G = P_5$ given in the Figure 3.4, $\gamma_g^+(G) = 3$, $\gamma_g^+(G-u) = 4$ and $deg(u) = 2 \neq 1$. Thus, $\gamma_g^+(G-u) \not\leq \gamma_g^+(G)$.*

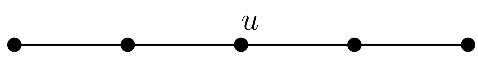

Figure 3.4: G

Remark 3.29. *Note that removing a vertex can increase the upper geodetic domination number by at least one, but can decrease it by at most one. For example, removing any vertex v from the complete graph K_p decreases the upper geodetic domination number by one. Similarly, removing a support vertex of a non-trivial connected tree T from $Cor(T)$ increases the upper geodetic domination number by at least one, but removing an end vertex a non-trivial connected tree T from $Cor(T)$ does not change the value of an upper geodetic domination number.*

Theorem 3.30. *For any connected tree T with $p \geq 3$, there exists a vertex $v \in V(T)$ such that $\gamma_g^+(T - v) = \gamma_g^+(T)$.*

Proof. Let T be any connected tree with $p \geq 3$. It can be verified that the result is true for $p = 3$ since if $p = 3$, then $T = P_3$. Now consider the case that $p > 3$. Since T has at least one vertex with degree greater than or equal to 2, there exists a vertex $v \in V(T)$ with $deg(v) \geq 2$ such that v is adjacent to at least one leaf and at most one non-leaf. If there exists a vertex v such that v is adjacent to at least one leaf and no non-leaf, then it is clear that $T = K_{1,p-1}$ and v is the support vertex. So that $\gamma_g^+(T - v) = p - 1 = \gamma_g^+(T)$. If there does not exist a vertex v such that v is adjacent to exactly one leaf, then it is clear that v is adjacent to two or more leaves. Assume that v is adjacent to exactly one non-leaf. By Theorem 3.3 every minimal geodetic dominating set of T contains its leaves. So it is clear that $\gamma_g^+(T - v) = \gamma_g^+(T)$. If there exists a vertex v such that v is adjacent to exactly one leaf u and one non-leaf, then $deg(u) = 1$ and $deg(v) = 2$. Let $T' = T - v - u$. Since $deg(u) = 1$, by Theorem 3.26 $\gamma_g^+(T - u) \leq \gamma_g^+(T)$. Hence, $\gamma_g^+(T') \leq \gamma_g^+(T - u) \leq \gamma_g^+(T)$. However, we have $\gamma_g^+(T') \geq \gamma_g^+(T) - 1$. If $\gamma_g^+(T') = \gamma_g^+(T) - 1$, then $\gamma_g^+(T) = \gamma_g^+(T - v)$. If $\gamma_g^+(T') > \gamma_g^+(T) - 1$, then $\gamma_g^+(T') = \gamma_g^+(T) = \gamma_g^+(T - u)$. Hence there exists a vertex $v \in V(T)$ such that $\gamma_g^+(T - v) = \gamma_g^+(T)$. □

Remark 3.31. *Theorem 3.30 is not true for any graph G. For the complete graph K_p, $\gamma_g^+(K_p - v) \neq \gamma_g^+(K_p)$ for every $v \in V(K_p)$.*

Theorem 3.32. *Let G be a connected graph of order p. If G' is a graph obtained by adding k, where $1 \leq k \leq p$, pendant edges to a graph G, then $\gamma_g^+(G) \leq \gamma_g^+(G') \leq \gamma_g^+(G) + k$.*

Proof. Let G be a connected graph of order p and let G' be the connected graph obtained from G by adding k pendant edges u_iv_i $(1 \leq i \leq k)$, where each $u_i \in V(G)$ and $v_i \notin V(G)$.

First, we show that $\gamma_g^+(G) \leq \gamma_g^+(G')$. Let S be a γ_g^+-set of G, so $\gamma_g^+(G) = |S|$. We now consider three cases.

Case (*i*): Let $u_i \notin S$ for all i, $1 \leq i \leq k$. Then let $S' = S \cup \{v_1, v_2, \ldots, v_k\}$. Since each $v_i \notin V(G)$ is an end vertex of G' and $u_i \notin S$, $v_i \notin I[S]$ and

$v_i \notin N[S]$. Hence S' is a minimal geodetic dominating set of G'. Therefore $\gamma_g^+(G) = |S| < |S'| \leq \gamma_g^+(G')$.
Case *(ii)*: Let $u_i \in S$ for some i, $1 \leq i \leq k$. Since S is a geodetic dominating set of G, there exists a vertex $x \notin S$ such that $x \in I[u_i, v] \subseteq I[S]$, $x \in N[S]$ and $d(u_i, v) \leq 3$ for some $v \in S$. If $d(u_i, v) = 3$, then consider the set $S' = (S - \{u_i\}) \cup \{v_i, x\}$. If $d(u_i, v) \leq 2$, then consider the set $S' = (S - \{u_i\}) \cup \{v_i\}$. It is striaghtforward to verify that S' is a minimal geodetic dominating set of G'. Therefore $\gamma_g^+(G) = |S| \leq |S'| \leq \gamma_g^+(G')$.
Case *(iii)* Let $u_i \in S$ for all i, $1 \leq i \leq k$. Then by the similar argument as in Case(ii), we can prove that $\gamma_g^+(G) \leq \gamma_g^+(G')$.

Hence, in all the three cases, $\gamma_g^+(G) \leq \gamma_g^+(G')$. To show the sharpness of the bound, consider the graph $G = K_p$. If we add $k \geq 1$ pendant edges u_iv_i, $1 \leq i \leq k$, to a complete graph G, we obtain a graph G'. It is clear that $\gamma_g^+(G) = p$ and $\gamma_g^+(G') = p$. Hence, $\gamma_g^+(G) = \gamma_g^+(G')$.

Next, we show that $\gamma_g^+(G') \leq \gamma_g^+(G) + k$. Let $S \subseteq V(G)$ and let $S' = S \cup \{v_1, v_2, \ldots, v_k\}$ be a minimal geodetic dominating set of G' with maximum cardinality. So that $\gamma_g^+(G') = |S'| = |S| + k$. Since S' is a minimal geodetic dominating set of G', $u_i \notin S$ for all i, where $1 \leq i \leq k$. We show that S is a minimal geodetic dominating set of G. If $u_i \in I[S]$ and $u_i \in N[S]$ for all $u_i \in V(G) - S$, then S is a minimal geodetic dominating set of G. If not, then there exists a vertex $u_i \in V(G)$ such that $u_i \notin I[S]$ or $u_i \notin N[S]$. Then the set $S \cup \{u_i\}$ is a minimal geodetic dominating set of G. Hence, $\gamma_g^+(G') = |S| + k \leq \gamma_g^+(G) + k$. □

4 Realization Results

In this section we give realization results concerning the upper geodetic domination number. We first establish the existence of a connected graph G with $\gamma_g^+(G) = a$ and $|V(G)| = p$ for any two integers a, p with $2 \leq a \leq p$.

Theorem 4.1. *For any two integers a and p with $2 \leq a \leq p$, there exists a connected graph G with $\gamma_g^+(G) = a$ and $|V(G)| = p$.*

Proof. It can be verified that the result is true for $2 \leq p \leq 3$ since if $p = 2$, then $G = K_2$ while if $p = 3$, then $G \in \{P_3, K_3\}$. Let us now consider the case that $p \geq 4$. If $a = p$, then $G = K_p$ and if $a = p - 1$, then $G = K_{1,p-1}$. For $a \leq p-2$, let K_2, K_{p-a} and K_{a-2} be complete graphs with vertex sets $V(K_2) = \{x, y\}$, $V(K_{p-a}) = \{u_1, u_2, \ldots, u_{p-a}\}$ and $V(K_{a-2}) = \{v_1, v_2, \ldots, v_{a-2}\}$. Let $G_1 = K_2 + K_{p-a}$ and $G_2 = K_{p-a} + \overline{K}_{a-2}$. Form a connected graph G, as shown in the Figure 4.1, consisting of G_1 and G_2. Then the vertex set of G is $V(G) = \{x,\ y,\ u_1,\ u_2,\ \ldots,\ u_{p-a},\ v_1,\ v_2,\ \ldots,\ v_{a-2}\}$ and the set of extreme vertices of G is $S = \{x, y,\ v_1, v_2,\ \ldots, v_{a-2}\}$.

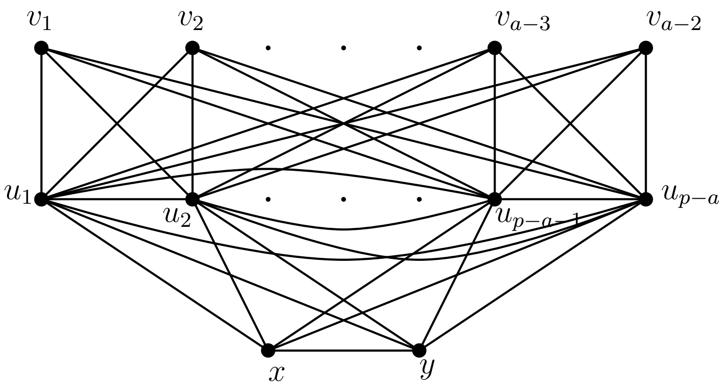

Figure 4.1: G

Therefore, $|V(G)| = p$ and $|S| = a$. By Observation 3.3, every minimal geodetic dominating set of G contains its extreme vertices. Since $u_i \in I[v_1, x] \subseteq I[S]$ and $u_i \in N[x] \subseteq N[S]$ for all $u_i \in V(G)$, S is the unique minimal geodetic dominating set of G. Thus, $\gamma_g^+(G) = |S| = a$. Hence, $\gamma_g^+(G) = a$ and $|V(G)| = p$. □

Theorem 4.2. *For any three integers a, b and p with $2 \leq a \leq b \leq p$, there exists a connected graph G with $\gamma_g(G) = a$, $\gamma_g^+(G) = b$ and $|V(G)| = p$.*

Proof. If $2 = a = b = p$, then consider the graph $G = K_2$. It is clear that $\gamma_g(K_2) = \gamma_g^+(K_2) = 2$. If $2 < a = b = p$, then consider the graph $G = K_p$ ($p > 2$). It is clear that $\gamma_g(K_p) = \gamma_g^+(K_p) = p$. If $2 < a = b < p$, then consider the graph $G = K_{1,a}$. Then by Observation 3.18, $\gamma_g(K_{1,a}) = \gamma_g^+(K_{1,a}) = a$. Now we consider $2 < a < b < p$. Take a copy of star $K_{1,a}$ with leaves $x_1, x_2, \ldots, x_a$ and the support vertex x. Subdivide the edges xx_i, where $1 \leq i \leq p - b - 1$, calling the new vertices $y_1, y_2, \ldots, y_{p-b-1}$ where x_i is adjacent to y_i and y_i is adjacent to x for all $i \in \{1, 2, \ldots, p - b - 1\}$. Let G be the graph obtained by adding $b - a$ new vertices $w_1, w_2, \ldots, w_{b-a}$ and joining each w_i ($1 \leq i \leq b - a$) with x and x_1. The graph G is shown in Figure 4.2. It is clear that $diam(G) = 4$, $|V(G)| = p$ and set of extreme vertices $S = \{x_2, x_3, \ldots, x_a\}$ is not a geodetic dominating set of G. Clearly $S \cup \{x_1\}$ is a minimum geodetic dominating set of G, so that $\gamma_g(G) = |S \cup \{x_1\}| = a$. Since every minimal geodetic dominating set contains S, let $T = \{y_1, w_1, w_2, \ldots, w_{b-a}\} \cup S$. Clearly T is a geodetic dominating set of G and $|T| = b$. We show that T is a minimal geodetic dominating set of G. Suppose there exists a proper subset U of T such that U is a geodetic dominating set of G, then there exists a vertex $v \in T$ such that $v \notin U$. Clearly $v \neq x_i$ for every $x_i \in S$. Suppose $v = w_j$, where $1 \leq j \leq b - a$, or $v = y_1$, then the vertex v does not lie on any geodesic joining any pair of vertices of U and $v \notin N[U]$, which is a contradiction. Therefore T is a minimal geodetic dominating set of G. Now, we show that there is no minimal geodetic dominating set of G with cardinality greater than b. Suppose there exists a minimal geodetic dominating set W of G with $|W| > b$.

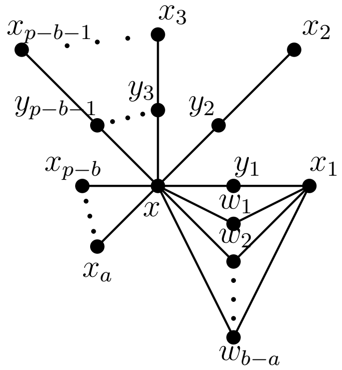

Figure 4.2: G

Since W contains S, the vertex u lies on some geodesic joining a pair of vertices of S and $u \in N[S]$ where $u \in \{x, y_2, y_3, \ldots, y_{p-b-1}\}$. Since W is a minimal geodetic dominating set of G and $|W| > b$, $x, y_2, y_3, \ldots, y_{p-b-1} \notin W$ and $W = \{y_1, w_1, w_2, \ldots, w_{b-a}, x_1\} \cup S$. Clearly $T \subset W$ is a minimal geodetic dominating set of G, which is a contradiction. Thus T is a γ_g^+-set of G. Hence, $\gamma_g^+(G) = |T| = b$. □

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