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Regularity Properties of p-Distance Transformations in Image Analysis

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Abstract

In this paper we extend the notion of infimal convolution to a p-infimal convolution, and show that it is commutative and associative operation on functions. We also use p-infimal convolution to get some p- norms (see [2]) that define p-distances and to get some class of p-convex neighborhoods which are more appropriate to handle with and to get better performance for image processing ($0 < p < 1$). Our results generalize those in [8] when *p=1*.

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1. Introduction

We introduced invariant distance results concerning the translation invariant distance on an abelian group X in section 2, (see [8]). We study in section 3 the positively p-homogeneous and p-midpoint convexity ($0 < p \leq 1$).We have proved convex set if and only if $f(a) \le \frac{1}{2}[f(a+b) + f(a-b)] \ \forall a,b \in X$. In addition, equivalent conditions to a function *g* being midpoint convex on a p-convex set are given in proposition 3.3. In section 4 we extend the notion of infimal convolution to a p-infimal convolution, and show that this p-infimal convolution is a commutative and associative operation on functions defined on an abelian group *X*.

We have introduced in section 5 the p-regularity ($0 < p < 1$). We give the conditions on a function *G* on an abelian group to be p-semi regular. Finally, some equivalent concepts on semiregularity are given in section 6 see, Theorem 6.1.

2. Basic definitions and Notations

Distance transformations of digital images are useful tools in image analysis. A distance transform of a shape is the set of distances from a given pixel to the shape, see [3, 4]. The distances can be measured in different ways, e.g., by approximating the Euclidean distance in the two-dimensional image, the Euclidean distance between two pixels $x = (x_1, x_2)$ and $y = (y_1, y_2)$ being $\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ .Other distances that have been used are the city-block distance $|x_1 - y_1| + |x_2 - y_2|$ and the chess-board distance $\max(|x_1 - y_1|, |x_2 - y_2|)$.

Definition 2.1.(Distances and Metrics) Let *X* be any nonempty set, we shall measure distance between points in *X*, which amounts to defining a real valued function on the Cartesian product *X*×*X*. Let us agree to call a function $d : X \times X \to R$, a *distance* if *d* is positive definite, that is, $d(x, y) \ge 0$ with equality precisely when *x*=*y*, (2.1)

and symmetric, $d(x, y) = d(y, x) \quad \forall x, y \in X$ (2.2)

A distance will be called a *metric* if, in addition, it satisfies the triangle inequality,

$$d(x,z) \le d(x,y) + d(y,z) \quad \forall x,y,z \in X \tag{2.3}$$

If *X* is an abelian group then the translation-invariant distances are those which satisfy, $d(x+a, y+a) = d(x,y) \quad \forall a,x,y \in X$ (2.4)

Definition 2.2.A *p-norm* on a linear space *E* is a mapping $\|.\|$ from *E* to *R* satisfying:

1) $\|x\| \ge 0$ and $\|x\| = 0$ if and only if x=0.

2) $\|\lambda x\| = |\lambda|^p \|x\|$, λ is a scalar.

3) $\|x + y\| \le \|x\| + \|y\| \ \forall x, y \in E$.

On the Euclidean R^n, a *p-norm* (0<p<1) is defined by $\|x\| = \sum_{i=1}^{n} |x_i|^p$ and so the *p-distance* between two points $x = (x_1,...,x_n)$, $y = (y_1,...,y_n)$ is given by

$$d(x,y) = \sum_{i=1}^{n} |x_i - y_i|^p. \tag{2.5}$$

Definition 2.3. A *quasi-norm* on a linear space *E* is a mapping $\|.\|$ from *E* to *R* satisfying:

1) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if x=0.

2) $\|\lambda x\| = |\lambda| \, \|x\|$

3) $\|x+y\| \leq \sigma(\|x\| + \|y\|)$ for all $x, y \in E$ with some constants $\sigma \geq 1$. For $\sigma = 1$ it is called a norm. see, [1,2] .

It is well known [9] that, for any *p>0*,

$$(\zeta + \eta)^p \leq \tau_p(\zeta^p + \eta^p), \forall \zeta, \eta \geq 0 \text{ with } \tau_p = \max(2^{p-1},1) \tag{2.6}$$

Remark 2.1. (I) If ||.|| is a p-norm with *0<p<1*, then $\|.\|^{\frac{1}{p}}$ is a quasi norm. In fact,

1) $\|x\|^{\frac{1}{p}} \geq 0$ and $\|x\|^{\frac{1}{p}} = 0$ if and only if *x=0*.

2) $\|\lambda x\|^{\frac{1}{p}} = (|\lambda|^p \|x\|)^{\frac{1}{p}} = |\lambda| \|x\|^{\frac{1}{p}}$.

Using (2.6) and noting that $\tau_{\frac{1}{p}} = 2^{\frac{1}{p}-1}$ we get,

3) $\|x+y\|^{\frac{1}{p}} \leq (\|x\| + \|y\|)^{\frac{1}{p}} \leq (2^{\frac{1}{p}-1})(\|x\|^{\frac{1}{p}} + \|y\|^{\frac{1}{p}})$.

(II) In a similar way if ||.|| is a p-norm with *p>1*, then $\|.\|^{\frac{1}{p}}$ is a norm.

Lemma2.1. [8]: There is a one to one correspondence between positive definite even functions and translation invariant distances. In fact, any translation invariant distance *d* on an abelian group *X* defines a function $g(x) = d(x,0)$ on *X* which is positive definite, i.e. $g(x) \geq 0$, $g(x) = 0$ when *x=0* (2.7)

And it is an even function i.e $g(-x) = g(x) \;\; \forall x \in X$ (2.8)

Conversely, a function *g* which satisfies (2.7), (2.8) defines a translation invariant distance $d(x,y) = g(x-y)$.

The following lemma will be used in section 3.

Lemma 2.2. [8]: Let *d* be a translation-invariant distance on an abelian group *X* and *g* a function on *X* related to *d* as in lemma (2.1).Then *d* is a metric if and only if *g* is subadditive. i.e. $g(x+y) \leq g(x) + g(y) \;\; \forall x, y \in X$. (2.9)

3. Positively p-homogeneous and p-midpoint convexity

Definition 3.1. We call a distance d on an abelian group X a *positively p-homogenous* if it satisfies,

$$d(mx, my) = m^p d(x, y) \quad \forall x, y \in X, m \in N, \quad 0 < p < 1 \tag{3.1}$$

Of course, if g is the even function $g(x) = d(x,0)$ related to a translation-invariant distance d, so being positively p-homogeneous, i.e.

$g(mx) = d(mx, 0) = m^p g(x) \forall x \in X, m \in N$ implies that,

$$g(mx) = |m|^p \; g(x) \forall x \in X, m \in Z \tag{3.2}$$

We call a function *f(x)* of one variable a positively p-homogeneous, if $f(mx) = m^p f(x)$ and *f(x, y)* of two variables is called a positively p-homogeneous if $f(mx, my) = m^p f(x, y)$.

We extend the concept of convex and midpoint convex function to *p-convex* and *p-midpoint convex function* for *0<p<1.*

Definition 3.2. let E be a vector space over the field R of real numbers. A subset A of E is said to be *p-convex* if the arc segment $[x, y] = \{(1-\lambda)^{\frac{1}{p}} x + \lambda^{\frac{1}{p}} y ; 1 \ge \lambda \ge 0\}$ is contained in A for every choice of $x, y \in A$. For *p=1* we get the definition of convex set [5, 10].

Definition 3.3. A function f on an abelian group X with values in the extended real number system $[-\infty, \infty]$ is called:

1. *Convex on a convex set* if it is satisfies

$$f((1-\lambda)x + \lambda y) \le (1-\lambda) f(x) + \lambda f(y) \quad \forall 0 \le \lambda \le 1$$

2. *Convex on a p-convex set,* $0 < p < 1$, if it is satisfies

$$f((1-\lambda)^{\frac{1}{p}} x + \lambda^{\frac{1}{p}} y) \le (1-\lambda) f(x) + \lambda f(y) \;\; \forall 0 \le \lambda \le 1,$$

3. *p-Convex on a convex set,* $0 < p < 1$, if it is satisfies

$$f((1-\lambda)x + \lambda y) \le (1-\lambda)^{\frac{1}{p}} f(x) + \lambda^{\frac{1}{p}} f(y) \quad \forall 0 \le \lambda \le 1,$$

4. *p-Convex on a p-convex set,* $0 < p < 1$, if it is satisfies

$$f((1-\lambda)^{\frac{1}{p}} x + \lambda^{\frac{1}{p}} y) \le (1-\lambda)^{\frac{1}{p}} f(x) + \lambda^{\frac{1}{p}} f(y) \;\; \forall 0 \le \lambda \le 1.$$

Definition 3.4. A function f on an abelian group X with values in the extended real number system $[-\infty, \infty]$ is called:

1. *Midpoint convex on a convex set* if it is satisfies

$$f\left(\frac{x+y}{2}\right) \le \left(\frac{1}{2}\right)[f(x) + f(y)] \quad \forall x, y \in X$$

2. *Midpoint convex on a p-convex set* if it is satisfies

$$f\left(\frac{x+y}{2^{\frac{1}{p}}}\right) \le \left(\frac{1}{2}\right)[f(x)+f(y)] \quad \forall x, y \in X$$

3. *p-Midpoint convex on a convex set* if it is satisfies

$$f\left(\frac{x+y}{2}\right) \le \left(\frac{1}{2}\right)^{\frac{1}{p}}[f(x)+f(y)] \quad \forall x, y \in X$$

4. *p-Midpoint convex on a p-convex set* if it is satisfies

$$f\left(\frac{x+y}{2^{\frac{1}{p}}}\right) \le \left(\frac{1}{2}\right)^{\frac{1}{p}}[f(x)+f(y)] \quad \forall x, y \in X \ .$$

Example3.1. In R^2 the set $E = \{(X,Y) : \sqrt{x} + \sqrt{y} \le 1\}$ is 1/2-convex.
For $(x_1, y_1), (x_2, y_2) \in E$ we show that
$\lambda^2(x_1, x_2) + (1-\lambda)^2(y_1, y_2) = (\lambda^2 x_1 + (1-\lambda)^2 y_1, \lambda^2 x_2 + (1-\lambda)^2 y_2) \in E$. In fact

$$\sqrt{|\lambda^2 x_1 + (1-\lambda)^2 x_2|} + \sqrt{|\lambda^2 y_1 + (1-\lambda)^2 y_2|} \le \lambda\sqrt{|x_1|} + (1-\lambda)\sqrt{|x_2|} + \lambda\sqrt{|y_1|} + (1-\lambda)\sqrt{|y_2|} \le$$

$$\le \lambda(\sqrt{|x_1|} + \sqrt{|y_1|}) + (1-\lambda)(\sqrt{|x_2|} + \sqrt{|y_2|}) \le \lambda + (1-\lambda) = 1$$

Example3.2. A similar argument shows that the set $B = \{x = (x_1, x_2) : |x_1|^p + |x_2|^p \le 1\}$ is a p-convex set.

The following theorems (3.1, 3.2, 3.3, and 3.4) give some properties concerning the different types of p-midpoint convexity functions.

Theorem3.1. A homogenous function *f* is a p-midpoint convex on a p-convex set if and only if $f(a) \le \frac{1}{2}[f(a+b)+f(a-b)] \ \forall a, b \in X$

Proof. Since *f* is a p-midpoint convex function on a p-convex set then

$$f\left(\frac{x+y}{2^{\frac{1}{p}}}\right) \le \left(\frac{1}{2}\right)^{\frac{1}{p}}[f(x)+f(y)] \quad \forall x, y \in X$$

Putting $\frac{x+y}{2^{\frac{1}{p}}} = a$ and $\frac{x-y}{2^{\frac{1}{p}}} = b$ then

$$x = 2^{\frac{1}{p}-1}(a+b), \ y = 2^{\frac{1}{p}-1}(a-b) \tag{3.3}$$

Hence, $f(a) \le \frac{1}{2^{\frac{1}{p}}}[f(2^{\frac{1}{p}-1}(a+b)) + f(2^{\frac{1}{p}-1}(a-b))] \le \frac{1}{2}[f(a+b)+f(a-b)]$.

Conversely, let $f(a) \le \frac{1}{2}[f(a+b)+f(a-b)]$ and using (3.3) we get,

$$f(\frac{x+y}{2^{\frac{1}{p}}}) \le \frac{1}{2}[f(\frac{1}{2^{\frac{1}{p}-1}}x)+f(\frac{1}{2^{\frac{1}{p}-1}}y)] \le$$

$$\le \frac{1}{2}\times\frac{1}{2^{\frac{1}{p}-1}}[f(x)+f(y)] \le \frac{1}{2^{\frac{1}{p}}}[f(x)+f(y)]$$

Then f is a p-midpoint function on a p-convex set. ■

Theorem 3.2. A p-homogeneous function f is midpoint convex on a p-convex set if and only if $f(a) \le \frac{1}{2^p}[f(a+b)+f(a-b)] \quad \forall a,b \in X$

Proof. Since f is midpoint convex on a p-convex set then

$$f(\frac{x+y}{2^{\frac{1}{p}}}) \le (\frac{1}{2})[f(x)+f(y)]$$

Using (3.3) we get,

$$f(a) \le \frac{1}{2}[f(2^{\frac{1}{p}-1}(a+b))+f(2^{\frac{1}{p}-1}(a-b))] \le \frac{1}{2}\times 2^{1-p}[f(a+b)+f(a-b)] \le$$

$$\le \frac{1}{2^p}[f(a+b)+f(a-b)]$$

Conversely, let $f(a) \le \frac{1}{2^p}[f(a+b)+f(a-b)]$ and using (3.3) we get,

$$f(\frac{x+y}{2^{\frac{1}{p}}}) \le \frac{1}{2^p}[f(\frac{1}{2^{\frac{1}{p}-1}}x)+f(\frac{1}{2^{\frac{1}{p}-1}}y)] \le \frac{1}{2^p}\times\frac{1}{2^{1-p}}[f(x)+f(y)] \le \frac{1}{2}[f(x)+f(y)]$$

Then f is midpoint convex on a p-convex set. ■

Theorem 3.3. (1) If $f^{\frac{1}{p}}$ is homogeneous and p-midpoint convex on p-convex set then f is midpoint convex on a p-convex set.

(2) If f is p-homogeneous and midpoint convex on a p-convex set then $f^{\frac{1}{p}}$ is midpoint convex on a p-convex set.

Proof: (1) Since $f^{\frac{1}{p}}$ is p- midpoint convex function on a p-convex set it follows

$$f^{\frac{1}{p}}(\frac{x+y}{2^{\frac{1}{p}}}) \le (\frac{1}{2})^{\frac{1}{p}}[f^{\frac{1}{p}}(x)+f^{\frac{1}{p}}(y)]$$

Using (3.3) we get, $f^{\frac{1}{p}}(a) \le \frac{1}{2^{\frac{1}{p}}}[f^{\frac{1}{p}}(2^{\frac{1}{p}-1}(a+b))+f^{\frac{1}{p}}(2^{\frac{1}{p}-1}(a-b))] \le$

$$\le \frac{1}{2^{\frac{1}{p}}}\times 2^{\frac{1}{p}-1}[f^{\frac{1}{p}}(a+b)+f^{\frac{1}{p}}(a-b)] \le \frac{1}{2}[f^{\frac{1}{p}}(a+b)+f^{\frac{1}{p}}(a-b)].$$

Rising to the p^{th} power we get: $f(a) \le \frac{1}{2^p}[f(a+b)+f(a-b)]$

2) Since f is a midpoint convex function on a p-convex set then

$$f(a) \le \frac{1}{2^p}[f(a+b)+f(a-b)].$$

Using (3.3) we get, $f(\frac{x+y}{2^{\frac{1}{p}}}) \le \frac{1}{2^p}[f(\frac{1}{2^{\frac{1}{p}-1}}x)+f(\frac{1}{2^{\frac{1}{p}-1}}y)]$. Rising to power 1/p we get, $f^{\frac{1}{p}}(\frac{x+y}{2^{\frac{1}{p}}}) \le (\frac{1}{2^p})^{\frac{1}{p}} \times 2^{\frac{1}{p}-1} \times \frac{1}{2^{\frac{1}{p}-1}}[f^{\frac{1}{p}}(x)+f^{\frac{1}{p}}(y)] \le \frac{1}{2}[f^{\frac{1}{p}}(x)+f^{\frac{1}{p}}(y)]$ ■

Theorem 3.4. For a non negative function *f* we get:

1) If $f^{\frac{1}{p}}$ is midpoint convex on convex set then *f* is midpoint convex on p-convex set.

2) If *f* is p-midpoint convex on p-convex set then $f^{\frac{1}{p}}$ is midpoint convex on convex set.

Proof. To claim (1), assume that $f^{\frac{1}{p}}$ is midpoint convex on convex set then,

$$f^{\frac{1}{p}}(x) \le \frac{1}{2}[f^{\frac{1}{p}}(x+y)+f^{\frac{1}{p}}(x-y)]$$

Rising to p^{th} power, and noticing that $0 \le p \le 1$ we get from (2.6):

$$f(x) \le \frac{1}{2^p}[f^{\frac{1}{p}}(x+y)+f^{\frac{1}{p}}(x-y)]^p \le \frac{1}{2^p}[f(x+y)+f(x-y)].$$

To claim (2), note that as *f* is p-midpoint convex on p-convex set it follows that

$$f(x) \le \frac{1}{2}[f(x+y)+f(x-y)]$$

Rising to $\frac{1}{p^{th}}$ power we get: $f^{\frac{1}{p}}(x) \le (\frac{1}{2})^{\frac{1}{p}} \times 2^{\frac{1}{p}-1}[f^{\frac{1}{p}}(x+y)+f^{\frac{1}{p}}(x-y)]$

$$\le (\frac{1}{2})[f^{\frac{1}{p}}(x+y)+f^{\frac{1}{p}}(x-y)]$$ ■

In what follows we give the relation between a positively p-homogeneous function and a p-midpoint convex function for $0 < p < 1$. For classical case of *p=1*, see [8].

Lemma 3.3. A positively p-homogeneous function *f* is a midpoint convex on a p-convex set if it is subadditive.

Proof. Let *f* be a subadditive and a positively p- homogeneous function then

$$2^p f(x) = f(2x) = f(x+x) = f((x+y)+(x-y)) \le f(x-y)+f(x+y).$$

Hence, $f(x) \le \frac{1}{2^p}[f(x+y)+f(x-y)]$.

The following result extends the classical one of [8].

Proposition 3.3. Let $g^{\frac{1}{p}}$ be a subadditive function on an abelian group *X* satisfying $0 \le g(x) < +\infty$ where *0<p<1*.

The following three conditions are equivalent:

(i) *g* is midpoint convex on p-convex set and *g(0)=0*

(ii) $g(2x) = 2^p g(x) \qquad \forall x \in X$

(iii) g is positively p-homogeneous.

Proof. $(i) \Rightarrow (ii)$ Let g be midpoint convex on p-convex set then, by lemma 2.3 $g^{\frac{1}{p}}$ is midpoint convex on p-convex set, i.e. $g(x) \le \frac{1}{2^p}[g(x+y)+g(x-y)]$.
Taking *x=y* and noting that *g(0)=0* we get,

$$g(x) \le \frac{1}{2^p}[g(x+x)+g(x-x)] = \frac{1}{2^p}[g(2x)+g(0)].$$

So, $2^p g(x) \le g(2x)$. On the other hand,
$g^{\frac{1}{p}}(2x) = g^{\frac{1}{p}}(x+x) \le g^{\frac{1}{p}}(x) + g^{\frac{1}{p}}(x) = 2g^{\frac{1}{p}}(x).$ Hence,
$g(2x) = 2^p g(x)$.

$(ii) \Rightarrow (i)$ Since $g^{\frac{1}{p}}$ is subadditive then from (ii) we get

$$2^p g(x) = g(2x) = g(x+x) \le g(x+y) + g(x-y)$$

Hence, $g(x) \le \frac{1}{2^p}[g(x+y)+g(x-y)]$. Also, $g(0) = g(2\times 0) = 2^p g(0) = 0$ so, $g^{\frac{1}{p}}(0) = 0$. Since g is positively p-homogeneous function then by taking *m=2* we get, $g(2x) = 2^p g(x)$.

$(ii) \Rightarrow (iii)$ From subadditivity of $g^{\frac{1}{p}}(x)$ we see that,
$g^{\frac{1}{p}}(mx) \le mg^{\frac{1}{p}}(x) \ \forall m \in N$.

We show that if, $g^{\frac{1}{p}}(mx) = mg^{\frac{1}{p}}(x)$ for certain m then it is true that:

$$g^{\frac{1}{p}}((m-1)x) = (m-1)g^{\frac{1}{p}}(x)$$

In fact, $g^{\frac{1}{p}}(mx) \le g^{\frac{1}{p}}(m-1)x + g^{\frac{1}{p}}(x) \le (m-1)g^{\frac{1}{p}}(x) + g^{\frac{1}{p}}(x) = mg^{\frac{1}{p}}(x)$
The first and last elements of this inequality are equal so we have,
$g^{\frac{1}{p}}(m-1)x = (m-1)g^{\frac{1}{p}}(x)$. By induction
$g^{\frac{1}{p}}((2^k - j)x) = (2^k - j)g^{\frac{1}{p}}(x)$ holds for all $k, j \in N$ ■

In the following section we extend the definition of infimal convolution to p-infimal convolution.

4. Metrics defined by p-infimal convolution

Definition4.1. Let *f*, *g* be two functions defined on an abelian group *X* with values in the extended real line $[-\infty, +\infty]$. For *0<p<1* we define the p-Infimal convolution $f \square_p g$ of *f* and *g* as follows:

$$(f \square_p g)(x) = \inf_{y \in X}[f^{\frac{1}{p}}(x-y) + g^{\frac{1}{p}}(y)]^p, \ x \in X. \qquad (4.1)$$

For *p=1* see [8] and [10].

Remark 4.1. One can define a p-infimal convolution for $p>1$ as follows,

$$(f \bullet_p g)(x) = \inf_{y\in x}[f^p(x-y)+g^p(y)]^{\frac{1}{p}}$$

Lemma 4.1. A function $f^{\frac{1}{p}}$ on an abelian group is subadditive if and only if it satisfies the inequality $f \square_p f \geq f^{\frac{1}{p}}$. In case $f^{\frac{1}{p}}(0)=0$, then $f \square_p f \geq f^{\frac{1}{p}}$ yields $f \square_p f = f^{\frac{1}{p}}$.

Proof. We claim: $f^{\frac{1}{p}}$ is sub additive $\Rightarrow f \square_p f \geq f^{\frac{1}{p}}$

If $f^{\frac{1}{p}}$ is sub additive then, $f^{\frac{1}{p}}(x) = f^{\frac{1}{p}}(x+y-y) \leq f^{\frac{1}{p}}(x-y)+f^{\frac{1}{p}}(y)$

Hence, $f^{\frac{1}{p}}(x) \leq \inf_y[f^{\frac{1}{p}}(x-y)+f^{\frac{1}{p}}(y)] = (f \square_p f)(x)$.

On the other hand, If $f \square_p f\,(x) \geq f^{\frac{1}{p}}(x)$ then,

$$\inf_y(f^{\frac{1}{p}}(x-y)+f^{\frac{1}{p}}(y))(x) \geq f^{\frac{1}{p}}(x)$$

In particular, $(f \square_p f)(x+y) \geq f^{\frac{1}{p}}(x+y)$ so,

$f^{\frac{1}{p}}(x)+f^{\frac{1}{p}}(y) \geq [\inf_y(f^{\frac{1}{p}}(x)+f^{\frac{1}{p}}(y))] \geq f^{\frac{1}{p}}(x+y)$ i.e. $f^{\frac{1}{p}}$ is subadditive.

Finally, we always have $(f \square_p f)(x) \leq f^{\frac{1}{p}}(x)+f^{\frac{1}{p}}(0)$, so if $f^{\frac{1}{p}}(0)=0$ it follows that $f \square_p f \leq f^{\frac{1}{p}}$ ■

A good example is obtained by applying lemma 4.1 for a p- norm $\|x\|$. If we take $f^{\frac{1}{p}}(x) = \|x\|^{\frac{1}{p}}$, then $\|x\|^{\frac{1}{p}} \leq \|x-y\|^{\frac{1}{p}}+\|y\|^{\frac{1}{p}} = \inf_y(f^{\frac{1}{p}}(x-y)+f^{\frac{1}{p}}(y))$

i.e. $f \square_p f \geq f$.

Theorem 4.1.p-infimal convolution is a commutative and associative oration on functions.

Proof.(I) Commutativity: Letting *z=x-y* we get,

$$(f \square_p g)(x) = \inf_y(f^{\frac{1}{p}}(x-y)+g^{\frac{1}{p}}(y))^p$$

$$= \inf_z(f^{\frac{1}{p}}(z)+g^{\frac{1}{p}}(x-z))^p = (g \square_p f)(x)$$

(II) Associativity:

$$(f \square_p (g \square_p h))(x) = \inf_y[f^{\frac{1}{p}}(x-y)+(g \square_p h)^{\frac{1}{p}}(y)]^p =$$

$$= \inf_y[f^{\frac{1}{p}}(x-y)+\inf_z(g^{\frac{1}{p}}(y-z)+h^{\frac{1}{p}}(z))]^p =$$

$$= \inf_y \inf_z[f^{\frac{1}{p}}(x-y)+g^{\frac{1}{p}}(y-z)+h^{\frac{1}{p}}(z)]^p =$$

$$=\inf_z \inf_y [h^{\frac{1}{p}}(z)+(g^{\frac{1}{p}}(y-z)+f^{\frac{1}{p}}(x-y))]^p =$$

$$=\inf_z [h^{\frac{1}{p}}(z)+\inf_{y\in x}(g^{\frac{1}{p}}(y-z)+f^{\frac{1}{p}}(x-y))]^p =$$

$$=\inf_z [h^{\frac{1}{p}}(z)+\inf_{y-z}(f^{\frac{1}{p}}((x-z)-(y-z))+g^{\frac{1}{p}}(y-z))]^p$$

Putting $y-z=\tilde{y}$,

$$=\inf_z [h^{\frac{1}{p}}(z)+\inf_{\tilde{y}}(f^{\frac{1}{p}}((x-z)-\tilde{y})+g^{\frac{1}{p}}(\tilde{y}))]^p =$$

$$=\inf_z [h^{\frac{1}{p}}(z)+(f\,\square_p\, g)^{\frac{1}{p}}(x-z)]^p = ((f\,\square_p\, g)\square_p\, h)(x) \quad \blacksquare$$

A k-fold convolution can be defined by

$$(f_1\square_p \ldots \square_p f_k)(x)=\inf[\sum_{i=1}^{k} f^{\frac{1}{p}}_i(x^i)]^p\ , \tag{3.2}$$

where the infimum is over all choices of elements $x^i \in X$ such that $x^1+\ldots+x^k=x$.

Theorem 4.2. Let $G: X \to [0,+\infty]$ be a function on an abelian group *X* satisfying *G(0)=0*. Define sequence of functions $(G_j)_{j=1}^{\infty}$ by putting, $G_1=G$, $G_j=(G_{j-1}\square_p G)^p$

Then the sequence (G_j) is decreasing and its limit $\lim G_j = g \geq 0$ is subadditive. Moreover, dom g=N.domG i.e., g is finite precisely in the semi group generated by *dom G*.

Proof. We will prove that the sequence G_j is decreasing if we take *y=0* in the definition of G_{j+1}.

$$G_{j+1}(x)=(G_j\square_p G)^p(x)=\inf_y (G_j^{\frac{1}{p}}(x-y)+G^{\frac{1}{p}}(y))^p \leq G_j(x)$$

Next we shall prove that $g^{\frac{1}{p}}(x+y)\leq g^{\frac{1}{p}}(x)+g^{\frac{1}{p}}(y)$.

Let *X, Y* be given with $g^{\frac{1}{p}}(x), g^{\frac{1}{p}}(y)<+\infty$ and fix a positive number ε . Then there exist numbers *j, k* such that $G_j^{\frac{1}{p}}\leq g^{\frac{1}{p}}(x)+\varepsilon$ and $G_k^{\frac{1}{p}}(y)\leq g^{\frac{1}{p}}(y)+\varepsilon$.

We have, $G_{j+1}^{\frac{1}{p}}(x+y)\leq G_j^{\frac{1}{p}}(x)+G^{\frac{1}{p}}(y)$

$$G_{j+2}^{\frac{1}{p}}(x+y)\leq G_{j+1}^{\frac{1}{p}}(x+z)+G^{\frac{1}{p}}(y-z)\leq G_j^{\frac{1}{p}}(x)+G^{\frac{1}{p}}(z)+G^{\frac{1}{p}}(y-z)\leq$$

$$\leq G^{\frac{1}{p}}(x)+\inf_z (G^{\frac{1}{p}}(y-z)+G^{\frac{1}{p}}(z))\leq G^{\frac{1}{p}}(x)+G_2^{\frac{1}{p}}(y)$$

So, $G_{j+k}^{\frac{1}{P}}(x+y)\leq G_j^{\frac{1}{p}}(x)+G_k^{\frac{1}{p}}(y)$

$$g^{\frac{1}{p}}(x+y)\leq G_{j+k}^{\frac{1}{p}}(x+y)\leq G_j^{\frac{1}{p}}(x)+G_k^{\frac{1}{p}}(y)\leq g^{\frac{1}{p}}(x)+g^{\frac{1}{p}}(y)+2\varepsilon.$$

Since ε is arbitrary, the inequality $g^{\frac{1}{p}}(x+y)\leq g^{\frac{1}{p}}(x)+g^{\frac{1}{p}}(y)$ follows. $\blacksquare$

Theorem4.3. [8]: There is a translation invariant metric d_1 on X such that $G(x) \geq d_1(x,0)$ for all $x \in X$ and G as in theorem 3.2. Then the limit g of the sequence G_1 also satisfies this inequality, $g(x) \geq d_1(x,0)$, so that it is positive definite. If G is symmetric, g is also symmetric and defines a metric $d(x,y) = g(x-y) \geq d_1(x,y)$ on the sub group *Z.A = N.A*

Corollary 4.1.[8]**:** Let W be a finite set in an abelian group X containing the origin and let G be a function on X with *G(0)=0*, taking the value $+\infty$ outside W and finite positive values at all points in *W/{0}*.Then $g = \lim G_j$ is a positive definite subadditive function. If W is symmetric and
G(-x)=G(x), then g defines a metric on the subgroup *Z.W=N.W* of X generated by A.

Example 4.1. Let $W = \{x \in Z^2, |x_j| \leq 1\}$, and define the prime distances as $G(\pm 1,0) = G(0,\pm 1) = a > 0$, $G(\pm 1,\pm 1) = b > 0$. Then if $b \geq a$ we get $g(x_1,0) = a|x_1|$. But if $b < a$, then $g(2,0) = 2b < 2a$, so that $g(2,0) < 2g(1,0) = 2a$. In fact, by the definition of p-infimal convolution, $g(2,0) \leq G_2(2,0) \leq (G^{\frac{1}{p}}(1,1) + G^{\frac{1}{p}}(1,-1))^P \leq G(1,1) + G(1,-1) = 2b$.
For any k,$G_K(2,0) \geq 2b$, so that actually $g(2,0) = 2b$. This is because if we take $k \geq 2$ non zero steps to go from the origin to (2,0), the distance assigned to the path is at least $G(x^1) + ... + G(x^k) \geq k^P b$.

Example4.2. We always have $g \leq G$, and it may happen that $g(x) < G(x)$ for some pixel $x \in W$. Let for instance $G(1,0) = a, G(2,1) = c$, and extend G by reflection and permutation of the coordinates. Then
$g(1,0) \leq G_3(1,0) \leq (G^{\frac{1}{p}}(2,1) + G^{\frac{1}{p}}(1,-2) + G^{\frac{1}{p}}(-2,1))^p = 3c$, So if $3c < a$ we get $g(1,0) \leq 3c < a = G(1,0)$. this is undesirable, because we expect the prime distance originally defined between the origin and $(1,0) \in W$ to survive and to be equal to the distance defined by the minimum over all paths. It is therefore natural to require that *g=G* everywhere in *W*. For *p=1* see, [6] and also, [3], [4].

Proposition4.1. [8]: let G be as in corollary 4.1. Then the sequence $G_j(x)$ is pointwise stationary, i.e., for every $x \in X$ there is an index j_x such that $G_j(x) = g(x)$ for all $j \geq j_x$.

5. p-Regularity Properties

We say that an arbitrary function is *p-semiregular* if $G(x) < +\infty$, $mx = y^1 + ... + y^j$ implies $m^p G(x) \leq G(y^1) + ... + G(y^j)$. (5.1)

G is said to be *p-regular* if for any point mx with $x \in W$ and $m \in N$ and any representation $mx = y^1 + ... + y^j$ with $y^i \in A$ but not all equal to x or 0 we have a strict inequality $m^p G(x) < G(y^1) + ... + G(y^j)$.

Thus p-regularity means that the arc segment is the unique minimal path from 0 to mx, whereas p-semiregularity means that the arc segment from 0 to mx is minimal, but not necessarily the only minimal path.
For $p=1$ we get the concept regularity due to [8].
In what follows we give a sufficient condition to have a p-semi regular function.

Proposition 5.2. Let $G : X \to [-\infty, \infty]$ be a function on an abelian group X. Assume that there exists a function f such that,
1. f is a positively p-homogeneous
2. f is a p-midpoint convex
3 . f agrees with G wherever G is less than $+\infty$
Then G is p-semi regular.
Proof. Let $mx = y^1 + ... + y^j$. We shall prove that $m^p G(x) \le G(y^1) + ... + G(y^j)$ when $x \in dom G$. If one of the $G(y^i) = +\infty$, this inequality certainly holds; on the other hand, if $y^i \in dom G$ for all i, we know that $G(x) = f(x)$ and $G(y^i) = f(y^i)$, so that the inequality follows from the subadditivity of f:
$m^p G(x) = m^p f(x) = f(mx) \le f(y^1) + ... + f(y^j) = G(y^1) + ... + G(y^j)$ ■

6. p-Semiregularity of Distances in Two Dimensions

In this section we let X be the image plane Z^2, we can then embed it into the Euclidean plane R^2 and use also functions defined there. We define a prime distance G on all rays $R^+ w^i$ as follows: $f(sw^i) = s^p G(w^i)$ for $s \in R^+$ and w^i an element of W. This makes sense if two different rays $R^+ w^i$ and $R^+ w^j$ intersect only at the origin, so we assume this to be true. Here we denote by R^+ the set of all nonnegative real numbers, so that $R^+ w$ is the ray from the origin through w:
$R^+ w = \{tw; t \in R, t \ge 0\}$.
We then extend f to all of R^2 so that it becomes linear in each sector defined by two neighboring vectors in W. To make this precise we let the nonzero elements of W be $w^1, ..., w^k$ enumerated in the R^2 counterclockwise direction, so that w^1 and w^2 define a sector free from elements of W, and so on, until the sector defined by w^k and w^1. In the sector defined by w^i and w^{i+1} we define

$$f(sw^i + tw^{i+1}) = s^p G(w^i) + t^p G(w^{i+1}), \quad s, t \in R^+ \tag{6.1}$$

Here, of course, w^{i+1} shall be understood as w^1 if $i=k$.

The function f will then actually be piecewise linear on R^2, and defines a distance there as well as on Z^2. The following lemma will be used to prove theorem 6.1.

Lemma 6.1. Let f be a real valued function defined on an abelian group X such that

$$f(-x) = f(x) \geq 0 \text{ with equality only for } x = 0; \tag{6.2}$$

And, $f(2x) = 2^p f(x) \quad \forall x \in X$ (6.3)

Define a distance on X by $d(x,y) = f(x-y)$. Then d is a metric if and only if f is p-midpoint convex on a convex set.

Proof. The properties (2.1) and (2.2) of a metric being obviously fulfilled, the only question can be whether d satisfies the triangle inequality (2.3). If f is p-midpoint convex we get

$$\begin{aligned} d(x,z) = f(x-z) &\leq \frac{1}{2^p} f(2x-2y) + \frac{1}{2^p} f(2y-2z) \\ &= f(x-y) + f(y-z) = \\ &= d(x,y) + d(y,z) \end{aligned}$$

So the triangle inequality is true.

Conversely, suppose now that the triangle inequality holds, then we claim that f is p-midpoint convex. We note that,

$$2^p f(x) = f(2x) = d(2x,0) \leq d(2x, x-y) + d(x-y, 0) = f(x+y) + f(x-y) \blacksquare$$

The following result gives some equivalent concepts on semiregularity.

Theorem 6.1: Let a finite symmetric set W in Z^2 be given, $W = \{w^0, w^1, ..., w^k\}$, where $w^0 = 0$. Assume that two rays $R^+ w^i$ and $R^+ w^j$ intersect only at the origin and that W contains two linearly independent vectors. Let a symmetric function G be given with finite positive values in $W/\{0\}$, $G(0) = 0$ and the value $+\infty$ outside W. Then define f on R^2 to be equal to G on W, to be a positively p-homogeneous and piecewise linear in each sector which does not contain any point from W in its interior. Explicitly, this means that we define f by (6.1) above. The following five conditions are equivalent:

A. The prime distance function G is p-semiregular.

B. The function f is (p-midpoint convex) p-convex in R^2;

C. The restriction of f to Z^2 is p-midpoint convex;

D. The distance $d_f(x,y) = f(x-y)$ is a metric on R^2;

E. The distance $d_f(x,y) = f(x-y)$ is a metric on Z^2.

Proof. In our case the function is continuous, so p-midpoint convexity is equivalent to p-convexity. We also note that $f \geq g$ on $Nw^{i} + Nw^{i+1}$ if g is the function constructed from *G* as in Corollary 4.1. Indeed,

$$f(kw^{j} + mw^{j+1}) = k^{p}G(w^{j}) + m^{p}G(w^{j+1}) \geq G_{k+m}(kw^{j} + mw^{j+1}) \geq g(kw^{j} + mw^{j+1})$$

It may happen that $g(x) > f(x)$ for certain pixels x.

In fact B implies C by taking the restriction from R^{2} to Z^{2}. If C holds, it follows from the p-homogeneity that $f(x) \leq \frac{1}{2^{p}}(f(x+y) + f(x-y))$ for all vectors with rational components, and then for all vectors by continuity. The proof that D and E are equivalent is of course similar.

Next we shall prove the equivalence of B and D (and of C and E; the proof is the same).

Having established the equivalence of B, C, D and E, we shall now see that B implies A. Indeed. If *f* is p-convex, then *G* is p-semi regular by Proposition 4.2 (in this case we have $g(x) \geq f(x)$ everywhere in $\boldsymbol{R}^{2}$).

Finally we shall prove that A implies B. Thus, assume that the prime distance function *G* is semiregular. We have to prove that *f* is p-convex, but since it is piecewise linear in the sectors defined by the vectors in *W*, it is enough to prove that the linear interpolation $f_{i-1,i+1}$ of *f* between the rays $R^{+}w^{i-1}$ and $R^{+}w^{i+1}$ lies above *f* on the ray, $\boldsymbol{R}^{+}\boldsymbol{p}^{i}$. The function $f_{i-1,i+1}$ is given by

$$f_{i-1,i+1}(sw^{i-1} + tw^{i+1}) = s^{p}G(w^{i-1}) + t^{p}G(w^{i+1}), \quad s,t \in R^{+}$$

Now the value of $f_{i-1,i+1}$ at a point $x = sw^{j-1} + tw^{i+1}$ in the sector defined by w^{i-1} and w^{i+1} with $s,t \in N$ is precisely the length assigned to the path from *0* to sw^{i-1} followed by the segment from that point to $x = sw^{j-1} + tw^{i+1}$. Suppose now that the latter point is on the ray $\boldsymbol{R}^{+}\boldsymbol{w}^{i}$; thus $sw^{j-1} + tw^{i+1} = rw^{j}$ for some $r \in \boldsymbol{R}^{+}$ .By semiregularity, the value of that length is not smaller than the value of *f* at rw^{j}, assuming *s, t* and *r* to be integers. This means that $f \leq f_{i-1,i+1}$ at the point rw^{j} . In general, if $s,t \in N$, *r* will be a rational number, Therefore *r* will be an integer if we choose *s* and *t* as multiples of some integer. In view of the positive homogeneity of *f* and $f_{i-1,i+1}$ we must then have $f \leq f_{i-1,i+1}$ on the whole ray $\boldsymbol{R}^{+}w^{i}$, which, as we remarked, means that *f* is p-convex.

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