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An Explication of the Category $\mathbf{C}hu(\mathbf{S}et, 2)$

Dasharath Singh[1]

Department of Mathematics
Ahmadu Bello University, Zaria, Nigeria

Muideen Aliyu

Department of Mathematics
Ahmadu Bello University, Zaria, Nigeria

Abstract

In this paper, Chu spaces and Chu categories, in particular, $\mathbf{C}hu(\mathbf{S}et, 2)$, are briefly described and a couple of examples are constructed.

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1 Introduction

The idea that mathematics involves different categories and their relationships has been implicit since long. However, it explicitly came into existence with 1945 publication of the theory of *categories, functors, and natural transformations* by Samuel Eilenberg and Saunders Mac Lane while they were specifically engaged in formalizing algebraic topology. In fact, a broader objective underlying their work was to develop an axiomatic foundation which could provide an adequate characterization of the relations between a large number of mathematical structures developed in 1930's and the processes preserving them (See

[1]Corresponding author

[8, 9] for details).

Informally, a category $\mathfrak{C}$ is an algebraic structure consisting of a class of *objects*, denoted $Ob(\mathfrak{C})$, linked together by a set of *arrows*(*morphisms* or $\mathfrak{C}$-*morphisms*), denoted $\mathcal{M}(X, Y)$ or $Mor_{\mathfrak{C}}(X, Y)$, for every pair of objects X, Y of $\mathfrak{C}$, satisfying the fundamental properties: Arrows compose *associatively* and there exists an *identity* arrow for each object. For example, *Set* is the category in which sets are objects and functions are morphisms. The fundamental idea of *mutability of precise mathematical structures by morphisms* [8] as well applied to categories themselves taken as structures and functors as mappings between them and further, functors taken as structures and natural transformations as mappings.

In view of the fact that *objects* and *arrows*, the two basic ingredients for defining a category, may be abstract entities of any kind, a large number of familiar mathematical structures can be viewed as categories. Category theory has come to occupy a central position in generalizing most of the branches of mathematics, some areas of theoretical computer science, and mathematical physics (Continuum physics, essentially).

In the sequel, the notion of *Chu spaces* emerged in 1979. Historically, it was Mackey [10] who formalized the idea of representing *duality* as a contravariant pair of morphisms. Micheal Barr abstracted this idea to introduce the notion of *Chu spaces enriched* (in the sense of *enriched category* [6] in a *symmetric monoidal closed category* V, which was elaborately studied by Po-Hsiang Chu, a student of Barr, in his master's thesis. A major contribution of Chu's master's thesis has come to be known as *Chu constructions* which first appeared in print as an appendix in Barr's book titled **- Autonomous Categories* [1]. In fact, Chu constructions formed the basis for constructing *- autonomous categories from autonomous categories. Note that an autonomous category is a *clossed symmetric monoidal category*, while a *-autonomous category is an autonomous category with a *dualizing object*.

Informally, a Chu space over an arbitrary set K, abbreviated as K-*Chu space* or $\mathbf{C}hu_{\mathbf{K}}$, is a structured mathematical object consisting of a transformable matrix whose entries are drawn from K and, rows transform forwards and columns transform backwards. Essentially, there underly a simple form of structure and suitable morphisms, called *Chu transforms*. The discovery of Chu spaces, owing to their matrix orientation and hence being amenable to automation, can be considered as an outstanding development in the field of category theory.

The application of Chu spaces abound [1, 2, 3, 7, 11, 12]. Chu spaces are recognized as universal mathematical objects and turn out to be a unifying thread for a wide range of mathematical structures: algebraic, topological, and ordered. For example, relational structures such as sets, directed graphs (networks), posets, etc., algebraic structures such as groups, rings, fields, vector spaces, lattices and Boolean algebras, etc., and topological structures such as topological spaces, compact Hausdorff spaces, topological Boolean algebras, etc., are all represented as Chu constructs. Moreover, applying powerful tools provided by category theory, a number of categorical models of Chu spaces have been developed (see [1, 11] for details).

At this end, it may be emphasized that, in all these developments, the elementary theory of category of Cantor's structureless sets, developed by Lawvere [8], and, by extension, $\mathbf{C}hu(\mathbf{S}et, K)$ (in particular, $\mathbf{C}hu(\mathbf{S}et, 2)$)[2, 11], can be considered as the core of the *paradise* of Categories.

MacLane observed, which appeared in his Autobiography, as follows:

The most radical aspect is Lawvere's notion of using axioms for the Category of Sets as a foundation of mathematics ...

The reason being that it is *simple* and *close* since the objects are sets and morphisms are functions. Note that certain morphisms may not even be functions and hence the notions like *injectivity* or *surjectivity* would not apply. Moreover, it has the same proof-theoretic strength as that of the theory of category of categories, an appropriate framework for mathematics. The difference is that the former is defined by *discrete functor* whereas the latter by a *structure functor*, called *adjoint* which was developed by Kan in the mid 1950's (see [8] for details).

It is remarkable of $\mathbf{C}hu(\mathbf{S}et, K)$, in particular, $\mathbf{C}hu(\mathbf{S}et, 2)$, that it realizes a rich variety of mathematical structures (see [11] for details). The case $V = \mathbf{S}et$ was first explicitly treated by Lafont and Streicher[7]. It was pointed out that Chu spaces over $K = \{0, 1\} = 2$ realizes both *toplogical* spaces and *coherent* spaces (introduced by J-Y Girard to model linear logic [4]). Also, $\mathbf{C}hu(\mathbf{S}et, 2)$, via logical relations, yields a fully complete model for multiplicative linear logic [3].

In the following, Chu spaces and Chu categories, in particular, $\mathbf{C}hu(\mathbf{S}et, 2)$, are briefly described, and a couple of examples are constructed.

2 Chu space

A $K-$*Chu space* is a triple $\mathcal{A} = (A,\ r,\ X)$ consisting of sets A *and* X and a function $r : A \times X \longrightarrow K$. Essentially, this amounts to defining a Chu space as an $A \times X$ matrix whose entries are drawn from K,or a $K-$*valued* binary relation between A and X. The function r is intended to provide a structure to the object Chu it defines by relating the elements of A with the elements of X. The entries at (a, x) is written as $r(a, x)$ or arx or $a(x)$. The elements of A are called *points, subjects or individuals* depending on the context, while the elements of X are considered as *states or predicates*. Note, however, that the roles of the sets A and X could be conveniently swapped (see [12] for its first introduction).

The Chu space $\mathcal{A}$ may be organized either into rows or columns. If $\mathcal{A}$ is organized into rows, the set A is regarded as the *domain* or the *carrier* of the structure and the row, indexed by a, as the complete description of the element a. In this case, the function $\hat{r} :\ A \ \rightarrow\ K^X$ is used for sorting the row entries satisfying the condition $\hat{r}(a)(x) \ =\ r(a,\ x)$, and it assigns to each point a its description. Notationally, $\hat{a}$ is written for the description of the point a, namely $\hat{r}(a)$, and$\hat{A}$ for the set $\{\hat{a} : a \in A\}$ of all rows of the Chu space where $\{\hat{a} : a \in A\}$ is the set of all descriptions of the point a *in* A. Whenever $\hat{r}$ is injective, that is, when there is no repeated rows, the Chu space $\mathcal{A}$ is called *separable* or *coextensional*.

On the other hand, if $\mathcal{A}$ is organized into columns, A is viewed as consisting of variables with values ranging over the set K and the column indexed by x as one of the permitted assignments to these variables. The function $\check{r} :\ X \ \rightarrow\ K^A$ satisfies $\check{r}(x)(a) \ =\ r(x,\ a)$ and it is used for sorting the columns' entries. These entries appear as duals of $\hat{a}$ and $\hat{A}$. However, if $\check{r}$ is injective, that is, when there is no repeated columns, the Chu space $\mathcal{A}$ is called extensional.
A Chu space is said to be *biextensional* if it is both extensional and coextensional.

It may be noted that the ground set K could be as meagre as the empty set or a singleton, or as big as the set of all complex numbers (while representing Hilbert spaces), or something in between such as the set of 2^n elements of an $n-$*set* while representing topological groups (see[11] for details).

3 Chu Category

Definition 3.1 Initial and terminal objects in a category

An object I in a category $\mathfrak{C}$ is called an *initial* object of $\mathfrak{C}$ if for every object A of $\mathfrak{C}$, there exists a unique morphism $I \longrightarrow\ A$. An object T of a category $\mathfrak{C}$ is called a *terminal* object of $\mathfrak{C}$ if for every object A in $\mathfrak{C}$, there exists a unique morphism $A \longrightarrow\ T$.

Definition 3.2 Monic, coretraction, epi, retraction, and isomorphism
A morphism $f : A \longrightarrow B$ in a category $\mathfrak{C}$ is called

1. *a monomorphism* (*or monic*) if it is left cancellable i.e., for every pair of morphisms $g, h : C \longrightarrow\ A$ if $fog = foh\ \Rightarrow g = h$;

2. *an epimorphism* if it is right cancellable i.e.,for every pair of morphisms $g, h : B \longrightarrow\ C,\ gof = hof\ \Rightarrow g = h$;

3. *a split monomorphism* (*or section or coretraction*) if it is left invertible i.e., there exists a morphism $g : B \longrightarrow\ A$ such that $gof = 1_A$;

4. *a split epimorphism*(*or retraction*) if it is right invertible i.e., there exists a morphism $g : B \longrightarrow\ A$ such that $fog = 1_B$; and

5. *an isomorphism* if it is both coretraction and retraction i.e., if it is invertible.

Definition 3.3 Chu Transform
Let $\mathcal{A} = (A, r, X)$ and $\mathfrak{B} = (B, s, Y)$ be any two Chu spaces. The *Chu transform* or *morphism of Chu spaces* $\mathcal{A}$ and $\mathfrak{B}$ is a pair (f, g) of functions $f : A \longrightarrow\ B\ and\ g : Y \longrightarrow\ X$ satisfying the *adjointness* condition: $s\,(f\,(a)\,, y) = r\,(a, g\,(y))\,,\ for\ all\ a \in\ A\ and\ y\ \in\ Y.$

Let $\mathcal{C} = (C, t, Z)$ be another Chu space. Then the Chu transform of $\mathfrak{B}$ *and* $\mathcal{C}$ is a pair (f', g') of functions $f' : B \longrightarrow\ C\ and\ g' : Z \longrightarrow\ Y$ satisfying the *adjointness* condition: $t(f'(b), z)\ =\ s(b, g'(z)),\ for\ all\ b\ in\ B\ and\ z\ in\ Z.$

Since $f\ :\ A \longrightarrow\ \ B,\ \ \ g\ :\ Y\ \longrightarrow\ \ X,\ f'\ :\ B \longrightarrow\ \ C\ , and\ g'\ :\ Z \longrightarrow\ \ Y,$ we have $f'f\ :\ A \longrightarrow\ \ C\ and\ gg'\ :\ Z \longrightarrow X$, and hence their composition $(f'f, gg') : \mathcal{A} \longrightarrow\ \mathcal{C}$.

It is straightforward to see that the composition satisfies the adjointness condition; $r\,(a, gg'\,(z)) = s\,(f\,(a)\,, g'\,(z)) = t\,(f'f\,(a)\,, z)\,,$ for *all* $a \in A$ *and* $z\ \in\ Z$, and hence it is a Chu transform.

The associativity of composition defined above is inherited from the one in **Set**. The pair (I_A, I_X) of the identity maps on A and X respectively, is an adjoint pair: $I_A : A \longrightarrow A$ and $I_X : X \longrightarrow X$ satisfying $(I_A(a), x) = r(a, I_X(x))$ $for\ all\ a\ in\ A\ and\ x\ in\ X$. Thus the pair (I_A, I_X) is a Chu transform and serves as the identity on $\mathcal{A}$.

Definition 3.4 Chu Category
It follows that Chu spaces over K together with Chu transforms form a *Category*, denoted *Chu(Set*,K) or simply $\mathbf{C}hu_{\mathbf{K}}$, where K is a fixed object of the category $\mathbf{S}et$. In other words, the category $\mathbf{C}hu(\mathbf{S}et, K)$ has *Chu spaces* over the set K as *objects* and *Chu transforms* as *morphisms*.

The initial objects of the category $\mathbf{C}hu(\mathbf{S}et, K)$ are all Chu spaces with the empty set of points and the empty set of states. Its monics are those Chu transforms (f, g) for which f is an injection and g a surjection, and dually for its epis. Moreover, its isomorphisms are those (f, g) for which f and g are both bijections i.e., isomorphisms in the **Set** (see [11], for details).

Recall that when K is a two-element set $\{0, 1\}$, we have $\mathbf{C}hu(\mathbf{S}et, 2)$. The category $\mathbf{C}hu(\mathbf{S}et, 2)$ realizes a large number of mathematical structures such as preordered sets, Stone spaces, ordered Stone spaces, topological spaces, locales, complete semillatices, distributive lattices (but not general lattices), algebraic lattices, frames, profinite (Stone) distributive lattices, Boolean algebras, and complete atomic Boolean algebras, and so on (see[11] for details). For instance, a topological space can be represented as a Chu space over the set $K = 2 = \{0, 1\}$: if X stands for the set of open subsets of A, then for any $a \in A$ and $x \in X$, the expression $r(a, x)$ is equal to one if a belongs to x, and zero otherwise, where $r : A \times X \longrightarrow 2$. Also, Chu spaces over the set $K = 2 = \{0, 1\}$ realizes a *poset* by imposing an order relation on the set K i.e., by defining $a \leq b$ to hold just when it holds in every column by taking $0 \leq 1$, as usual, in each row.

4 Examples of Chu spaces and related Chu categories on $K = \{0, 1\}$

At the outset, let us recall a motivating example of Chu space provided in [5]: $(A, r, \wp(A))$, where $\wp(A)$ is the power set of A, is a Chu space over $\{0, 1\}$ for any set A. The function r is defined as $r(a, x) = 1\ if\ and\ only\ if\ a \in x$ and zero otherwise, where $x \in \wp(A)\ and\ a \in A$.

In the following, adapting from [5, 11], two new examples of Chu spaces and related Chu categories are constructed.

Example 4.1 $(A, r, tc(R_A))$ **is a Chu space over** $\{0, 1\}$**, where** $tc(R_A)$ **is the transitive closure of an arbitrary relation** R_A **on a set** A
The structure $(A, r, tc(R_A))$ defines a Chu space over the set $K = \{0, 1\}$ with $r\,(a, x) = 1$ if and only if $a \in x$ and zero otherwise, $for\ all\ a \in A, x \in tc(R_A)$.

The Chu transform of the aforesaid Chu spaces is defined as follows: Let $\mathcal{A} = (A, r, tc(R_A))$, $\mathfrak{B} = (B, s, tc(R_B))$, and $\mathcal{C} = (C, r, tc(R_C))$ be Chu spaces. Then the Chu transform between $\mathcal{A} = (A, r, tc(R_A))$ and $\mathfrak{B} = (B, s, tc(R_B))$ is the pair (f, g) of functions $f : A \longrightarrow B$ and $g : tc\,(R_B) \longrightarrow tc(R_A)$ satisfying the adjointness condition:

$$s\,(f\,(a)\,, y) = r\,(a, g\,(y))\ \ for\ all\ a \in A,\ y \in tc(R_B).$$

The identity maps on $A\ and\ tc(R_A)$, respectively, are given by the pair $\left(I_A, I_{tc(R_A)}\right)$. This pair also satisfies the adjointness condition and it is the Chu transform on A.
Moreover, for a pair of functions (f', g') between the Chu spaces $\mathcal{B}\ and\ \mathcal{C}$, it can be likewise shown that this pair meets the adjointness condition and hence is a Chu transform between $\mathcal{B}\ and\ \mathcal{C}$.

The composition $(f, g) \circ (f', g') = (f'f, gg')$ follows from $r\,(a, gg'\,(z)) = s\,(f\,(a)\,, g'\,(z)) = t(f'f\,(a)\,, z)\ for\ all\ a \in A,\ z \in tc(R_c)$.
Thus the Chu spaces obtained in this way over the set $K = \{0, 1\}$ with their adjoint pairs form a category.

For example, let $A = \{a, b, c, d\}$ and $R_A = \{(a, a)\,, (b, a)\,, (c, b)\,, (d, c)\}$. Hence, $tc(R_A) = \{(a, a), (b, a), (c, a), (c, b), (d, b), (d, c), (d, a)\}$.The function r is defined as $r\,(a, x) = 1$ if and only if $a \in x$ and zero otherwise $for\ all\ a \in A, x \in tc(R_A)$. The resulting Chu space over $K = \{0, 1\}$ is as follows: .

Table 1: Chu matrix

	(a,a)	**(b,a)**	**(c,a)**	**(c,b)**	**(d,b)**	**(d,c)**	**(d,a)**
a	1	1	1	0	0	0	1
b	0	1	0	1	1	0	0
c	0	0	1	1	0	1	0
d	0	0	0	0	1	1	1

Note that the Chu space in Table(1) is biextensional.
It is straightforward to see that if the columns corresponding to $(a, a), (c, b), (d, b), (d, c)$ are deleted (see[5, 11] for details), then another Chu space is obtained whose first row is the *join* (bitwise disjunction, denoted $\vee$) of the other rows viz.,

Table 2: Chu matrix

	(b,a)	**(c,a)**	**(d,a)**
a	1	1	1
b	1	0	0
c	0	1	0
d	0	0	1

Let R_i, $i = \overline{1,4}$, denote the rows of the matrix in Table(2). An order $\leq$, as usual, can be defined on Table(2) as follows: $R_j \leq R_k$ (component-wise), $j \geq k$, $j, k = \overline{1,4}$.
It is easy to see that the Chu space in Table(2) is a *Poset* under $\leq$. Also this Chu space under $\vee$ (bitwise disjunction) satisfies commutativity, idempotency and associativity. Hence it is a *join-semilattice* i.e., a semigroup that is commutative and idempotent (see[10] for a detailed description of operations on Chu spaces).
Note, however, that this Chu space is not a *meet-semilattice.*

Example 4.2 $(\mathbb{Z}_n \backslash \mathbf{0}, \mathbf{r}, \mathbb{Z}_n)$ **over** $\{0, 1\}$ **is a Chu Space**

The matrix $r : A \times X \longrightarrow 2$, under the operation $a \mid (a + x)$, read as *a divides* $(a + x)$ where $a \in A,\ x \in \mathbb{Z}_n$, defines a Chu space.

The Chu transform of the aforesaid Chu spaces is defined as follows: Let $\mathcal{A} = (\mathbb{Z}_n \backslash \mathbf{0}, \mathbf{r}, \mathbb{Z}_n)$, $\mathcal{B} = (\mathbb{Z}_m \backslash \mathbf{0}, \mathbf{s}, \mathbb{Z}_m)$, and $\mathcal{C} = (\mathbb{Z}_l \backslash \mathbf{0}, \mathbf{t}, \mathbb{Z}_l)$ be Chu spaces. Then the Chu transform between $\mathcal{A}$ and $\mathcal{B}$ is the pair (f, g) of functions $f : \mathbb{Z}_n \backslash \mathbf{0} \longrightarrow \mathbb{Z}_m \backslash \mathbf{0}$ and $g : \mathbb{Z}_m \longrightarrow \mathbb{Z}_n$ satisfying the adjointness condition: $s(f([a]), [y]) = r([a], g([y]))$ $for\ all$ $[a] \in \mathbb{Z}_n \backslash \mathbf{0}, [y] \in \mathbb{Z}_m$. The identity maps on $A = \mathbb{Z}_n \backslash \mathbf{0}$ and $X = \mathbb{Z}_n$, respectively, are given by the pair $(I_{A=\mathbb{Z}_n \backslash \mathbf{0}}, I_{X=\mathbb{Z}_n})$. This pair also satisfies the adjointess condition and it is the Chu transform on $\mathcal{A}$.
Moreover, for a pair of functions (f', g') between the Chu spaces $\mathcal{B}$ *and* $\mathcal{C}$, it can be likewise shown that this pair meets the adjointness condition and hence is a Chu transform between $\mathcal{B}$ *and* $\mathcal{C}$.

The composition $(f, g) \circ (f', g')$ is given by

$$r\left(a, gg'\left(z\right)\right) = s\left(f\left(a\right), g'\left(z\right)\right) = t(f'f\left(a\right), z)$$

$for\ all$ $[a] \in \mathbb{Z}_n \backslash \mathbf{0}, [z] \in \mathbb{Z}_l$.
Thus the Chu spaces obtained in this way over the set $K = \{0, 1\}$ with their adjoint pairs form a category.

For example, let $\mathbb{Z}_4 = \{[\mathbf{0}], [\mathbf{1}], [\mathbf{2}], [\mathbf{3}]\}$, in standard notation.
Let $X = \mathbb{Z}_4,\ \ A = \mathbb{Z}_4 \backslash 0$, and $K = 2 = \{0, 1\}$. Then the matrix $r : A \times X \longrightarrow 2$, under the operation $a \mid (a + x)$, read as *a divides* $(a + x)$ where $a \in A,\ x \in X$, defines a biextensional Chu space:

$$\begin{matrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{matrix} \tag{1}$$

If the structure of the Chu space (1) is pruned by *deleting the column* corresponding to $[\mathbf{1}]$ of $\mathbb{Z}_4$, another Chu space is obtained whose first row is the join (bitwise disjunction, denoted $\vee$) of the other two rows viz.,

$$\begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{matrix} \tag{2}$$

Let R_i, $i = \overline{1, 3}$, denote the rows of the matrix (2). An order $\leq$, as usual, can be defined on (2) as follows: $R_j \leq R_k$ (component-wise), $j \geq k$, $j, k = \overline{1, 3}$.
It is easy to see that the Chu space (2) is a *Poset* under $\leq$.
The Chu space (2) under $\vee$ satisfies commutativity, idempotency and associativity. That is, if

$p = 1 \ \ 1 \ \ 1$,
$q = 1 \ \ 1 \ \ 0$, and
$t = 1 \ \ 0 \ \ 1$,

then $p \vee p = p$, etc., $p \vee q = q \vee p$, etc., $p \vee (q \vee r) = (p \vee q) \vee r$, etc., hold.
Hence,the Chu space (2) represents a *semilattice* i.e., a semigroup that is commutative and idempotent.

Note that the Chu space (2) under the *meet* operation is not a semilattice.

It is observed that $\mathbb{Z}_n,\ n = 2, 3, 4$, and $K = \{0, 1\}$, give rise to *biextensional* Chu spaces. For $\mathbb{Z}_n,\ n > 4$, under the aforesaid operation and $K = \{0, 1\}$, the resulting Chu spaces are not biextensional.

Concluding Remarks

The examples constructed in 4.1 and 4.2 are typical in a sense that some such other useful operations and relational structures could be formulated to generate suitable Chu spaces as the case may be.

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