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On the Common Criteria Shared by the de la Vallee Poussin Means Source Sequences

Ziad S. Ali

P.O. Box 5, Vouvry CH-1896, Switzerland

Abstract

The move from the Norlund means to the de la Vallee Poussin means was considered by Ziad S. Ali in [5].Such a move was possible by introducing the combinatorial sequence $q_j^{(n)} = \frac{(2n-2j+1)}{(2n-j+1)}\binom{2n}{j}$ $j = 0, 1, 2, ..., n.$
.
An interesting question is: Are there any other sequences that can make the move ? First we answer this question and show that the sequence $q_j^{(n)} = \frac{(2n-2j+1)}{(2n-j+1)}\binom{2n}{j}$ $j = 0, 1, 2, ..., n.$ is the unique sequence making the move.
For this reason the sequence $q_j^{(n)} = \frac{(2n-2j+1)}{(2n-j+1)}\binom{2n}{j}$ $j = 0, 1, 2, ..., n.$ is called a de la Vallee Poussin means source sequence.Accordingly when a de la Vallee Poussin means source sequence is applied to the Norlund means, it transforms Norlund means to de la Vallee Poussin means confirming that the Norlund means are generalizations of the de la Vallee Poussin means.
Second we use this new sequence $q_j^{(n)}$ to give a different equal statement of the Theorem of Polya, and Schonberg in [9] .
Third we show that the sequence $q_j^{(n)}$ is a unimodal sequence,then use this fact to answer the converse of a Theorem of Ziad S. Ali in [6] showing that monoticity of the sequence defining the Norlund means is not a necessity to have the convexity of the Norlund means. A role of the unimodality of the sequence $q_j^{(n)}$ on the convexity of the Norlund means is also given.
Fourth we consider de la Vallee Poussin means source squences $q_{(j,r)}^{n}$ of two variables.We show in this case that there are infintely many de la

Vallee Poussin means source sequences that can make the move.
Fifth we give applications of some of the de la Vallee Poussin means source sequences $q^n_{(j,r)}$ stating them as Collораries of the Theorem of Polya, and Schonberg on the convexity of the de la Vallee Poussin means in [9] .We further use some of the ideas of these Colloraries to give a new definition of the extended Norlund means, and hence giving a more global generalization of the typical defintion of the Norlund means which say for example appears in the classical book of G.H.Hardy [8].

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1 Introduction

Let $\sum_{k=0}^{\infty} u_k$ be a given series, and let $\{S_n\}_0^\infty$ denote the sequence of its partial sums. Let $\{q_n\}_0^\infty$ be a sequence of real numbers with $q_0 > 0$, and $q_n \geq 0$ for all $n > 0$, and let $Q_n = \sum_{k=0}^{n} q_k$. By G.H. Hardy [8] The sequence-to-sequence transformation

$$T_n = \frac{1}{Q_n} \sum_{k=0}^{n} q_{n-k} S_k = \frac{1}{Q_n} \sum_{k=0}^{n} Q_{n-k} a_k z^k$$

is called the Norlund means of $\{S_n\}_0^\infty$, and is denoted by (N, q_n).

The (N, q_n) is regular if and only if $q_n = o(Q_n)$ as $n \to \infty$; By P.L. Duren [7] a function f analytic in a domain D is said to be simple, schlicht, or univalent if f is one-to-one mapping of D onto another domain. A domain E of the complex plane is said to be convex if and only if the line segment joining any two points of E lies entirely in E. A function f which is analytic, univalent in the unit disc $D = \{z : z < 1\}$, and is normalized by $f(0) = f'(0) - 1 = 0$ is said to belong to the class S. Now $f \in S$ is said to belong to the class K if and only if it is a conformal mapping of the unit disc $D = \{z : z < 1\}$ onto a convex domain.
A sequence q^n_j is said to be unimodal iff:

$$q^n_0 \leq q^n_1 \leq q^n_2 \leq .... \leq q^n_{m-1} \leq q^n_m \geq q^n_{m+1} \geq q^n_{m+2} \geq ..... \geq q^n_n$$

i.e- the sequence q^n_j increases up q^n_m, and then decreases to q^n_n. If the sequence q^n_{j+1}/q^n_j is decreasing, then the sequence q^n_j is unimodal.

2 Means connected with power series

Suppose that $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is regular for $|z| < 1$. Let

$S_n(z,f) = \sum_{k=0}^{n} a_k z^k$ be the sequence of partial sums of f,

$\sigma_n(z,f) = \frac{1}{n+1} \sum_{k=0}^{n} S_k(z,f)$ be the Cesaro means or the $(C,1)$ means of f,

$T_n(z,f) = \frac{1}{Q_n} \sum_{k=0}^{n} q_{n-k} S_k(z,f)$ be the Norlund means of f, and let

$V_n(z,f) = \frac{1}{\binom{2n}{n}} \sum_{k=1}^{n} \binom{2n}{n+k} a_k z^k$ be the de la Vallee Poussin means of f.

3 Known results

In [9] G. Polya, and I.J. Schonberg proved the following theorem:

Theorem 3.1: For $f(z) \in K$, it is necessary and sufficient that $V_n(z,f) \in K$ for $n = 1, 2 \ldots$

In [6] Ziad S. Ali proved the following theorem:

Theorem 3.2:

(i) Let (N, q_n) be a regular Norlund transformation such that $\{q_n\}_0^{\infty}$ is a non-decreasing sequence of positive numbers. Suppose that the values taken by $f(z)$, for z in D, lie in a convex domain D_w, then the values taken by $T_n(z,f)$, also lie in D_w for all n, and all z in D.

(ii) Conversely, suppose that the values taken by $T_n(z,f)$ lie in a convex domain D_w; then the values taken by $f(z)$ lie in D_w for all z in D.

4 The de la Vallee Poussin means source sequence of the type q_j^n, and the formation of the system of linear equations associated with it

We now give our definition of the de la Vallee Poussin means source sequence. :

Definition 4.1:

A sequence $q_j^{(n)}$ is said to be a de la Vallee Poussin means source sequence if and only if:

$$\sum_{j=0}^{(n-k)} q_j^{(n)} = \binom{2n}{n-k}, \ 0 \le k \le n$$

Clearly the following sequence $q_j^{(n)}$ is a de la Valle Poussin means source sequence

$$q_j^{(n)} = \frac{(2n-2j+1)}{(2n-j+1)} \binom{2n}{j} \ ,$$

furthermore the Norlund means can be explained by considering the product of the following two matrices

$$Q_{(n+1)\times(n+1)} Z_{(n+1)\times 1} = \begin{pmatrix} q_0^{(n)} & q_0^{(n)} & q_0^{(n)} & \dots & q_0^{(n)} & q_0^{(n)} & q_0^{(n)} \\ q_1^{(n)} & q_1^{(n)} & q_1^{(n)} & \dots & q_1^{(n)} & q_1^{(n)} & 0 \\ q_2^{(n)} & q_2^{(n)} & q_2^{(n)} & \dots & q_2^{(n)} & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 & 0 \\ q_{n-2}^{(n)} & q_{n-2}^{(n)} & q_{n-2}^{(n)} & \dots & 0 & 0 & 0 \\ q_{n-1}^{(n)} & q_{n-1}^{(n)} & 0 & \dots & 0 & 0 & 0 \\ q_n^{(n)} & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 z \\ a_2 z^2 \\ \vdots \\ a_{n-2} z^{n-2} \\ a_{n-1} z^{n-1} \\ a_n z^n \end{pmatrix}$$

Note that all the non-zero entries in any row of the above matrix $Q_{(n+1)\times(n+1)}$ are equal.
We now indicate how the system of linear equations associated to the sequence $q_j^{(n)}$ is formed.
Since we want the sum of all the entries in the zero column of $Q_{(n+1)\times(n+1)}$ above to be $\binom{2n}{n}$, and the sum of entries of the first column of $Q_{(n+1)\times(n+1)}$ above to be $\binom{2n}{n-1}$......etc. , then clearly solving for $q_0^{(n)}$, $q_1^{(n)}$, $q_2^{(n)}$,.........,$q_{(n-1)}^{(n)}$, and $q_n^{(n)}$, is the same as solving a system of $(n+1)$ linear equations with $(n+1)$ unknowns,which is the case of a unique solution . Accordingly we have the following Theorems:

Theorem 4.2:

Suppose that $f(z) = \sum\limits_{k=0}^{\infty} a_k z^k$ is regular for $|z| < 1$.
Let $T_n(z, f)$, and $V_n(z, f)$ respectively be the Norlund means of f , and the de la Vallee Poussin means of f, then

$$T_n(z, f) = V_n(z, f),\ if\ and\ only\ if$$

$$q_j^{(n)} = \frac{(2n-2j+1)}{(2n-j+1)}\binom{2n}{j},\ j = 0, 1, 2, ..., n.$$

Proof of Theorem 4.2: Assume that

$$q_j^{(n)} = \frac{(2n-2j+1)}{(2n-j+1)}\binom{2n}{j},\ j = 0, 1, 2, ..., n.$$

Then clearly

$$T_n(z, f) = V_n(z, f)$$

Now assume

$$T_n(z, f) = V_n(z, f) \quad ,$$

then due to the fact that the system of linear equations defined above has a unique solution it follows that

$$q_j^{(n)} = \frac{(2n-2j+1)}{(2n-j+1)}\binom{2n}{j},\ j = 0, 1, 2, ..., n.$$

Theorem 4.3:

Suppose that $f(z) = \sum\limits_{k=0}^{\infty} a_k z^k$ is regular for $|z| < 1$.
Let $T_n(z, f)$ be the Norlund means of f ,and let

$$q_j^{(n)} = \frac{(2n-2j+1)}{(2n-j+1)}\binom{2n}{j},\ j = 0, 1, 2, ..., n., Then$$

$T_n(z, f)$ is convex if and only if f is convex

Proof of Theorem 4.3:

Follows by Theorem 4.2 , and by Theorem 2.1 given above, and which is by G. Polya, I.J. Schonberg in [9]

Corrolary 4.4:

Let $q_j^{(n)}$ be a solution of the system of linear equations associated with $q_j^{(n)}$, and defining the Norlund means $T_n(z,f)$. Then $T_n(z,f)$ is convex.

5 An alternative statement of Theorem 3.1 given above, and which is by G. Polya, I.J. Schonberg in [9]

We use the basic idea of the de la Vallee Poussin means source sequence to restate in a different way Theorem 3.1 given above, and which is by G. Polya, I.J. Schonberg in [9] We have :

Theorem 5.1: f is convex if and only if

$$V_n(z,f) = T_n(z,f) = \frac{1}{\binom{2n}{n}} \sum_{j=0}^{n} q_{n-j} S_j(z,f) = \frac{1}{\binom{2n}{n}} \sum_{j=0}^{n} \frac{2j+1}{n+j+1} \binom{2n}{n-j} S_j(z,f)$$

6 The sequence $q_j^{(n)} = \frac{(2n-2j+1)}{(2n-j+1)}\binom{2n}{j}$, j=0,1,2,......,n. is a unimodal sequence

We show that the sequence $q_j^{(n)} = \frac{(2n-2j+1)}{(2n-j+1)}\binom{2n}{j}$, j=0,1,2,......,n. is a unimodal sequence by showing that the sequence

$$\frac{q_{j+1}^{n}}{q_j^{n}}$$

is a decreasing sequence. This follows by noting that the derivative of

$$f(x) = \frac{(2n-2x-1)(2n-x+1)}{(2n-2x+1)(x+1)}$$

is given by :

$$f`(x) = -\frac{2(2n+1)(2x^2-4xn+2n^2+n+1)}{(x+1)^2(-2x+2n+1)^2}$$

We note that f‘(x) is negative. Hence the sequence

$$\frac{q_{j+1}^{n}}{q_j^{n}}$$

is decreasing. Accordingly $q_j^{(n)} = \frac{(2n-2j+1)}{(2n-j+1)}\binom{2n}{j}$ is not monotone in one direction.
With the above we can answer to a Theorem of Ziad S. Ali (Theorem 2.2: above) confirming that monoticity is not a necessity.
Therefore we have the following improved version of the Theorem 2.2 above by Ziad S.Ali in :

Theorem 6.1:

(i) Let (N, q_n) be a regular Norlund transformation defined by the sequence $q_j^{(n)} = \frac{(2n-2j+1)}{(2n-j+1)}\binom{2n}{j}$, j=0,1,2,......,n. of positive numbers. Suppose that the values taken by $f(z)$, for z in D, lie in a convex domain D_w, then the values taken by $T_n(z, f)$, also lie in D_w for all n, and all z in D.

(ii) Conversely, suppose that the values taken by $T_n(z, f)$ lie in a convex domain D_w; then the values taken by $f(z)$ lie in D_w for all z in D.

Theorem 6.2: Suppose that $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is regular for $|z| < 1$.
Let $T_n(z, f)$, and $V_n(z, f)$ respectively be the Norlund means of f , and the de la Vallee Poussin means of f, then

$$T_n(z, f) = V_n(z, f),\ if\ and\ only\ if$$

the sequence $q_j^{(n)}$ which is the solution of the system of linear equations formed by $q_j^{(n)}$ is a unimodal sequence.

7 Odd, even criteria of the de la Vallee Poussin means source sequence $q_j^{(n)} = \frac{(2n-2j+1)}{(2n-j+1)}\binom{2n}{j}$, j=0,1,2,......,n, and the generation of a new combinatorial identity

One easily found crieteria of the sequence $q_j^{(n)}$ is the following criteria:

$$\sum_{jodd}^{n} q_j^{(n)} = \sum_{jeven}^{n} q_j^{(n)} = \frac{1}{2}\binom{2n}{n}$$

Now it follows easily that the de la Vallee Poussin means source sequence $q_j^{(n)}$ satisfies the following newly generated combinatorial identity :

$$\frac{1}{\sum_{j=0}^{n}\frac{(2n-2j+1)}{(2n-j+1)}\binom{2n}{j}}\sum_{j=1}^{n}(-1)^j(\sum_{i=0}^{n-j}\frac{(2n-2i+1)}{(2n-i+1)}\binom{2n}{i})=1$$

Note that

$$\sum_{jodd}^{n}q_j^{(n)}=\sum_{jeven}^{n}q_j^{(n)}$$

has the same meaning as

$$\frac{1}{\sum_{j=0}^{n}q_j^{(n)}}\sum_{j=1}^{n}(-1)^j(\sum_{i=0}^{n-j}q_j^{(n)})=1$$

8 The de la Vallee Poussin means source sequence of the type $q^n_{(j,r)}$, and the system of linear equations formed by it.

We begin by noting that the extended Norlund means can be defined by considering the product of the two matrices given below. We have

$$Q^e_{n+1\times n+1}Z_{n\times 1}=\begin{pmatrix} q_0^{(n,0)} & q_0^{(n,1)} & q_0^{(n,2)} & \dots & q_0^{(n,n-2)} & q_0^{(n,n-1)} & q_0^{(n,n)} \\ q_1^{(n,0)} & q_1^{(n,1)} & q_1^{(n,2)} & \dots & q_1^{(n,n-2)} & q_1^{(n,n-1)} & 0 \\ q_2^{(n,0)} & q_2^{(n,1)} & q_2^{(n,2)} & \dots & q_2^{(n,n-2)} & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 & 0 \\ .q_{n-2}^{(n,0)} & q_{n-2}^{(n,1)} & q_{n-2}^{(n,2)} & \dots & 0 & 0 & 0 \\ q_{n-1}^{(n,0)} & q_{n-1}^{(n,1)} & 0 & \dots & 0 & 0 & 0 \\ q_n^{(n,0)} & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1z \\ a_2z^2 \\ \vdots \\ a_{n-2}z^{n-2} \\ a_{n-1}z^{n-1} \\ a_nz^n \end{pmatrix}$$

We note that the non-zero entries in any row of the matrix $Q^e_{n+1\times n+1}$ are generaly not equal,while the entries of the matrix $Q_{n+1\times n+1}$ defined by $q_j^{(n)}$ in the previous section are equal. Hence $Q^e_{n+1\times n+1}$ is an extention of $Q_{n+1\times n+1}$.
The system of linear equations formed in this case is similar to the previous system in the sense that the sum of the entries of the zero column is $\binom{2n}{n}$, the sum of the entries of the first column is $\binom{2n}{n-1}$, and so on.
We now define the extended Norlund means as follows :

Definition 8.1:The extended Norlund means T_e .The extended Norlund means T_e is defined by:

$$T_e(n,z,f)=\frac{1}{Q^n_{(n+1,0)}}\sum_{r=1}^{n}Q^n_{(n-r)}a_r z^r$$

where $Q^n_{(n+1,0)}$ is the sum of all the (n+1) entries of the zero column of $Q^e_{n+1\times n+1}$, and $Q^n_{(n-r)}$ is the sum of (n-r+1) entries of the rth column.

Note now that when all the non-zero entries in any row of $Q^e_{n+1\times n+1}$ are equal,then in this case the above extended definition of the Norlund means coinsides with the regular definition of the Norlund means.

Now we like to get to the point of de la Vallee Poussin means source sequence. In the ordinary Norlund means we indicated that there were $(n+1)$ linear equations and $(n+1)$ unknowns, and that was the case of the unique solution which we called the de la Valle Poussin means source sequence of the system of linear equations, and which was $q_j^{(n)}=\frac{(2n-2j+1)}{(2n-j+1)}\binom{2n}{j},\ j=0,1,2,......,n.$

Now every entry in the matrix $Q^e_{n+1\times n+1}$ is a variable. We have $\frac{(n+2)(n+1)}{2}$ variables, but we have only $(n+1)$ equations, which is the case of infintly many solutions.

Definition 8.2: $q^n_{(j,r)}$ as a de la Vallee Poussin means source sequence The sequence $q^n_{(j,r)}$ is a de la Vallee Pousin means source sequence if and only if

$$\sum_{j=0}^{n}q^n_{(j,r)}=\binom{2n}{n-r}$$

Note that r is playing the role of the column in the matrix $Q^e_{n+1\times n+1}$ above, and for each r we are taking the sum of a column. For example the sum of the zero column(r=0) is $\binom{2n}{n}$, and so on.

Now we have the following Theorem :

Theorem 8.3:

With the sequence $q^n_{(j,r)}$ the linear system defned above has infinitely many solutions. These solutions represent de la Valle Poussin means source sequences of the system of linear equations formed by the matrix $Q^e_{n+1\times n+1}$ above, and in which the sum of the entries of the zero column is $\binom{2n}{n}$,the sum of the entries of the first column is $\binom{2n}{n-1}$, and so on.

Now since the above system of linear equations have an infinite number of solution we will exhibit a sample or some model applications or illustrations each representing a solution in closed form to the system of linear equations formed by $q^n_{(j,r)}$.
Each $q^n_{(j,r)}$ given in the following illustrations represent a de la Vallee Poussin means source sequence taking us from the extended Norlund means, to the de la Vallee Poussin means, we then use the famous theorem of Polya, and Schonberg to demonstrate the application.
For this reason we refer to these applications as collorraries of Theorem 2.1 given above, and which is by Polya, and Schonerg in [9]We have :

9 Applications of the de la Vallee Poussin means source sequences as solutions to the system of linear equations with an infinite number of solutions.

Collorary 9.1:

(i) Let $f(z) = \sum\limits_{r=1}^{\infty} a_r z^r, (a_1 = 1)$ be regular in the unit disc $|z| < 1$.

(ii) Let T_e be an extended transformation of the Norlund type defined by

$$T_e(n,z,f) = \frac{1}{Q^n_{(n+1,0)}} \sum_{r=1}^{n} Q^n_{n-r} a_r z^r, where$$

$$q^n_{(j,r)} = \binom{n}{j}\binom{n}{n-r-j} j = 0,1,2,...,n. \, and \; r = 0,1,2,....,n.$$

Let

$$Q^n_{n-r} = \sum_{j=0}^{n-r} q^n_{(j,r)} = \sum_{j=0}^{n-r} \binom{n}{j}\binom{n}{n-r-j}, then$$

$$T_e(n,z,f) \in K \quad \text{if and only if } f \in K.$$

proof of Collorary 9.1 Follows by the basic Vandermonde combinatorial identity.

Example 9.2: Consider the linear system of equations given by :

$x_{0,0} + x_{1,0} + x_{2,0} + x_{3,0} = 20$
$x_{0,1} + x_{1,1} + x_{2,1} = 15$
$x_{0,2} + x_{1,2} = 6$
$x_{0,3} = 1$

We shall hear after denote the above system of linear equations by sytem A .By noting that each equation above is the sum of the entries of columns, then the following matrix is a first solution of system A obtained by leting n = 3, and j =0,1,2,3., and r = 0,1,2,3. in the general combinatorial formula

$$q^n_{(j,r)} = \binom{n}{j}\binom{n}{n-r-j} \quad j = 0, 1, 2, ..., n. \, and \; r = 0, 1, 2, ..., n.$$

$$\begin{pmatrix} 1 & 3 & 3 & 1 \\ 9 & 9 & 3 & 0 \\ 9 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Note that the determinant of the above matrix is 9.

Collorary 9.3:

(i) Let $f(z) = \sum\limits_{r=1}^{\infty} a_r z^r, (a_1 = 1)$ be regular in the unit disc $|z| < 1$.

(ii) Let T_e be an extended transformation of the Norlund type defined by

$$T_e(n, z, f) = \frac{1}{Q^n_{(n+1,0)}} \sum_{r=1}^{n} Q^n_{n-r} a_r z^r, where$$

$$q^n_{(j,r)} = (-1)^j \binom{2n+1}{n-r-j} j = 0, 1, 2, ..., n. \, and \; r = 0, 1, 2, ..., n.$$

Let

$$Q^n_{n-r} = \sum_{j=0}^{n-r} (-1)^j \binom{2n+1}{n-r-j}, then$$

$$T_e(n, z, f) \in K \quad \text{if and only if } f \in K.$$

proof of Collorary 9.3 Clearly the sequence

$$q^n_{(j,r)} = (-1)^j \binom{2n+1}{n-r-j} j = 0, 1, 2, ..., n. \, and \; r = 0, 1, 2, ..., n.$$

is a second de la Vallee Poussin means sourcce sequence which is a solution to the system of linear equations above.Note further that by the Vandermonde identity we have :
for j=0,1,2,...,n., and r = 0,1,2,....,n

$$\sum_{j=0}^{(n-r)} q^{(n-r)}_{(j,r)} = \sum_{j=0}^{(n-r)} \binom{-1}{j} \binom{2n+1}{n-r-j} = \binom{2n}{n-r}$$

Example 9.4: We note now that the following matrix represent a second solution of system A given above, and whose entries are obtained by using

$$q^n_{(j,r)} = (-1)^j \binom{2n+1}{n-r-j} j = 0, 1, 2, ..., n. \, and \; r = 0, 1, 2, ..., n.$$

$$\begin{pmatrix} 35 & 21 & 7 & 1 \\ -21 & -7 & -1 & 0 \\ 7 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

Note that the determinant of the above matrix is 1. We like now to stress the fact that the de la Vallee Poussin means source sequence $q^n_{(j,r)}$ defined above is an alternating sequence.

Collorary 9.5:

(i) Let $f(z) = \sum_{r=1}^{\infty} a_r z^r, (a_1 = 1)$ be regular in the unit disc $|z| < 1$.

(ii) Let T_e be an extended transformation of the Norlund type defined by

$$T_e(n, z, f) = \frac{1}{Q^n_{(n+1,0)}} \sum_{r=1}^{n} Q^n_{n-r} a_r z^r, where$$

$$q^n_{(j,r)} = (-1)^j \binom{n-r}{j}\binom{3n-r-j}{2n} j = 0, 1, 2, ..., n. \, and \, r = 0, 1, 2, ..., n.$$

Let

$$Q^n_{n-r} = \sum_{j=0}^{n-r} (-1)^j \binom{n-r}{j}\binom{3n-r-j}{2n}, then$$

$$T_e(n, z, f) \in K \quad \text{if and only if } f \in K.$$

proof of Collorary 9.5 Clearly the sequence

$$q^n_{(j,r)} = (-1)^j \binom{n-r}{j}\binom{3n-r-j}{2n} j = 0, 1, 2, ..., n. \, and \, r = 0, 1, 2, ..., n.$$

is a third de la Vallee Poussin means sourcce sequence which is a solution to the system of linear equations above.
Note that by an extended version of the Vandermonde identity we have

$$\sum_{j=0}^{n-r} \binom{[-(n-r)-1]+j}{j}\binom{2n+[(n-r)-j]}{[(n-r)-j]} = \binom{2n}{n-r} j = 0, 1, 2, ..., n,$$

$$and \; r = 0, 1, 2, ..., n.$$

Example 9.6: We note now that the following matrix represent a third solution system A given above, and whose entries are obtained by using

$$q^n_{(j,r)} = (-1)^j \binom{n-r}{j}\binom{3n-r-j}{2n} j = 0, 1, 2, ..., n,$$

$$and \; r = 0, 1, 2, ..., n.$$

$$\begin{pmatrix} 84 & 28 & 7 & 1 \\ -84 & -14 & -1 & 0 \\ 21 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

Note that the determinant of the above matrix is 1.

Collorary 9.7:

(i) Let $f(z) = \sum_{r=1}^{\infty} a_r z^r, (a_1 = 1)$ be regular in the unit disc $|z| < 1$.

(ii) Let T_e be an extended transformation of the Norlund type defined by

$$T_e(n, z, f) = \frac{1}{Q^n_{(n+1,0)}} \sum_{r=1}^{n} Q^n_{n-r} a_r z^r, where$$

$$q^n_{(j,r)} = \binom{j+r}{r}\binom{2n-j-r-1}{n-j-r}.$$

Let

$$Q^n_{n-r} = \sum_{j=0}^{n-r} \binom{j+r}{r}\binom{2n-j-r-1}{n-j-r}, then$$

$$T_e(n, z, f) \in K \quad \text{if and only if } f \in K.$$

proof of Collorary 9.7 Clearly the sequence

$$q^n_{(j,r)} = \binom{j+r}{r}\binom{2n-j-r-1}{n-j-r} j = 0, 1, 2, ..., n. \, and \; r = 0, 1, 2, ..., n.$$

is a fourth de la Vallee Poussin means sourcce sequence which is a solution to the system of linear equations above.Note that

$$q^n_{(j,r)} = \binom{r+j}{j}\binom{n-1+[(n-r)-j]}{[(n-r)-j]} j = 0, 1, 2, ..., n. \, and \; r = 0, 1, 2, ..., n.$$

Now by an extended version of the Vandermonde identity we have the following:

$$\sum_{j=0}^{n-r} \binom{r+j}{j}\binom{n-1+[(n-r)-j]}{[(n-r)-j]} = \binom{2n}{n-r}$$

Accordingly the sequence $q^n_{(j,r)}$ is a de la Vallee Poussin means source sequence

Example 9.8: We note now that the following matrix represent fourth solution of system A given above, and whose entries are obtained by using

$$q^n_{(j,r)} = \binom{j+r}{r}\binom{2n-j-r-1}{n-j-r} j = 0, 1, 2, ..., n. \, and \; r = 0, 1, 2, ..., n.$$

$$\begin{pmatrix} 10 & 6 & 3 & 1 \\ 6 & 6 & 3 & 0 \\ 3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Note that the determinant of the above matrix is 9.

10 Remark on the definition of the regular Norlund means

From the applications given above we note for example that the de la Vallee Poussin means source sequence given by

$$C = (-1)^j \binom{2n+1}{n-r-j} j = 0, 1, 2, ..., n. \, and \, r = 0, 1, 2, ..., n.$$

is an alternating sequence, while in the typical definition of the Norlund means say for example in the classical book of G.H.Hardy [8] the the sequence q_j^n is non-negative with $q_0 > 0$. Now for the generalized extended Norlund means we have the following definition :

Definition 10.1:Generalized extended Norlund means Let $q_{(j,r)}^n$ be a sequence of real numbers such that $\sum_{j=0}^{n-r} q_{(j,r)}^n > 0$, then

$$T_e(n, r, z) = \frac{1}{Q_{(n+1,0)}^n} \sum_{r=1}^{n} Q_{(n-r)}^n a_r z^r$$

is called the generalized extended Norlund means of f.

11 Concluding remark

The case of the ordinary Norlund means shows that there exists a unique unimodal sequence defined by $q_j^{(n)} = \frac{(2n-2j+1)}{(2n-j+1)} \binom{2n}{j}$ $j = 0, 1, 2, ..., n.$, and which transforms the ordinary Norlund means to the de la Vallee Poussin means. The reason we call the sequence $q_j^{(n)}$ the unique de la Vallee Poussin means source sequence making the move.
The case of the extended Norlund means shows that there exists infintely many sequences $q_{(j,r)}^n$, which transforms the extended Norlund means to the

de la Vallee Poussin means. The sequences $q^{n}_{(j,r)}$ we talk about are obviously combinatorial sequences. These sequences are interesting, important utilizing combinatorial theory, and serving as extremely good applications of combinatorial theory.More specifically we have used the ordinary version of the Vandermonde identity as well as its extended version. Hence we have made combinatorial identities lively by showing their role in the study of systems of linear equations and the theory of convexity in complex analysis.
the $q^{(n)}_{j} = \frac{(2n-2j+1)}{(2n-j+1)}\binom{2n}{j}$ $j = 0, 1, 2, ..., n.$, as well as the $q^{n}_{(j,r)}$ all share the common property that they all are de la Valle Poussin means source sequences transfomjng the Norlund means to the de la Vallee Poussin means.
Even though the topic of the de la Vallee Poussin means source sequences,which we started is new, we hope that it will find its correct path in the field of mathematics.

References

[1] Ziad S. Ali , On a Combinatorial multiplier of the Progressive means and convex maps of the unit disc. Int. J. Contemp. Math. Sciences, Vol 9, 2014, no. 16, pages 97-131. http://dx.doi.org/10.12988/ijcms.2014.312134

[2] Ziad S. Ali, On a Combinatorial multiplier of the extended Nörlund means and convex maps of the unit disc. Int. J. Contemp. Math. Sciences, Vol 8, 2013, no. 16, pages 749-778. http://dx.doi.org/10.12988/ijcms.2013.3782

[3] Ziad S.Ali, "On a multiplier of the Progressive means and convex maps of the unit disc", Acta Mathematica Academiae Paedagogicae Nyiregyhaziensis, Volume 27, 2011, pages 89-104.

[4] Ziad S.Ali,.'A short note on a multiplier of the Nörlund means and convex maps of the unit disc"Fasciculi Mathematici, Volume 46, 2011, pages 17-24.

[5] Ziad S.Ali, "A note on a multiplier of the Norlund means and convex maps of the unit disc", International Journal of Mathematics and Mathematical Sciences (IJMMS), 2005, issue 20, December 11, pages 3351–3357. http://dx.doi.org/10.1155/ijmms.2005.3351

[6] Ziad S. Ali, "Convexity of Norlund and progressive means", Indian Journal of mathematics, Vol. 15, No. 2, pages 133–136, May 1973.

[7] P.L. Duren, Univalent Functions, Springer-Verlag, Berlin and NewYork, (1983).

[8] G.H. Hardy, Divergent Series, Oxford Uni. Press, 1949.

[9] G. Polya and I. J. Schonberg, "Remarks on the de la Vallee Poussin means and convex conformal maps of the circle", Pacific J. Math. 8, 1958, 295–334. http://dx.doi.org/10.2140/pjm.1958.8.295

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