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On the Group of Isometries of the Generalized Taxicab Plane

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Abstract

In this paper, we show that the group of isometries of the plane with generalized taxicab metric, is either the semi direct product of the group D_4 and $T(2)$ or the semi-direct product of Dihedral group D_2 and $T(2)$ where D_4 and D_2 are the (Euclidean) symmetry groups of the square and the line segment, respectively; and T(2) is the group of all translations of the plane.

Mathematics Subject Classification: 20N25

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1 Introduction

Metric spaces are among the most important widely studied topics in mathematics. Mathematicians began to investigate using other metrics different from Euclidean metric [2,4,7]. One of them is the taxicab metric. Today there is a wide application and great interest of the taxicab metric.

Taxicab plane geometry is one of the non-Euclidean geometry which has been introduced by Menger [8] and developed by Krause [6]. The taxicab plane geometry has been studied and improved by some mathematicians. The Taxicab plane $\mathbb{R}^2_T$ is almost the same as the Euclidean plane $\mathbb{R}^2$.

The points and lines are the same, and the angles are measured the same way, but the distance function is different. The taxicab metric is defined using the distance function

$$d_T(A,B) = |x_1 - x_2| + |y_1 - y_2|$$

where $A = (x_1, y_1)$, $B = (x_2, y_2)$ in $\mathbb{R}^2$. The unit circle in $\mathbb{R}^2_T$ is the set of points (x, y) in plane which satisfy the equation $|x| + |y| = 1$.

In [9] , Lawrance J. Wallen altered taxicab distance by redefining in order to get rid of possibly misleading symmetry. For two points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ in $\mathbb{R}^2$, the (slightly generalized) taxicab distance function is defined $d_{T_g}(A,B) = a\,|x_2 - x_1| + b\,|y_2 - y_1|$,where $a, b > 0$. Also, the distance d_{T_g} determines a metric. According to d_{T_g}, the norm of a vector $v = \overrightarrow{OP}$ (O is the origin) is defined by

$$\|v\|_{T_g} = d_{T_g}(O,P) = a\,|x| + b\,|y|\,,$$

where $v = (x, y)$ in $\mathbb{R}^2$. The analytical plane with the distance d_{T_g} will be denoted by $\mathbb{R}^2_{T_g}$.

The unit circle in $\mathbb{R}^2_{T_g}$ is the set of points (x, y) in the plane which satisfy the equation $a\,|x| + b\,|y| = 1$ that its vertex vectors are $v_1 = (\frac{1}{a}, 0)$, $v_2 = (0, \frac{1}{b})$, $v_3 = (-\frac{1}{a}, 0)$ and $v_4 = (0, -\frac{1}{b})$. Without lost the generality, v_i vertex vector can be written as $\left(\frac{1}{a}\cos(i-1)\frac{\pi}{2}, \frac{1}{b}\sin(i-1)\frac{\pi}{2}\right)$, $i = 1, 2, 3, 4$. Also, This unit circle is the set of vectors x in $\mathbb{R}^2_{T_g}$ satisfying $u_i.x = 1$ and the side s_i of the unit circle has the equation $u_i.x = 1$ which $u_1 = (a, b)$, $u_2 = (-a, b)$, $u_3 = (-a, -b)$ and $u_4 = (a, -b)$. If the vector v lies in the sector determined by v_i and v_{i+1}, then $\|v\|_{T_g} = u_i.v = t_i + t_{i+1}$ where t_i, t_{i+1} are nonnegative real numbers such that $v = t_i v_i + t_{i+1} v_{i+1}$.

The shortest distance d_{T_g} between the points A and B is the union of the line segments with the same slopes as v_i and v_{i+1} , $i = \{1, 2, 3, 4\}$, when the vector AB is in the sector obtained by extending the vectors v_i and v_{i+1}. If the slope of the line segment AB is equal to the slope of v_i, $i = \{1, 2, 3, 4\}$, then the d_{T_g}-distance of AB is constant a or b multiple of the Euclidean distance between A and B.

The following lemma gives a functional relation between $d_{T_g}-$distance and d_E-distance (Euclidean distance).

Lemma 1 *Let m be the slope of the line through the points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ in $\mathbb{R}^2$, and let d_E denote the Euclidean metric. Then*

$$d_{T_g}(A,B) = \rho(m) d_E(A,B)$$

where

$$\rho(m) = \begin{cases} \frac{a+b|m|}{\sqrt{1+m^2}}, & m \neq 0 \\ b, & m \to \infty \\ a & m = 0 \end{cases} .$$

Proof. If the line through the points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ is parallel to $x-$axis or $y-$axis, then $m = 0$ and $\frac{a+b|m|}{\sqrt{1+m^2}} = a$ or $m \to \infty$ and $\frac{a+b|m|}{\sqrt{1+m^2}} = b$. If the line through the points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ is not parallel to $x-$axis or $y-$axis, then $x_1 \neq x_2$ and $y_1 \neq y_2$, and

$$d_{T_g}(A, B) = a\,|x_2 - x_1| + b\,|y_2 - y_1| = |x_2 - x_1|\,(a + b\,|m|).$$

Similarly,

$$d_E(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = |x_2 - x_1|\,\sqrt{1 + m^2},$$

for all $m \in \mathbb{R}$. Consequently, given equality is valid. ■

Furthermore, one can immediately obtain from the previous lemma the following:

Corollary 2 *If $A,\ B,\ X$ are any three collinear points in $\mathbb{R}^2$, then $d_E(X, A) = d_E(X, B)$ iff $d_{T_g}(X, A) = d_{T_g}(X, B)$.*

Corollary 3 *If $A,\ B,\ X$ are any three distinct collinear points in $\mathbb{R}^2$, then*

$$\frac{d_{T_g}(X, A)}{d_{T_g}(X, B)} = \frac{d_E(X, A)}{d_E(X, B)}.$$

That is, the ratios of the Euclidean and $d_{T_g}-$distances along a line are the same. Notice that, the latter corollary gives us the validity of the Theorems of Menelaus and Ceva in $\mathbb{R}^2_{T_g}$.

One of the basic problems in geometric investigations for a given space S with a d is to determine the group G of isometries [3, 5, 11, 12, 13]. If S is the Euclidean plane with d_E - metric, then it is well-known that G consists of all translations, rotations, reflections and glide reflections of the plane. The Euclidean Group $G = E(2)$ is the semi-direct product of the symmetry group of the unit circle, $O(2)$, and the translation group consisting of all translations of the plane, $T(2)$, [5, 10, 14].

In the remaining part of this work, we will study the isometries of $\mathbb{R}^2_{T_g}$, and determine its group of isometries.

2 Isometries of the plane $\mathbb{R}^2_{T_g}$

An isometry of a plane is defined to be a transformation which preserves the distances in the plane. Therefore, an isometry of $\mathbb{R}^2_{T_g}$ is an isometry of the real plane with respect to the d_{T_g} metric. Note that T is an isometry for $\|.\|_{T_g}$ if and only if T transforms the unit circle to the unit circle [1].

Proposition 4 *Every Euclidean translation is an isometry of* $\mathbb{R}^2_{T_g}$.

Proof. Let $T_a : \mathbb{R}^2_{T_g} \to \mathbb{R}^2_{T_g}$ such that $T_a(u) = a + u$ be translation as in the real plane $\mathbb{R}^2$, where a is a translation vector in $\mathbb{R}^2_{T_g}$. For any vectors u and v in $\mathbb{R}^2_{T_g}$, we have

$$\begin{aligned} d_{T_g}(T_a(u), T_a(v)) &= d_{T_g}(a+u, a+v) \\ &= \|a+u-(a+v)\|_{T_g} \\ &= \|a+u-(a+v)\|_{T_g} = d_{T_g}(u,v). \end{aligned}$$

That is, every translation T_a is an isometry of $\mathbb{R}^2_{T_g}$ ■

Now, the reflections which preserves the distance in $\mathbb{R}^2_{T_g}$ are determined .in the following proposition.

Proposition 5 *The set of reflections preserving the* $d_{T_g}-$ *distances,* S, *is* $\left\{ f|\ f \text{ is defined by } \begin{pmatrix} \cos\frac{k\pi}{2} & \sin\frac{k\pi}{2} \\ \sin\frac{k\pi}{2} & -\cos\frac{k\pi}{2} \end{pmatrix},\ k \in \{2,4\} \text{ if } a \neq b \text{ or } k \in \{1,2,3,4\} \text{ if } a = b \right\}$.

Proof. It is sufficient to study the reflections of $\mathbb{R}^2$ preserving the diamond, the unit circle of $d_{Tg}-$metric, since every isometric reflection of $\mathbb{R}^2_{T_g}$ preserves the unit circle of $\mathbb{R}^2_{T_g}$. So, we must show that the set, S, preserves the $d_{T_g}-$distances. If the vector $v = \overrightarrow{OA}$ is in the sector obtained by extending the vectors v_i and v_{i+1} in the plane $\mathbb{R}^2_{T_g}$, then $v = t_i v_i + t_{i+1} v_{i+1}$ such that $t_i,\ t_{i+1} > 0$ and $d_{T_g}(O,A) = \|v\|_{T_g} = t_i + t_{i+1}$, $i \in \{1,2,3,4\}$. Firstly, we calculate $f(v)$ for $a = b$ as

$$\begin{aligned} f(v) &= \begin{pmatrix} \cos\frac{k\pi}{2} & \sin\frac{k\pi}{2} \\ \sin\frac{k\pi}{2} & -\cos\frac{k\pi}{2} \end{pmatrix} \begin{pmatrix} t_i(\frac{1}{a}\cos(i-1)\frac{\pi}{2}) + t_{i+1}(\frac{1}{a}\cos\frac{\pi i}{2}) \\ t_i(\frac{1}{a}\sin(i-1)\frac{\pi}{2}) + t_{i+1}(\frac{1}{a}\sin\frac{\pi i}{2}) \end{pmatrix} \\ &= t_i \begin{pmatrix} \frac{1}{a}\cos(k-i+1)\frac{\pi}{2} \\ \frac{1}{a}\sin(k-i+1)\frac{\pi}{2} \end{pmatrix} + t_{i+1} \begin{pmatrix} \frac{1}{a}\cos(k-i)\frac{\pi}{2} \\ \frac{1}{a}\sin(k-i)\frac{\pi}{2} \end{pmatrix} \\ &= t_i v_{k-i+2} + t_{i+1} v_{k-i+1}. \end{aligned}$$

In case $a = b$, we see that $f(v)$ is in the sector obtained by extending v_{k-i+2} and v_{k-i+1} , $k \in \{1,2,3,4\}$. Now, we calculate $f(v)$ for $a \neq b$ as

$$\begin{aligned} f(v) &= \begin{pmatrix} \cos\frac{k\pi}{2} & \sin\frac{k\pi}{2} \\ \sin\frac{k\pi}{2} & -\cos\frac{k\pi}{2} \end{pmatrix} \begin{pmatrix} t_i(\frac{1}{a}\cos(i-1)\frac{\pi}{2}) + t_{i+1}(\frac{1}{a}\cos\frac{\pi i}{2}) \\ t_i(\frac{1}{b}\sin(i-1)\frac{\pi}{2}) + t_{i+1}(\frac{1}{b}\sin\frac{\pi i}{2}) \end{pmatrix} \\ &= t_i \begin{pmatrix} \frac{1}{a}\cos\frac{(i-1)\pi}{2}\cos\frac{k\pi}{2} + \frac{1}{b}\sin\frac{(i-1)\pi}{2}\sin\frac{k\pi}{2} \\ \frac{1}{a}\cos\frac{(i-1)\pi}{2}\sin\frac{k\pi}{2} - \frac{1}{b}\sin\frac{(i-1)\pi}{2}\cos\frac{k\pi}{2} \end{pmatrix} + t_{i+1} \begin{pmatrix} \frac{1}{a}\cos\frac{i\pi}{2}\cos\frac{k\pi}{2} + \frac{1}{b}\sin\frac{i\pi}{2}\sin\frac{k\pi}{2} \\ \frac{1}{a}\cos\frac{i\pi}{2}\sin\frac{k\pi}{2} - \frac{1}{b}\sin\frac{i\pi}{2}\cos\frac{k\pi}{2} \end{pmatrix}. \end{aligned}$$

For $k \in \{2,4\}$, we obtain that $f(v) = t_i v_{k-i+2} + t_{i+1} v_{k-i+1}$, $i \in \{1,2,3,4\}$. In case $a \neq b$, we see that $f(v)$ is in the sector obtained by extending two consecutive vertex vectors. Then

$$\begin{aligned} d_{T_g}\left(f(O), f(A)\right) &= d_{T_g}\left(O, f(v)\right) = \|f(v)\|_{T_g} \\ &= u_{k-i+1}.(t_i v_{k-i+2} + t_{i+1} v_{k-i+1}) \\ &= t_i(u_{k-i+1}.v_{k-i+2}) + t_{i+1}(u_{k-i+1}.v_{k-i+1}) \\ &= t_i + t_{i+1} = \|v\|_{T_g} \end{aligned}$$

It is seen that the norm of $f(v)$ is preserved according to $d_{T_g}-$distances in $\mathbb{R}^2_{T_g}$ in cases $a = b$ and $a \neq b$.This result completes the proof. ■

In the following proposition, the rotations which preserves $d_{T_g}-$distances in $\mathbb{R}^2_{T_g}$ are determined.

Proposition 6 *The set of isometric rotations in $\mathbb{R}^2_{T_g}$, R, is*

$$\left\{ r_\theta |\ r_\theta \text{ is a rotation with the angle } \theta = \frac{k\pi}{2},\ k \in \{2,4\} \text{ if } a \neq b \text{ or } k \in \{1,2,3,4\} \text{ if } a = b \right\}.$$

Proof. It is enough to determine the rotations preserving the unit circle of $\mathbb{R}^2_{T_g}$, since every isometric rotation of $\mathbb{R}^2_{T_g}$ must preserve the unit circle of $\mathbb{R}^2_{T_g}$. Therefore it will be shown that the set of rotations preserving the unit circle of $\mathbb{R}^2_{T_g}$, R , preserves $d_{T_g}-$distances. When the vector $v = \overrightarrow{OA}$ is in the sector obtained by extending the vectors v_i and v_{i+1} in the plane $\mathbb{R}^2_{T_g}$, it is known that $v = t_i v_i + t_{i+1} v_{i+1}$, $t_i, t_{i+1} \geqslant 0$ and $d_{T_g}\left(O, A\right) = \|v\|_{T_g} = t_i + t_{i+1}$, $i \in \{1,2,3,4\}$. $r_\theta(v)$ is calculated for $a = b$ as

$$\begin{aligned} r_\theta(\overrightarrow{OA}) = r_\theta(v) &= \begin{pmatrix} \cos\frac{k\pi}{2} & -\sin\frac{k\pi}{2} \\ \sin\frac{k\pi}{2} & \cos\frac{k\pi}{2} \end{pmatrix} \begin{pmatrix} t_i(\frac{1}{a}\cos(i-1)\frac{\pi}{2}) + t_{i+1}(\frac{1}{a}\cos\frac{\pi i}{2}) \\ t_i(\frac{1}{a}\sin(i-1)\frac{\pi}{2}) + t_{i+1}(\frac{1}{a}\sin\frac{\pi i}{2}) \end{pmatrix} \\ &= t_i \begin{pmatrix} \frac{1}{a}\cos(k+i-1)\frac{\pi}{2} \\ \frac{1}{a}\sin(k+i-1)\frac{\pi}{2} \end{pmatrix} + t_{i+1} \begin{pmatrix} \frac{1}{a}\cos(k+i)\frac{\pi}{2} \\ \frac{1}{a}\sin(k+i)\frac{\pi}{2} \end{pmatrix} \\ &= t_i v_{k+i} + t_{i+1} v_{k+i+1}. \end{aligned}$$

Thus, $r_\theta(v)$ is in the sector obtained by extending v_{k+i} and v_{k+i+1}, $k \in \{1,2,3,4\}$. $r_\theta(v)$ is calculated for $a \neq b$ as

$$r_\theta(\overrightarrow{OA}) = \begin{pmatrix} \cos\frac{k\pi}{2} & -\sin\frac{k\pi}{2} \\ \sin\frac{k\pi}{2} & \cos\frac{k\pi}{2} \end{pmatrix} \begin{pmatrix} t_i(\frac{1}{a}\cos(i-1)\frac{\pi}{2}) + t_{i+1}(\frac{1}{a}\cos\frac{\pi i}{2}) \\ t_i(\frac{1}{b}\sin(i-1)\frac{\pi}{2}) + t_{i+1}(\frac{1}{b}\sin\frac{\pi i}{2}) \end{pmatrix}$$

$$= t_i \begin{pmatrix} \frac{1}{a}\cos\frac{(i-1)\pi}{2}\cos\frac{k\pi}{2} - \frac{1}{b}\sin\frac{(i-1)\pi}{2}\sin\frac{k\pi}{2} \\ \frac{1}{a}\cos\frac{(i-1)\pi}{2}\sin\frac{k\pi}{2} + \frac{1}{b}\sin\frac{(i-1)\pi}{2}\cos\frac{k\pi}{2} \end{pmatrix} + t_{i+1} \begin{pmatrix} \frac{1}{a}\cos\frac{i\pi}{2}\cos\frac{k\pi}{2} - \frac{1}{b}\sin\frac{i\pi}{2}\sin\frac{k\pi}{2} \\ \frac{1}{a}\cos\frac{i\pi}{2}\sin\frac{k\pi}{2} + -\frac{1}{b}\sin\frac{i\pi}{2}\cos\frac{k\pi}{2} \end{pmatrix}.$$

It is obtain that $r_\theta(v) = t_i v_{k+i} + t_{i+1} v_{k+i+1}$, $k = \{2,4\}$, $i \in \{1,2,3,4\}$. In case $a \neq b$, we see that $r_\theta(v)$ is in the sector obtained by extending two consecutive vertex vectors. Then

$$\begin{aligned} d_{T_g}\left(r_\theta(O), r_\theta(A)\right) &= d_{T_g}\left(O, r_\theta(v)\right) = \left\|r_\theta(v)\right\|_{T_g} \\ &= u_{k+i}.(t_i v_{k+i} + t_{i+1} v_{k+i+1}) \\ &= t_i(u_{k+i}.v_{k+i}) + t_{i+1}(u_{k+i}.v_{k+i+1}) \\ &= t_i + t_{i+1} = \|v\|_{T_g}. \end{aligned}$$

So, the norm of $r_\theta(v)$ is preserved according to $d_{T_g}-$distances in $\mathbb{R}^2_{T_g}$ for $a \neq b$ and $a = b$. This result completes the proof.

Thus we have the orthogonal group, $O_{T_g}(2)$, consisting of four reflections and four rotations when $a = b$ and consisting of two reflections and two rotations when $a \neq b$

$$O_{T_g}(2) = R \cup S,$$

which gives us *Dihedral group* D_4 for $a = b$ and *Dihedral group* D_2 for $a \neq b$, that is, the Euclidean symmetry group of a square for $a = b$ and the Euclidean symmetry group of the a line segment for $a \neq b$. Now, it will be shown that all isometries of $\mathbb{R}^2_{T_g}$ are in $T(2).O_{T_g}(2)$. ■

Definition 7 *Let $A = (a_1, a_2)$, $B = (b_1, b_2)$ be two points in $\mathbb{R}^2_{T_g}$. The minimum distance set of A, B is defined by*

$$\left\{X \mid d_{T_g}(A, X) + d_{T_g}(B, X) = d_{T_g}(A, B)\right\}$$

and denoted by $\overset{\square}{AB}$.

Let m_{AB} denotes the slope of the line through the points A and B. If the slope of AB is the same slope with v_i, $i \in \{1, 2, 3, 4\}$, the set $\overset{\square}{AB}$ is the line segment joining A and B, that is, $\overset{\square}{AB} = \overline{AB}$. It is called that $\overset{\square}{AB}$ is the standard rectangle with diagonal $\overline{AB}$. If the vector AB in the sector joining v_i and v_{i+1} vectors, $\overset{\square}{AB}$ is the standard rectangle with diagonal $\overline{AB}$ and the sides of it are parallel to v_i and v_{i+1}.

Proposition 8 *Let $\phi : \mathbb{R}^2_{T_g} \to \mathbb{R}^2_{T_g}$ be an isometry and let $\overset{\square}{AB}$ be the standard rectangle. Then*

$$\phi(\overset{\square}{AB}) = \phi(A)\overset{\square}{\phi}(B).$$

Proof. Let $Y \in \phi(\overset{\square}{AB})$. Then,

$$\begin{aligned} Y \in \phi(\overset{\square}{AB}) &\Leftrightarrow \exists X \in \overset{\square}{AB} \text{ such that } Y = \phi(X) \\ &\Leftrightarrow d_{T_g}(A, X) + d_{T_g}(B, X) = d_{T_g}(A, B) \\ &\Leftrightarrow d_{T_g}(\phi(A), \phi(X)) + d_{T_g}(\phi(X), \phi(B)) = d_{T_g}(\phi(A), \phi(B)) \\ &\Leftrightarrow Y = \phi(X) \in \phi(A)\overset{\square}{\phi}(B) \end{aligned}$$

■

Corollary 9 *Let $\phi : \mathbb{R}^2_{T_g} \to \mathbb{R}^2_{T_g}$ be an isometry and let $\overset{\square}{AB}$ be the standard rectangle. Then ϕ maps vertices to vertices and preserves the lengths of sides of $\overset{\square}{AB}$.*

Proposition 10 *Let $\phi : \mathbb{R}^2_{T_g} \to \mathbb{R}^2_{T_g}$ be an isometry such that $\phi(O) = O$. Then $\phi \in R$ or $\phi \in S$.*

Proof. It is known that the vertices of unit circle in $\mathbb{R}^2_{T_g}$ are $A_1 = (\frac{1}{a}, 0)$, $A_2 = (0, \frac{1}{b})$, $A_3 = (-\frac{1}{a}, 0)$ and $A_4 = (0, -\frac{1}{b})$.Let the standard rectangle $\overset{\square}{OD}$ be considered that its vertices are $A_1 = (\frac{1}{a}, 0)$, $A_2 = (0, \frac{1}{b})$, $D = (\frac{1}{a}, \frac{1}{b})$.

It is clear that $\phi(A_1) \in \overline{A_i A_{i+1}}$, $i \in \{1, 2, 3, 4\}$. Since ϕ is an isometry by Corollary 9, $\phi(A_1)$ and $\phi(A_2)$ must be the vertices of the standard rectangle $\overset{\square}{O\phi(D)}$. Therefore, if $\phi(A_1) \in \overline{A_i A_{i+1}}$, then $\phi(A_1) = A_i$ or $\phi(A_1) = A_{i+1}$. Similarly $\phi(A_2) = A_i$ or $\phi(A_2) = A_{i+1}$. In case $a \neq b$,when $\phi(A_1) = A_i$ and $\phi(A_2) = A_{i+1}$, ϕ is an rotation with the angle $\theta = \frac{(i-1)\pi}{2}$, $i \in \{1, 3\}$. If $\phi(A_1) = A_{i+1}$ and $\phi(A_2) = A_i$, ϕ is an reflection in the line with the angle $\theta = \frac{i\pi}{4}$, $i \in \{2, 4\}$ (which are $y-$axes and $x-$axes). In case $a = b$, when $\phi(A_1) = A_i$ and $\phi(A_2) = A_{i+1}$, ϕ is an rotation with the angle $\theta = \frac{(i-1)\pi}{2}$, $i \in \{1, 2, 3, 4\}$. If $\phi(A_1) = A_{i+1}$ and $\phi(A_2) = A_i$, ϕ is an reflection in the line with the angle $\theta = \frac{i\pi}{4}$, $i \in \{1, 2, 3, 4\}$ (which are $y = x$, $y-$axes , $y = -x$ and $x-$axes). Consequently, $\phi \in R$ or $\phi \in S$. ■

Theorem 11 *Let $f : \mathbb{R}^2_{T_g} \to \mathbb{R}^2_{T_g}$ be an isometry. Then there exists a unique $T_a \in T(2)$ and $\phi \in O_{T_g}(2)$ such that $f = T_a \circ \phi$.*

Proof. Let $f(O) = A$ where $A = (a_1, a_2)$. Define $\phi = T_{-a} \circ f$. It is known that ϕ is an isometry and $\phi(O) = O$. Thus, $\phi \in O_{T_g}(2)$ and $f = T_a \circ \phi$ by Proposition 10. The proof of uniqueness is trivial. ■

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