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# Cvijović's Addition Formula for the Lah Numbers

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**Abstract**

Recently, Cvijović found a new addition formula for the partial Bell polynomials $B_{n,k}$. In this note, we consider the unsigned Lah numbers $L(n,k)$ and derive algebraically Cvijović's addition formula for $L(n,k)$. Also, we obtain a couple of vertical recurrence relations for $L(n,k)$.

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## 1 Introduction

Recently, Cvijović [4, Eq. (1.4)] obtained the following new addition formula (with respect to $k$) for the partial Bell polynomials $B_{n,k}$

$$B_{n,k_1+k_2}(x) = \frac{k_1!k_2!}{(k_1+k_2)!}\sum_{\alpha=0}^{n}\binom{n}{\alpha}B_{\alpha,k_1}(x)B_{n-\alpha,k_2}(x), \tag{1}$$

where $B_{n,k}(x) \equiv B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$ depends on the variables $x_1, x_2, \ldots, x_{n-k+1}$ (cf. Equation (6)). In this note (Section 2), we derive algebraically Cvijović's addition formula for the unsigned Lah numbers $L(n,k)$. Specifically, we show the identity

$$L(n,k_1+k_2) = \frac{k_1!k_2!}{(k_1+k_2)!}\sum_{j=k_1}^{n-k_2}\binom{n}{j}L(j,k_1)L(n-j,k_2), \tag{2}$$

| $n\backslash k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | | |
| 2 | 2 | 1 | | | | | | |
| 3 | 6 | 6 | 1 | | | | | |
| 4 | 24 | 36 | 12 | 1 | | | | |
| 5 | 120 | 240 | 120 | 20 | 1 | | | |
| 6 | 720 | 1800 | 1200 | 300 | 30 | 1 | | |
| 7 | 5040 | 15120 | 12600 | 4200 | 630 | 42 | 1 | |
| 8 | 40320 | 141120 | 141120 | 58800 | 11760 | 1176 | 56 | 1 |

Table 1: The first few unsigned Lah numbers $L(n,k)$.

which holds for $n \geq k_1 + k_2$, $k_1, k_2 \geq 0$, and where the unsigned Lah numbers are given explicitly by[1]

$$L(n,k) = \frac{n!}{k!}\binom{n-1}{k-1}. \tag{3}$$

Table 1 displays the first eight rows of the Lah number triangle, where it is understood that $L(n,0) = 0$ for $n > 0$, and $L(n,k) = 0$ for $k > n$. Note also the special cases $L(n,1) = n!$, $L(n,n-1) = n(n-1)$, and $L(n,n) = 1$. As a simple example, for $n = 7$, $k_1 = 1, k_2 = 3$, applying formula (2) we get

$$\begin{aligned} L(7,4) &= \frac{1}{4}\big(7L(1,1)L(6,3) + 21L(2,1)L(5,3) \\ &\qquad + 35L(3,1)L(4,3) + 35L(4,1)L(3,3)\big) \\ &= \frac{1}{4}(7 \cdot 1200 + 21 \cdot 2 \cdot 120 + 35 \cdot 6 \cdot 12 + 35 \cdot 24) = 4200. \end{aligned}$$

In Section 3, we briefly discuss the partial Bell polynomials $B_{n,k}$ and their relationship with the Lah numbers $L(n,k)$. Invoking the well-known identity connecting $B_{n,k}$ and $L(n,k)$, the identity (2) follows as an immediate consequence of Cvijović's addition formula (1). Moreover, by employing two different recurrence relations for $B_{n,k}$, we deduce a couple of vertical recurrence relations for $L(n,k)$ (see Equations (9) and (10) below).

[1]The numbers $L(n,k)$ are called the (unsigned) Lah numbers after the Slovenian mathematician and actuary Ivo Lah (1896-1979) who first investigated their properties in the middle of the 1950's. Combinatorially, $L(n,k)$ is the number of ways a set of $n$ elements can be partitioned into $k$ non-empty (linearly ordered) tuples (see, for instance, [6]). For further well-known properties of the Lah numbers, especially their relationship with the successive derivatives of the function $e^{1/x}$, we refer the reader to Daboul et al.'s paper [5].

## 2 Algebraic proof of the addition formula

To prove algebraically the addition formula (2) we rely on the following binomial identity, which corresponds to the identity 137 in the book of Benjamin and Quinn [2], namely,

$$\sum_{j=r}^{n+r-k} \binom{j-1}{r-1}\binom{n-j}{k-r} = \binom{n}{k}, \quad \text{for } 1 \le r \le k. \tag{4}$$

By definition, the right-hand side of identity (4) denotes the number of size $k$ subsets contained in the set $\{1, 2, \ldots, n\}$. On the other hand, in a size $k$ subset, there are $r-1$ (smaller) elements below the $r$th element and $k-r$ (greater) elements above it. Therefore, the number of size $k$ subsets with $r$th element $j$ is $\binom{j-1}{r-1}\binom{n-j}{k-r}$. Since $j$ can range from $r$ to $n+r-k$, the above identity follows.

Now we specialize the identity (4) to $r = k_1$ and $k = k_1 + k_2$, that is,

$$\sum_{j=k_1}^{n-k_2} \binom{j-1}{k_1-1}\binom{n-j}{k_2} = \binom{n}{k_1+k_2}.$$

Next, for our purpose, we slightly modify the last identity by means of the transformations $n \to n-1$ and $k_2 \to k_2 - 1$. Notice that both the lower and upper bound of the summation are left unchanged by these transformations. Hence, we arrive at the desired binomial identity

$$\sum_{j=k_1}^{n-k_2} \binom{j-1}{k_1-1}\binom{n-j-1}{k_2-1} = \binom{n-1}{k_1+k_2-1}, \tag{5}$$

which holds for $k_1, k_2 \ge 1$, and $n \ge k_1 + k_2$.

With identity (5) at hand, and using the explicit expression (3) for the Lah numbers, we can evaluate the right-hand side of formula (2) as

$$\begin{aligned}
&\frac{k_1!k_2!}{(k_1+k_2)!}\sum_{j=k_1}^{n-k_2}\binom{n}{j}L(j,k_1)L(n-j,k_2)\\
&\quad= \frac{k_1!k_2!}{(k_1+k_2)!}\sum_{j=k_1}^{n-k_2}\binom{n}{j}\frac{j!}{k_1!}\binom{j-1}{k_1-1}\frac{(n-j)!}{k_2!}\binom{n-j-1}{k_2-1}\\
&\quad= \frac{n!}{(k_1+k_2)!}\sum_{j=k_1}^{n-k_2}\binom{j-1}{k_1-1}\binom{n-j-1}{k_2-1} = \frac{n!}{(k_1+k_2)!}\binom{n-1}{k_1+k_2-1},
\end{aligned}$$

which corresponds to $L(n, k_1 + k_2)$. It is to be noted that identity (2) applies to either $k_1 = 0$ or $k_2 = 0$ (or to both $k_1 = k_2 = 0$) since, by convention, $L(0, 0) = 1$.

# 3 Partial Bell polynomials and Lah numbers

The $(n,k)$th partial Bell polynomial $B_{n,k}$ in the variables $x_1, x_2, \ldots, x_{n-k+1}$ is given by [3, p. 134, Eqs. (3d) and (3e)]

$$\begin{aligned} B_{n,k}(x) &\equiv B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) \\ &= \sum_{P_k(n)} \frac{n!}{j_1! j_2! \cdots j_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{j_1} \left(\frac{x_2}{2!}\right)^{j_2} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}}, \end{aligned} \tag{6}$$

where the summation extends effectively over all partitions $P_k(n)$ of a positive integer $n$ into exactly $k$ parts, that is, over all sets of non-negative integers $j_1, j_2, \ldots, j_{n-k+1}$ such that $j_1 + j_2 + \cdots + j_{n-k+1} = k$ and $j_1 + 2j_2 + \cdots + (n-k+1)j_{n-k+1} = n$ (note that $j_i$ is of course the number of parts of size $i$, $i = 1, 2, \ldots, n-k+1$). For example, for $n = 5$ and $k = 2$, the solutions $(j_1, j_2, j_3, j_4)$ of the equations $j_1 + j_2 + j_3 + j_4 = 2$ and $j_1 + 2j_2 + 3j_3 + 4j_4 = 5$ are $(1, 0, 0, 1)$ and $(0, 1, 1, 0)$. Thus, from Equation (6), we have

$$B_{5,2}(x_1, x_2, x_3, x_4) = 5! \left(\frac{x_1}{1!}\frac{x_4}{4!} + \frac{x_2}{2!}\frac{x_3}{3!}\right) = 5x_1x_4 + 10x_2x_3.$$

In [1], Abbas and Bouroubi showed (among other identities) that

$$B_{n,k}(1!a^0, 2!a^1, 3!a^2, \ldots, (n-k+1)!a^{n-k}) = \frac{n!}{k!}\binom{n-1}{k-1}a^{n-k}.$$

In particular, letting $a = 1$ gives the well-known identity [3, p. 135, Eq. (3h)]

$$B_{n,k}(1!, 2!, 3!, \ldots, (n-k+1)!) = \frac{n!}{k!}\binom{n-1}{k-1} = L(n,k). \tag{7}$$

Therefore, by virtue of the relationship (7) between $B_{n,k}$ and $L(n,k)$, and noting that $B_{n,k} = 0$ for $k > n$, the identity (2) follows upon setting $x_i = i!$ in Cvijović's addition formula (1). Incidentally, it is worth noting that combining Equations (6) and (7) yields the following alternative definition of the Lah numbers

$$L(n,k) = \sum_{P_k(n)} \frac{n!}{j_1! j_2! \cdots j_{n-k+1}!}.$$

Let us next consider the basic recurrence formula for $B_{n,k}$ [3, p. 136, Eq. (3k)]

$$B_{n,k}(x) = \frac{1}{k}\sum_{j=k-1}^{n-1}\binom{n}{j}x_{n-j}B_{j,k-1}(x), \quad n \geq 1.$$

By putting $x_{n-j} = (n-j)!$ in this formula, replacing $B_{n,k}(1!, 2!, 3!, \ldots)$ by $L(n,k)$, and observing that

$$L(n,k+1) = \frac{n-k}{k(k+1)}L(n,k), \tag{8}$$

one readily gets the following vertical recurrence relation for the Lah numbers

$$L(n,k) = \frac{n!k}{n-k} \sum_{j=k}^{n-1} \frac{L(j,k)}{j!}, \quad n \geq k+1. \tag{9}$$

It is to be noted that the recurrence (9) can equally be obtained by setting $k_1 = k$ and $k_2 = 1$ in the addition formula (2), and then employing relation (8).

In the same fashion, starting from the recurrence for $B_{n,k}$ [4, Eq. (1.3)]

$$B_{n,k}(x) = \frac{1}{x_1} \frac{1}{n-k} \sum_{\alpha=1}^{n-k} \binom{n}{\alpha} \left[ (k+1) - \frac{n+1}{\alpha+1} \right] x_{\alpha+1} B_{n-\alpha,k}(x),$$

and taking into account the previous relation (9), one arrives at the following alternative vertical recurrence relation for $L(n,k)$

$$L(n,k) = \frac{(n-1)!(k+1)}{n-k} \sum_{j=k}^{n-1} \frac{jL(j,k)}{j!}, \quad n \geq k+1. \tag{10}$$

Finally, we point out that using the explicit expression (3) for the Lah numbers in the recurrence (10) gives rise to the binomial identity

$$\sum_{j=k}^{n} (j+1) \binom{j}{k} = (k+1) \binom{n+2}{k+2}, \quad 0 \leq k \leq n,$$

from which we deduce the identity

$$\sum_{j=k}^{n} j \binom{j}{k} = \frac{n+k+nk}{k+2} \binom{n+1}{k+1}, \quad 0 \leq k \leq n.$$

In particular, for $k = 1$ we recover the familiar formula for the sum of the squares of the first $n$ positive integers $\sum_{j=1}^{n} j^2 = \frac{1}{6} n(n+1)(2n+1)$.

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