Notes on the Non Self-adjoint Elliptic Differential Operators

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Abstract

Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^n$. Let the space $H_\ell = W_{2,\alpha}^\ell(\Omega)$ of vector functions $u(x) = (u_1(x), \ldots, u_\ell(x))$ defined on Ω with finite norm: $|u, W_{2,\alpha}^\ell(\Omega)| = \left( \sum_{i=1}^n \int \rho^{2\alpha}(x)|u'_{x_i}(x)|^2_C dx + \int_\Omega |u(x)|^2_C \right)^{1/2}$. In this paper we will find the asymptotic distribution of eigenvalues of the operator $(Au)(x) = -\sum_{i,j=1}^n (\rho^{2\alpha}(x)a_{ij}(x)q(x)u_{x_i}(x))_{x_j}'$ in the space $H_\ell = L^2(\Omega)^\ell$.

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1 Introduction

Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^n$, i.e., $\partial \Omega \in C^\infty$. We introduced the space $\mathcal{H}_\ell = W_{2,\alpha}^\ell(\Omega) = W_{2,\alpha}^1(\Omega) \times \cdots \times W_{2,\alpha}^\ell(\Omega)$ ($\ell$-times) as the space of vector functions $u(x) = (u_1(x), \ldots, u_\ell(x))$ defined on Ω with finite norm:

$$|u, W_{2,\alpha}^\ell(\Omega)| = \left( \sum_{i=1}^n \int \rho^{2\alpha}(x)|u'_{x_i}(x)|^2_C dx + \int_\Omega |u(x)|^2_C \right)^{1/2}.$$

By $\overset{\circ}{\mathcal{H}}_\ell$ we denote the closure of $C_0^\infty(\Omega)^\ell$ in $\mathcal{H}_\ell$, for $\ell = 1$ we set $\mathcal{H} = \mathcal{H}_1$ and $\overset{\circ}{\mathcal{H}} = \mathcal{H}_1$. In this article, we investigate the asymptotic formula for distribution

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of the eigenvalues (ev) of a non-self adjoint elliptic differential operator $A$ defined by:

$$(Au)(x) = -\sum_{i,j=1}^{n} (\rho^{2\alpha}(x) a_{ij}(x) q(x) u_{x_i}'(x))_{x_j}' \quad \text{in space } H_\ell = L_2(\Omega)^\ell \text{ here},$$

$$\rho(x) = \text{dist}\{x, \partial \Omega\}, \quad \alpha \in [1,0), \quad a_{ij}(x) = a_{ji}(x) \quad (i, j = 1, 2, \ldots, n),$$

$$a_{ij}(x) \in C^2(\overline{\Omega}) \quad (i, j = 1, 2, \ldots, n), \quad q(x) \in C^2(\overline{\Omega}, \text{ End } \mathbb{C}^\ell).$$

Furthermore assume that for $\forall x \in \overline{\Omega}$, the matrix function $q(x)$ has simple eigenvalues $\mu_1(x), \ldots, \mu_\ell(x)$ arranged in the complex plane in the following way:

$$\arg \mu_j(x) = 0 \quad (j = 1, 2, \ldots, \nu), \quad \mu_j(x) \in \Phi \quad (j = \nu + 1, \ldots, \ell),$$

where $\Phi = \{z \in \mathbb{C} : |\arg z| < \varphi\}, \quad \varphi \in (0, \pi)$. We also assume that the matrix function $a_{ij}(x)$ satisfies the uniformly elliptic condition: i.e., there exist $M > 0$ such that for every $s = (s_1, \ldots, s_n) \in \mathbb{C}^n$, $x \in \Omega$ we have $|s|^2 \leq M \sum_{i,j=1}^{n} a_{ij}(x) s_i \overline{s_j}$.

Now for a closed extension of the operator $A$ we need to extend its domain to the:

$$D(A) = \{u \in H_\ell \cap W_{2,loc}^2(\Omega)^\ell : \sum_{i,j=1}^{n} (\rho^{2\alpha} a_{ij} u_{x_i}'(x))_{x_j}' \in H_\ell\},$$

where $W_{2,loc}^2(\Omega) = \{u : \sum_{i=0}^{2} \int_{J} |u^{(i)}(x)|^2 \, dx < \infty \text{ is open set in } \Omega\}$

Here and in the sequel the value of the function $\arg z \in (-\pi, \pi]$, and $\|T\|$ denotes the norm of the bounded operator $T : H_\ell \to H_\ell$.

## 2 Resolvent Estimate

**Theorem 2.1.** Let $S \subset \Phi \setminus R_+$ be some closed sector with vertex at 0, and as in Section 1, we recall that for $\forall x \in \overline{\Omega}$, the matrix function $q(x)$ has simple eigenvalues $\mu_1(x), \ldots, \mu_\ell(x)$ which are different from zero and $\arg \mu_j(x) = 0$ for $j = 1, 2, \ldots, \nu$ and $\mu_j(x) \in \Phi$ for $j = \nu + 1, \ldots, \ell$ where $\Phi = \{z \in \mathbb{C} : |\arg z| < \varphi\}, \quad \varphi \in (0, \pi)$ (i.e., the eigenvalues $\mu_1(x), \ldots, \mu_\nu(x)$ lie on the positive real line inside the angle $\Phi$, and the rest of the eigenvalues $\mu_{\nu+1}(x), \ldots, \mu_\ell(x)$ lie outside of the angle $\Phi$ in view of $S \subset \Phi \setminus R_+$, implies that all the eigenvalues $\mu_1(x), \ldots, \mu_\ell(x)$ lie on the complex plane and outside of the closed sector $S$).

Then the operator $A$ has a discrete spectrum, and for sufficiently large in modules $\lambda \in S$ the inverse operator $(A - \lambda I)^{-1}$ exists and is continuous, and the following estimate is valid:

$$\|(A - \lambda I)^{-1}\| \leq M_S |\lambda|^{-1} \quad (\lambda \in S, \quad |\lambda| \geq C_S), \quad (2.1)$$
where $M_S$, $C_S$ are sufficiently large numbers. The symbol $\| \cdot \|$ stands for the norm of a bounded operator in $H$ or $H_\ell$.

Let the sequence of (ev) eigenvalues of the operator $A$ in the angle $\Phi = \{ z \in \mathbb{C} : |\arg z| \leq \varphi \}$, $\varphi \in (0, \pi)$, be denoted by $\lambda_1, \lambda_2, \ldots$ enumerated in the non-decreasing order of their absolute values and taking into account their multiplicity zero in the case $\omega = \max\{ \frac{n}{2}, \frac{n-1}{2-2\alpha} \} \in \mathbb{N}$.

Let $N(t) = \text{card}\{ j : |\lambda_j| \leq t \}$, $t > 0$. According to theorem 1, the operator $A$ has a finite number of (ev) in every set $\varphi_\psi$, $\psi > 0$ of the form (1). Therefore, it is easy to proof that:

$$\lim_{j \to \infty} \arg \lambda_j = 0. \quad (1.1')$$

The main result of this paper is formulated in the following Theorem.

\textbf{Theorem 1.2.} the following statement are hold:

(i) if $\alpha \in [0, 1)$, $\alpha < \frac{1}{n}$ then we have

\[
N(t) \sim (2\pi)^{-n} v_n t^{n/2} \sum_{k=1}^{\nu} \int_{\Omega} \rho^{-\alpha}(x) \mu_k^{-\frac{n}{2}}(x)(\det(a(x)))^{-\frac{1}{2}} dx,
\]

where $v_n$ denotes the volume of the unit ball in $R^n$ and $\alpha(x) = (a_{ij}(x))_{i,j=1}^n$.

Now let the sequence of eigenvalues of the operator $A$ in the angle $\Phi = \{ z \in \mathbb{C} : |\arg z| \leq \varphi \}$, $\varphi \in (0, \pi)$, be denoted by $\{ \lambda_i \}$ enumerated in the non-decreasing order of their absolute values and taking into account their multiplicities. Then based on the above assertions the following result holds. Since the operator $A$ has a finite number of eigenvalues in every set $\Phi_\psi$, $\psi > 0$ of the form (2.1), it is easy to prove that

$$\lim_{i \to \infty} \arg \lambda_i = 0. \quad (2.1')$$

We define $N(t) = \text{card}\{ j : |\lambda_j| \leq t \}$, $t > 0$. For our later work we define $\omega = \max\{ \frac{n}{2}, \frac{n-1}{2-2\alpha} \}$.

\textbf{Theorem 3.1.} (i) if $\alpha \in [0, 1)$, $\alpha < \frac{1}{n}$ then we have

\[
N(t) \sim (2\pi)^{-n} v_n t^{n/2} \sum_{k=1}^{\nu} \int_{\Omega} \rho^{-\alpha}(x) \mu_k^{-\frac{n}{2}}(x)(\det(a(x)))^{-\frac{1}{2}} dx,
\]

where $v_n$ denotes the volume of the unit ball in $R^n$ and $\alpha(x) = (a_{ij}(x))_{i,j=1}^n$.

Hence, for every natural number $S$, such that

$$S \in \mathbb{N}, \; S > \omega = \max\{ \frac{n}{2}, \frac{n-1}{2-2\alpha} \}, \quad S \in (\omega, \omega + 1]$$

we will have:

$$|\text{tr}(A - \lambda I)^{-s} - \text{tr} UB(\lambda)U^{-1}| \leq M|\lambda|^{-\frac{1}{2}}|\Gamma(\lambda)|_s^s$$
where the symbols tr, 1, 1, denote the trace of a trace-class operators and the \( \sigma \)-norm of the operator respectively [4].

By using the representation of the operator \( B(\lambda) \) we obtain:

\[
\text{tr} \ U B(\lambda)^* U^{-1} = \text{tr} \ B(\lambda)^* = \text{tr} \ B_s(\lambda)
\]

where \( B_s(\lambda) = \text{diag}\{(P_1 - \lambda I)^{-s}, \ldots, (P_\ell - \lambda I)^{-s}\} \).

If we estimate \( |\Gamma(s)|_s \) by using (7) and (8) we obtain this relation:

\[
| \sum_{i=1}^{+\infty} (\lambda'_i - \lambda)^{-s} - \sum_{j=1}^{\ell} \sum_{i=1}^{+\infty} (\lambda_{ij} - \lambda)^{-s} | \leq M \psi |\lambda|^{\omega-s-\frac{1}{4}}, \quad (\lambda \in \phi \psi, \ |\lambda| \geq C \psi)
\]

where \( \lambda'_1, \lambda'_2, \ldots, \lambda_{1j}, \lambda_{2j}, \ldots \) are denoted respectively the (ev.) of the operators \( A \) and \( P_j \). Notice that according to (8), there are finite number of (ev.) of the using the counter integral method in the same way as in [5, §4] for \( \omega \neq 1, 2, \ldots \) we can again obtain like the above relation by counter integral: i.e.

\[
\sum_{i=1}^{+\infty} (\lambda_i + \tau)^{-S} = \sum_{j=1}^{\nu} \sum_{i=1}^{+\infty} (\lambda_{ij} + \tau)^{-S} + O(\tau^{\omega-s-\frac{1}{4}}), \quad \tau \to +\infty.
\]

From here and from (2) kipping in mind that \( \lambda_{ij} > 0 \) (for \( i = 1, 2, \ldots, j = 1, \ldots, \nu \)). Now it is easy to establish the asymptotical formula:

\[
\int_0^{+\infty} \frac{dN_i(t)}{(t+\tau)^S} \sim \sum_{j=1}^{\nu} \sum_{i=1}^{\nu} \int_0^{+\infty} \frac{dN_i(t)}{(t+\tau)^S}, \quad \tau \to +\infty
\]

where \( N_i(t) = \text{card}\{ j : \lambda_{ij} \leq t \} \), \( i = 1, \ldots, \nu \) which are well known asymptotical formulas for functions-\( N_i(t) \) (see for example [6]), after apply of M. V. Keldish’s theorem of tasker-type we establish the assertion of theorem 2 in applying the multidimensional Tauberian Theorems of A. A. Shkalikov [7].

**References**


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