On the Group of Isometries of the Generalized Taxicab Plane

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Abstract
In this paper, we show that the group of isometries of the plane with generalized taxicab metric, is either the semi direct product of the group $D_4$ and $T(2)$ or the semi-direct product of Dihedral group $D_2$ and $T(2)$ where $D_4$ and $D_2$ are the (Euclidean) symmetry groups of the square and the line segment, respectively; and $T(2)$ is the group of all translations of the plane.

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1 Introduction
Metric spaces are among the most important widely studied topics in mathematics. Mathematicians began to investigate using other metrics different from Euclidean metric [2,4,7]. One of them is the taxicab metric. Today there is a wide application and great interest of the taxicab metric.

Taxicab plane geometry is one of the non-Euclidean geometry which has been introduced by Menger [8] and developed by Krause [6]. The taxicab plane geometry has been studied and improved by some mathematicians. The Taxicab plane $\mathbb{R}_T^2$ is almost the same as the Euclidean plane $\mathbb{R}^2$. 
The points and lines are the same, and the angles are measured the same way, but the distance function is different. The taxicab metric is defined using the distance function

\[ d_T(A, B) = |x_1 - x_2| + |y_1 - y_2| \]

where \( A = (x_1, y_1) \), \( B = (x_2, y_2) \) in \( \mathbb{R}^2 \). The unit circle in \( \mathbb{R}^2 \) is the set of points \((x, y)\) in plane which satisfy the equation \(|x| + |y| = 1\).

In [9], Lawrance J. Wallen altered taxicab distance by redefining in order to get rid of possibly misleading symmetry. For two points \( A = (x_1, y_1) \) and \( B = (x_2, y_2) \) in \( \mathbb{R}^2 \), the (slightly generalized) taxicab distance function is defined \( d_{T_g}(A, B) = a |x_2 - x_1| + b |y_2 - y_1| \), where \( a, b > 0 \). Also, the distance \( d_{T_g} \) determines a metric. According to \( d_{T_g} \), the norm of a vector \( v = \overrightarrow{OP} \) (\( O \) is the origin) is defined by

\[ \|v\|_{T_g} = d_{T_g}(O, P) = a |x| + b |y|, \]

where \( v = (x, y) \) in \( \mathbb{R}^2 \). The analytical plane with the distance \( d_{T_g} \) will be denoted by \( \mathbb{R}^2_{T_g} \).

The unit circle in \( \mathbb{R}^2_{T_g} \) is the set of points \((x, y)\) in the plane which satisfy the equation \( a |x| + b |y| = 1 \) that its vertex vectors are \( v_1 = (\frac{1}{a}, 0) \), \( v_2 = (0, \frac{1}{b}) \), \( v_3 = (-\frac{1}{a}, 0) \) and \( v_4 = (0, -\frac{1}{b}) \). Without lost the generality, \( v_i \) vertex vector can be written as \( (\frac{1}{a} \cos(i - 1)\frac{\pi}{2}, \frac{1}{b} \sin(i - 1)\frac{\pi}{2}) \), \( i = 1, 2, 3, 4 \). Also, This unit circle is the set of vectors \( x \) in \( \mathbb{R}^2_{T_g} \) satisfying \( u_i.x = 1 \) and the side \( s_i \) of the unit circle has the equation \( u_i.x = 1 \) which \( u_1 = (a, b) \), \( u_2 = (-a, b) \), \( u_3 = (-a, -b) \) and \( u_4 = (a, -b) \). If the vector \( v \) lies in the sector determined by \( v_i \) and \( v_{i+1} \), then \( \|v\|_{T_g} = u_i.v = t_i + t_{i+1} \) where \( t_i, t_{i+1} \) are nonnegative real numbers such that \( v = t_i v_i + t_{i+1} v_{i+1} \).

The shortest distance \( d_{T_g} \) between the points \( A \) and \( B \) is the union of the line segments with the same slopes as \( v_i \) and \( v_{i+1} \), \( i = \{1, 2, 3, 4\} \), when the vector \( AB \) is in the sector obtained by extending the vectors \( v_i \) and \( v_{i+1} \). If the slope of the line segment \( AB \) is equal to the slope of \( v_i \), \( i = \{1, 2, 3, 4\} \), then the \( d_{T_g} \)-distance of \( AB \) is constant \( a \) or \( b \) multiple of the Euclidean distance between \( A \) and \( B \).

The following lemma gives a functional relation between \( d_{T_g} \)-distance and \( d_E \)-distance (Euclidean distance).

**Lemma 1** Let \( m \) be the slope of the line through the points \( A = (x_1, y_1) \) and \( B = (x_2, y_2) \) in \( \mathbb{R}^2 \), and let \( d_E \) denote the Euclidean metric. Then

\[ d_{T_g}(A, B) = \rho(m) d_E(A, B) \]

where

\[ \rho(m) = \begin{cases} \frac{a + b|m|}{\sqrt{1 + m^2}}, & m \neq 0 \\ b, & m \to \infty \\ a, & m = 0 \end{cases} \]
**Proof.** If the line through the points \( A = (x_1, y_1) \) and \( B = (x_2, y_2) \) is parallel to \( x \)-axis or \( y \)-axis, then \( m = 0 \) and \( \frac{a+bm}{\sqrt{1+m^2}} = a \) or \( m \to \infty \) and \( \frac{a+bm}{\sqrt{1+m^2}} = b \).

If the line through the points \( A = (x_1, y_1) \) and \( B = (x_2, y_2) \) is not parallel to \( x \)-axis or \( y \)-axis, then \( x_1 \neq x_2 \) and \( y_1 \neq y_2 \), and

\[
d_{T_g}(A, B) = a|x_2 - x_1| + b|y_2 - y_1| = |x_2 - x_1|(a + b|m|).
\]

Similarly,

\[
d_E(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = |x_2 - x_1|\sqrt{1 + m^2},
\]

for all \( m \in \mathbb{R} \). Consequently, given equality is valid. \( \blacksquare \)

Furthermore, one can immediately obtain from the previous lemma the following:

**Corollary 2** If \( A, B, X \) are any three collinear points in \( \mathbb{R}^2 \), then \( d_E(X, A) = d_E(X, B) \) iff \( d_{T_g}(X, A) = d_{T_g}(X, B) \).

**Corollary 3** If \( A, B, X \) are any three distinct collinear points in \( \mathbb{R}^2 \), then

\[
\frac{d_{T_g}(X, A)}{d_{T_g}(X, B)} = \frac{d_E(X, A)}{d_E(X, B)}.
\]

That is, the ratios of the Euclidean and \( d_{T_g} \)-distances along a line are the same. Notice that, the latter corollary gives us the validity of the Theorems of Menelaus and Ceva in \( \mathbb{R}^2_{T_g} \).

One of the basic problems in geometric investigations for a given space \( S \) with a \( d \) is to determine the group \( G \) of isometries \([3, 5, 11, 12, 13]\). If \( S \) is the Euclidean plane with \( d_E \)-metric, then it is well-known that \( G \) consists of all translations, rotations, reflections and glide reflections of the plane. The Euclidean Group \( G = E(2) \) is the semi-direct product of the symmetry group of the unit circle, \( O(2) \), and the translation group consisting of all translations of the plane, \( T(2) \), \([5, 10, 14]\).

In the remaining part of this work, we will study the isometries of \( \mathbb{R}^2_{T_g} \), and determine its group of isometries.

## 2 Isometries of the plane \( \mathbb{R}^2_{T_g} \)

An isometry of a plane is defined to be a transformation which preserves the distances in the plane. Therefore, an isometry of \( \mathbb{R}^2_{T_g} \) is an isometry of the real plane with respect to the \( d_{T_g} \) metric. Note that \( T \) is an isometry for \( \| \cdot \|_{T_g} \) if and only if \( T \) transforms the unit circle to the unit circle \([1]\).
Proposition 4 Every Euclidean translation is an isometry of $\mathbb{R}^2_{T_g}$.

Proof. Let $T_a : \mathbb{R}^2_{T_g} \to \mathbb{R}^2_{T_g}$ such that $T_a(u) = a + u$ be translation as in the real plane $\mathbb{R}^2$, where $a$ is a translation vector in $\mathbb{R}^2_{T_g}$. For any vectors $u$ and $v$ in $\mathbb{R}^2_{T_g}$, we have

$$d_{T_g}(T_a(u), T_a(v)) = d_{T_g}(a + u, a + v) = \|a + u - (a + v)\|_{T_g} = \|a + u - (a + v)\|_{T_g} = d_{T_g}(u, v).$$

That is, every translation $T_a$ is an isometry of $\mathbb{R}^2_{T_g}$.

Now, the reflections which preserves the distance in $\mathbb{R}^2_{T_g}$ are determined in the following proposition.

Proposition 5 The set of reflections preserving the $d_{T_g}$-distances, $S$, is

$$\{ f \mid f \text{ is defined by } \left( \begin{array}{c} \cos\frac{k\pi}{2} \\ \sin\frac{k\pi}{2} \\ \sin\frac{k\pi}{2} \\ -\cos\frac{k\pi}{2} \end{array} \right), \ k \in \{2, 4\} \text{ if } a \neq b \text{ or } k \in \{1, 2, 3, 4\} \text{ if } a = b \}.$$ 

Proof. It is sufficient to study the reflections of $\mathbb{R}^2$ preserving the diamond, the unit circle of $d_{T_g}$-metric, since every isometric reflection of $\mathbb{R}^2_{T_g}$ preserves the unit circle of $\mathbb{R}^2_{T_g}$. So, we must show that the set, $S$, preserves the $d_{T_g}$-distances. If the vector $v = \overrightarrow{OA}$ is in the sector obtained by extending the vectors $v_i$ and $v_{i+1}$ in the plane $\mathbb{R}^2_{T_g}$, then $v = t_i v_i + t_{i+1} v_{i+1}$ such that $t_i, t_{i+1} > 0$ and $d_{T_g}(O, A) = \|v\|_{T_g} = t_i + t_{i+1}$, $i \in \{1, 2, 3, 4\}$. Firstly, we calculate $f(v)$ for $a = b$ as

$$f(v) = \left( \begin{array}{c} \cos\frac{k\pi}{2} \\ \sin\frac{k\pi}{2} \\ \sin\frac{k\pi}{2} \\ -\cos\frac{k\pi}{2} \end{array} \right) \left( \begin{array}{c} t_i \left( \frac{1}{a} \cos(i - 1) \frac{\pi}{2} \right) + t_{i+1} \left( \frac{1}{a} \cos \frac{3\pi}{2} \right) \\ t_i \left( \frac{1}{a} \sin(i - 1) \frac{\pi}{2} \right) + t_{i+1} \left( \frac{1}{a} \sin \frac{\pi}{2} \right) \end{array} \right) = t_i v_{k-i+2} + t_{i+1} v_{k-i+1}.$$

In case $a = b$, we see that $f(v)$ is in the sector obtained by extending $v_{k-i+2}$ and $v_{k-i+1}, \ k \in \{1, 2, 3, 4\}$. Now, we calculate $f(v)$ for $a \neq b$ as

$$f(v) = \left( \begin{array}{c} \cos\frac{k\pi}{2} \\ \sin\frac{k\pi}{2} \\ \sin\frac{k\pi}{2} \\ -\cos\frac{k\pi}{2} \end{array} \right) \left( \begin{array}{c} t_i \left( \frac{1}{a} \cos(i - 1) \frac{\pi}{2} \right) + t_{i+1} \left( \frac{1}{a} \cos \frac{3\pi}{2} \right) \\ t_i \left( \frac{1}{b} \sin(i - 1) \frac{\pi}{2} \right) + t_{i+1} \left( \frac{1}{b} \sin \frac{\pi}{2} \right) \end{array} \right) = t_i \left( \frac{1}{a} \cos \frac{(i-1)\pi}{2} \cos \frac{k\pi}{2} + \frac{1}{b} \sin \frac{(i-1)\pi}{2} \sin \frac{k\pi}{2} \right) + t_{i+1} \left( \frac{1}{a} \cos \frac{\pi}{2} \frac{k\pi}{2} + \frac{1}{b} \sin \frac{\pi}{2} \sin \frac{k\pi}{2} \right).$$

For $k \in \{2, 4\}$, we obtain that $f(v) = t_i v_{k-i+2} + t_{i+1} v_{k-i+1}, \ i \in \{1, 2, 3, 4\}$. In case $a \neq b$, we see that $f(v)$ is in the sector obtained by extending two
consecutive vertex vectors. Then
\[
d_{T_g}(f(O), f(A)) = d_{T_g}(O, f(v)) = \|f(v)\|_{T_g} = u_{k-i+1}(t_i v_{k-i+2} + t_{i+1} v_{k-i+1}) = t_i(u_{k-i+1} v_{k-i+2}) + t_{i+1}(u_{k-i+1} v_{k-i+1}) = t_i + t_{i+1} = \|v\|_{T_g}
\]

It is seen that the norm of \( f(v) \) is preserved according to \( d_{T_g} \)-distances in \( \mathbb{R}^2_{T_g} \), in cases \( a = b \) and \( a \neq b \). This result completes the proof. □

In the following proposition, the rotations which preserves \( d_{T_g} \)-distances in \( \mathbb{R}^2_{T_g} \) are determined.

**Proposition 6** The set of isometric rotations in \( \mathbb{R}^2_{T_g}, R, \) is

\[
\left\{ r_\theta \mid r_\theta \text{ is a rotation with the angle } \theta = \frac{k\pi}{2}, \ k \in \{2, 4\} \text{ if } a \neq b \text{ or } k \in \{1, 2, 3, 4\} \text{ if } a = b \right\}.
\]

**Proof.** It is enough to determine the rotations preserving the unit circle of \( \mathbb{R}^2_{T_g} \), since every isometric rotation of \( \mathbb{R}^2_{T_g} \) must preserve the unit circle of \( \mathbb{R}^2_{T_g} \). Therefore it will be shown that the set of rotations preserving the unit circle of \( \mathbb{R}^2_{T_g} \), \( R \), preserves \( d_{T_g} \)-distances. When the vector \( v = \overrightarrow{OA} \) is in the sector obtained by extending the vectors \( v_i \) and \( v_{i+1} \) in the plane \( \mathbb{R}^2_{T_g} \), it is known that \( v = t_i v_i + t_{i+1} v_{i+1} \), \( t_i, t_{i+1} \geq 0 \) and \( d_{T_g}(O, A) = \|v\|_{T_g} = t_i + t_{i+1}, \ i \in \{1, 2, 3, 4\} \). \( r_\theta(v) \) is calculated for \( a = b \) as

\[
r_\theta(\overrightarrow{OA}) = r_\theta(v) = \left( \begin{array}{c} \cos \frac{k\pi}{2} - \sin \frac{k\pi}{2} \\ \sin \frac{k\pi}{2} \cos \frac{k\pi}{2} \end{array} \right) \left( \begin{array}{c} t_i \left( \frac{1}{a} \cos(i - 1) \frac{\pi}{2} \right) + t_{i+1} \left( \frac{1}{a} \cos \frac{\pi}{2} \right) \\ t_i \left( \frac{1}{a} \sin(i - 1) \frac{\pi}{2} \right) + t_{i+1} \left( \frac{1}{a} \sin \frac{\pi}{2} \right) \end{array} \right)
\]

\[
= t_i \left( \frac{1}{a} \cos(k + i - 1) \frac{\pi}{2} \right) + t_{i+1} \left( \frac{1}{a} \sin(k + i) \frac{\pi}{2} \right)
\]

Thus, \( r_\theta(v) \) is in the sector obtained by extending \( v_{k+i} \) and \( v_{k+i+1}, \ k \in \{1, 2, 3, 4\} \). \( r_\theta(v) \) is calculated for \( a \neq b \) as

\[
r_\theta(\overrightarrow{OA}) = \left( \begin{array}{c} \cos \frac{k\pi}{2} - \sin \frac{k\pi}{2} \\ \sin \frac{k\pi}{2} \cos \frac{k\pi}{2} \end{array} \right) \left( \begin{array}{c} t_i \left( \frac{1}{a} \cos(i - 1) \frac{\pi}{2} \right) + t_{i+1} \left( \frac{1}{a} \cos \frac{\pi}{2} \right) \\ t_i \left( \frac{1}{b} \sin(i - 1) \frac{\pi}{2} \right) + t_{i+1} \left( \frac{1}{b} \sin \frac{\pi}{2} \right) \end{array} \right)
\]

\[
= t_i \left( \frac{1}{a} \cos(i - 1) \frac{\pi}{2} \right) - \frac{1}{b} \sin(i - 1) \frac{\pi}{2} \frac{\cos \frac{k\pi}{2}}{\sin \frac{k\pi}{2}} + t_{i+1} \left( \frac{1}{a} \cos \frac{\pi}{2} \frac{\cos \frac{k\pi}{2}}{\sin \frac{k\pi}{2}} - \frac{1}{b} \sin \frac{\pi}{2} \frac{\cos \frac{k\pi}{2}}{\sin \frac{k\pi}{2}} \right)
\]

It is obtain that \( r_\theta(v) = t_i v_{k+i} + t_{i+1} v_{k+i+1}, \ k \in \{2, 4\}, \ i \in \{1, 2, 3, 4\} \). In case \( a \neq b \), we see that \( r_\theta(v) \) is in the sector obtained by extending two consecutive
vertex vectors. Then
\[ d_{T_g}(r_{\theta}(O),r_{\theta}(A)) = d_{T_g}(O,r_{\theta}(v)) = \| r_{\theta}(v) \|_{T_g} = u_{k+i}.(t_i v_{k+i} + t_{i+1} v_{k+i+1}) = t_i(u_{k+i}.v_{k+i}) + t_{i+1}(u_{k+i}.v_{k+i+1}) = t_i + t_{i+1} = \| v \|_{T_g}. \]

So, the norm of \( r_{\theta}(v) \) is preserved according to \( d_{T_g} \)-distances in \( \mathbb{R}^2_{T_g} \) for \( a \neq b \) and \( a = b \). This result completes the proof.

Thus we have the orthogonal group, \( O_{T_g}(2) \), consisting of four reflections and four rotations when \( a = b \) and consisting of two reflections and two rotations when \( a \neq b \)

\[ O_{T_g}(2) = R \cup S, \]

which gives us Dihedral group \( D_4 \) for \( a = b \) and Dihedral group \( D_2 \) for \( a \neq b \), that is, the Euclidean symmetry group of a square for \( a = b \) and the Euclidean symmetry group of the a line segment for \( a \neq b \). Now, it will be shown that all isometries of \( \mathbb{R}^2_{T_g} \) are in \( T(2).O_{T_g}(2) \).

**Definition 7** Let \( A = (a_1,a_2), B = (b_1,b_2) \) be two points in \( \mathbb{R}^2_{T_g} \). The minimum distance set of \( A, B \) is defined by

\[ \{ X \mid d_{T_g}(A,X) + d_{T_g}(B,X) = d_{T_g}(A,B) \} \]

and denoted by \( \Box \).

Let \( m_{AB} \) denotes the slope of the line through the points \( A \) and \( B \). If the slope of \( AB \) is the same slope with \( v_i, i \in \{1,2,3,4\} \), the set \( \Box \) is the line segment joining \( A \) and \( B \), that is, \( \Box = \overline{AB} \). It is called that \( \Box \) is the standard rectangle with diagonal \( \overline{AB} \). If the vector \( AB \) in the sector joining \( v_i \) and \( v_{i+1} \) vectors, \( \Box \) is the standard rectangle with diagonal \( \overline{AB} \) and the sides of it are parallel to \( v_i \) and \( v_{i+1} \).

**Proposition 8** Let \( \phi : \mathbb{R}^2_{T_g} \to \mathbb{R}^2_{T_g} \) be an isometry and let \( AB \) be the standard rectangle. Then

\[ \phi(\Box) = \phi(A)\phi(B). \]

**Proof.** Let \( Y \in \phi(\Box) \). Then,

\[ Y \in \phi(\Box) \iff \exists X \in AB \text{ such that } Y = \phi(X) \iff d_{T_g}(A,X) + d_{T_g}(B,X) = d_{T_g}(A,B) \iff d_{T_g}(\phi(A),\phi(X)) + d_{T_g}(\phi(X),\phi(B)) = d_{T_g}(\phi(A),\phi(B)) \iff Y = \phi(X) \in \phi(A)\phi(B) \]

\[ \square \]
Corollary 9 Let \( \phi : \mathbb{R}_T^2 \to \mathbb{R}_T^2 \) be an isometry and let \( AB \) be the standard rectangle. Then \( \phi \) maps vertices to vertices and preserves the lengths of sides of \( AB \).

Proposition 10 Let \( \phi : \mathbb{R}_T^2 \to \mathbb{R}_T^2 \) be an isometry such that \( \phi(O) = O \). Then \( \phi \in R \) or \( \phi \in S \).

Proof. It is known that the vertices of unit circle in \( \mathbb{R}_T^2 \) are \( A_1 = (\frac{1}{a}, 0) \), \( A_2 = (0, \frac{1}{b}) \), \( A_3 = (-\frac{1}{a}, 0) \) and \( A_4 = (0, -\frac{1}{b}) \). Let the standard rectangle \( OD \) be considered that its vertices are \( A_1 = (\frac{1}{a}, 0), A_2 = (0, \frac{1}{b}), D = (\frac{1}{a}, \frac{1}{b}) \).

It is clear that \( \phi(A_1) \in A_iA_{i+1}, i \in \{1, 2, 3, 4\} \). Since \( \phi \) is an isometry by Corollary 9, \( \phi(A_1) \) and \( \phi(A_2) \) must be the vertices of the standard rectangle \( O\phi(D) \). Therefore, if \( \phi(A_1) \in A_iA_{i+1} \), then \( \phi(A_1) = A_i \) or \( \phi(A_1) = A_{i+1} \). Similarly \( \phi(A_2) = A_i \) or \( \phi(A_2) = A_{i+1} \). In case \( a \neq b \), when \( \phi(A_1) = A_i \) and \( \phi(A_2) = A_{i+1}, \phi \) is an rotation with the angle \( \theta = \frac{(i-1)\pi}{2}, i \in \{1, 3\} \). If \( \phi(A_1) = A_{i+1} \) and \( \phi(A_2) = A_i, \phi \) is an reflection in the line with the angle \( \theta = \frac{\pi}{4}, i \in \{2, 4\} \) (which are \( y \)-axes and \( x \)-axes ). In case \( a = b \), when \( \phi(A_1) = A_i \) and \( \phi(A_2) = A_{i+1}, \phi \) is an rotation with the angle \( \theta = \frac{(i-1)\pi}{2}, i \in \{1, 2, 3, 4\} \). If \( \phi(A_1) = A_{i+1} \) and \( \phi(A_2) = A_i, \phi \) is an reflection in the line with the angle \( \theta = \frac{\pi}{4}, i \in \{1, 2, 3, 4\} \) (which are \( y = x \), \( y \)-axes , \( y = -x \) and \( x \)-axes ). Consequently, \( \phi \in R \) or \( \phi \in S \).

Theorem 11 Let \( f : \mathbb{R}_T^2 \to \mathbb{R}_T^2 \) be an isometry. Then there exists a unique \( T_a \in T(2) \) and \( \phi \in O_{T_a}(2) \) such that \( f = T_a \circ \phi \).

Proof. Let \( f(O) = A \) where \( A = (a_1, a_2) \). Define \( \phi = T_{-a} \circ f \). It is known that \( \phi \) is an isometry and \( \phi(O) = O \). Thus, \( \phi \in O_{T_a}(2) \) and \( f = T_a \circ \phi \) by Proposition 10. The proof of uniqueness is trivial.

References


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