Operator Power Graph of a Group

A. Vethamanickam
Department of Mathematics
Kamarajar Government Arts College, Surandai
Tirunelveli, Tamilnadu, India

D. Premalatha
Department of Mathematics
Rani Anna Government Arts College (W), Gandhinagar - 627 008
Tirunelveli, Tamilnadu, India

Copyright © 2014 A. Vethamanickam and D. Premalatha. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

Let \((G, \ast)\) be a group with binary operation \('\ast'\). The Operator Power graph \(\Gamma_{OP}(G)\) of \(G\) is a graph with \(V(\Gamma_{OP}(G)) = G\) and two distinct vertices \(x\) and \(y\) are adjacent in \(\Gamma_{OP}(G)\) if and only if either \(x = (x \ast y)^n\) or \(y = (x \ast y)^m\), where \(n, m\) are positive integers. In this paper, we want to explore how the group theoretical properties of \(G\) can effect on the graph theoretical properties of \(\Gamma_{OP}(G)\). Some characterizations for fundamental properties of \(\Gamma_{OP}(G)\) have also been obtained.

Mathematics Subject Classification: 05C25, 20A05

Keywords: operator power graph, complete graph, star graph

1 Introduction

The study of algebraic structures, using the properties of graphs, becomes an exciting research topic in the last twenty years, leading to many fascinating
results and questions. There are many papers on assigning a graph to a ring or group and thereby investigating algebraic properties of the ring or group using the associated graph, for instance, see [1, 2, 3, 4, 6]. In the present article, to any group \(G\), we assign a graph and investigate algebraic properties of the group using the graph theoretical concepts. Before starting, let us introduce some necessary notation and definitions.

We consider simple graphs which are undirected, with no loops or multiple edges. For any graph \(\Gamma = (V, E)\), \(V\) denote the set of all vertices and \(E\) denote the set of all edges in \(\Gamma\). The degree \(\deg_\Gamma(v)\) of a vertex \(v\) in \(\Gamma\) is the number of edges incident to \(v\) and if the graph is understood, then we denote \(\deg_\Gamma(v)\) simply by \(\deg(v)\). The order of \(\Gamma\) is defined \(|V(\Gamma)|\) and its maximum and its minimum degrees will be denoted, respectively, by \(\Delta(\Gamma)\) and \(\delta(\Gamma)\). A graph \(\Gamma\) is regular if the degrees of all vertices of \(\Gamma\) are the same. A vertex of degree 0 is known as an isolated vertex of \(\Gamma\). A graph \(\Omega\) is called a subgraph of \(\Gamma\) if \(V(\Omega) \subseteq V(\Gamma)\), \(E(\Omega) \subseteq E(\Gamma)\). Let \(\Gamma = (V, E)\) be a graph and let \(S \subseteq V\). A subgraph \(\Omega\) of \(\Gamma\) is said to be an induced subgraph of \(\Gamma\) induced by \(S\), if \(V(\Omega) = S\) and each edge of \(\Gamma\) having its ends in \(S\) is also an edge of \(\Omega\). A simple graph \(\Gamma\) is said to be complete if every pair of distinct vertices of \(\Gamma\) are adjacent in \(\Gamma\). A graph \(\Gamma\) is said to be connected if every pair of distinct vertices of \(\Gamma\) are connected by a path in \(\Gamma\). A bipartite graph or bigraph is a graph whose vertex set \(V(\Gamma)\) can be partitioned into two subsets \(V_1\) and \(V_2\) such that every edge of \(\Gamma\) has one end in \(V_1\) and the other end in \(V_2\). \((V_1, V_2)\) is called a bipartition of \(\Gamma\). A complete bipartite graph is a bipartite graph with bipartition \((V_1, V_2)\) such that every vertex of \(V_1\) is joined to all the vertices of \(V_2\). It is denoted by \(K_{m,n}\), where \(|V_1| = m\) and \(|V_2| = n\). A star graph is a complete bigraph \(K_{1,n}\). Let \(G\) be a group with identity \(e\). The order of the group \(G\) is the number of elements in \(G\) and is denoted by \(O(G)\). The order of an element \(a\) in a group \(G\) is the smallest positive integer \(k\) such that \(a^k = e\). If no such integer exists, we say \(a\) has infinite order. The order of an element \(a\) is denoted \(O(a)\). Let \(p\) be a prime number. A group \(G\) with \(O(G) = p^k\) for some \(k \in \mathbb{Z}^+\), is called a \(p\)-group.

**Theorem 1.1.** (Lagrange) If \(G\) is a finite group and \(H\) is a subgroup of \(G\) then \(o(H)\) divides \(o(G)\).

**Theorem 1.2.** Order of a cyclic group is equal to the order of its generator.

**Theorem 1.3.** A group \(G\) of prime order must be cyclic and every element of \(G\) other than identity can be taken as its generator.

## 2 Operator Power Graph

In this section, we observe certain basic properties of operator power graphs.
Definition 2.1. Let $(G, *)$ be a group with binary operation ‘$*$’. The Operator Power graph $\Gamma_{OP}(G)$ of $G$ is a graph with $V(\Gamma_{OP}(G)) = G$ and two distinct vertices $x$ and $y$ are adjacent in $\Gamma_{OP}(G)$ if and only if either $x = (x*y)^n$ or $y = (x*y)^m$, where $n, m$ are positive integers.

Remark 2.2. Since $y = x^k \Rightarrow <y>\subseteq <x>$, $x$ and $y$ are adjacent in $\Gamma_{OP}(G)$ if and only if either $<x>\subseteq <x*y>$ or $<y>\subseteq <x*y>$.

Proposition 2.3. Let $(G, *)$ be a group with $n$ elements. In $\Gamma_{OP}(G)$, the identity element $e$ of $G$ has degree $n - 1$.

Proof. Let $(G, *)$ be a group with $n$ elements. Let $x \in G$ be any element. Clearly $x = (x*e)^1$. Hence the result follows. $\square$

Proposition 2.4. Let $(G, *)$ be a group. For any non self inverse element $x \in G$, $x$ and $x^{-1}$ are non adjacent in $\Gamma_{OP}(G)$.

Proof. Let $(G, *)$ be a group with identity element $e$. Let $x \in G$ be non identity element. Since $x*x^{-1} = e$, $x \neq (x*x^{-1})^n$. Hence the result follows. $\square$

Theorem 2.5. $\Gamma_{OP}(G)$ is complete if and only if $G \cong \mathbb{Z}_2$ or $\mathbb{Z}_1$.

Proof. Clearly if $G \cong \mathbb{Z}_2$ or $\mathbb{Z}_1$, then $\Gamma_{OP}(G)$ is complete. Conversely assume that $\Gamma_{OP}(G)$ is complete. It is enough to prove that $O(G) \leq 2$. Suppose $O(G) \geq 3$, we discuss the proof by the following two cases. Case(i): Suppose $G$ has at least one non self inverse element, by Proposition 2.3 $\Gamma_{OP}(G)$ can not be complete.

Case(ii): Suppose every element of $G$ is self inverse. Let $x, y \in G$ such that $a \neq e, b \neq e$. By assumption $O(x) = O(y) = O(x*y) = e$, where $e$ is the identity element of $G$. Since $x*y \neq x$ and $x*y \neq y$, $<x>\neq<y>\neq<x*y>$. Therefore $x$ and $y$ are non adjacent.

From both cases we conclude that $\Gamma_{OP}(G)$ can not be complete, which is a contradiction. Therefore $O(G) \leq 2$ and hence $G \cong \mathbb{Z}_2$ or $\mathbb{Z}_1$. $\square$

Theorem 2.6. Let $G$ a group. $\Gamma_{OP}(G)$ is a star graph if and only if every non identity element of $G$ has an order either 2 or 3.

Proof. Assume that every non identity element of $G$ has an order either 2 or 3. By Proposition 2.3, the identity element ‘$e$’ of $G$ has full degree so it is enough to prove that any two non identity elements are non adjacent. Let $x, y \in G$ such that $x \neq e \neq y$.

Case(i): $O(x) = 2, O(y) = 2$ and $O(x*y) = 2$

In this case, we have $<x>\neq<y>\neq<x*y>$. Therefore $x$ and $y$ are non adjacent.

Case(ii): $O(x) = 3, O(y) = 3$ and $O(x*y) = 3$
Suppose $x$ and $y$ are adjacent, $< x >= < x * y >$ or $< y >= < x * y >$. Without loss of generality assume that $< x >= < x * y >$. Since $O(x) = O(x * y) = 3$, $x * y = x$ or $x * y = x^{-1}$. If $x * y = x$, then $y = e$, which is a contradiction. If $x * y = x^{-1}$, then $x^2 = y^{-1}$. Since $O(x) = 3$, $x^2 = x^{-1}$. Therefore $x^{-1} = y^{-1}$ and hence $x = y$, which is a contradiction. Therefore $x$ and $y$ are non adjacent.

Case(iii): $O(x) = 2$, $O(y) = 3$ and $O(x * y) = 2$

By Lagrange’s theorem $< y > \not\subseteq < x * y >$ and in the view of case(i) $< x > \not\subseteq < x * y >$. Therefore $x$ and $y$ are non adjacent.

Case(iv): $O(x) = 2$, $O(y) = 3$ and $O(x * y) = 3$

By Lagrange’s theorem $< x > \not\subseteq < x * y >$ and in the view of case(ii) $< y > \not\subseteq < x * y >$. Therefore $x$ and $y$ are non adjacent.

Similarly we can prove other cases. Therefore $x$ and $y$ are non adjacent and hence $\Gamma_{OP}(G) = K_{1,n}$.

Conversely assume that $\Gamma_{OP}(G) = K_{1,n}$. Since the identity element $'e'$ has a full degree, any two non identity elements are non adjacent in $\Gamma_{OP}(G)$. We prove that every non identity element of $G$ has an order either 2 or 3. Suppose $G$ has an element $x$ of order $k$ such that $k > 3$. $< x > = \{e, x, x^2, x^3, \ldots, x^{k-2}, x^{k-1}\}$. Since $x * x^{k-2} = x^{k-1} = x^{-1}$, $< x > = < x * x^{k-2} >$. Therefore $x$ and $x^{k-2}$ are adjacent, which is a contradiction. Hence every non identity element of $G$ has an order either 2 or 3.

**Proposition 2.7.** Let $G$ be a finite group of order $n$ with no self inverse element and $q$ be number of edges in $\Gamma_{OP}(G)$. Then $q \leq \frac{(n-1)^2}{2}$. Moreover, this bound is sharp.

**Proof.** By Proposition 2.3, $\deg_{\Gamma_{OP}(G)}(e) = n - 1$, where $e$ is the identity element of $G$. Since $G$ has no self inverse element, by Proposition 2.4, for all $x \in G - e$, $\deg_{\Gamma_{OP}(G)}(x) \leq n - 2$. From this we get the degree sum $\leq (n - 1) + (n - 1)(n - 2) = (n - 1)^2$. Hence $q \leq \frac{(n-1)^2}{2}$. Moreover, for the group $\mathbb{Z}_3$, $\Gamma_{OP}(\mathbb{Z}_3) \cong K_{1,2}$ and for this graph the bound is sharp.

We now characterize the groups $G$ for which the associated graph $\Gamma_{OP}(G)$ attains this bound.

**Theorem 2.8.** Let $G$ be a finite group of order $n$ and $q$ be number of edges in $\Gamma_{OP}(G)$. Then $q = \frac{(n-1)^2}{2}$ if and only if $G$ is group of prime order.

**Proof.** Assume that $\Gamma_{OP}(G)$ is a graph with $\frac{(n-1)^2}{2}$ edges. In view of Proposition 2.7, we get that $\deg_{\Gamma_{OP}}(a) = n - 2$ for all vertices $a \in G - e$ and $\deg_{\Gamma_{OP}}(e) = n - 1$. Let $a \in G - e$ be any element of order $k$.

**Claim:** $k$ be a prime number

By the assumption, $a$ is adjacent to $a^2, a^3, \ldots, a^{k-2}$. Consider $a$ is adjacent to $a^2$. Then by definition either $< a > \subseteq < a^3 >$ or $< a^2 > \subseteq < a^3 >$.

**Case(i):** $< a > \subseteq < a^3 >$
Always $<a^3> \subseteq <a>$. Therefore $<a> = <a^3>$. Also $a$ is adjacent to $a^3$, we have $<a> \subseteq <a^4>$. Always $<a^4> \subseteq <a>$. Therefore $<a> = <a^4>$. We know that $<a^4> \subseteq <a^2>$ which implies that $<a> \subseteq <a^2>$ and so $<a> = <a^2>$. Similarly we can prove that $<a> = <a^2> = <a^3> = \ldots = <a^{k-1}>$.

**Case (ii):** $<a^2> \subseteq <a^3>$

Since $a$ is adjacent to $a^3$, $<a^3> \subseteq <a^4>$. Similarly we can prove that $<a^2> \subseteq <a^3> \subseteq <a^4> \subseteq \ldots$. Also $a^{-1}$ is adjacent to $a^2, a^3, \ldots$. We can prove that $<a^2> \subseteq <a^3> \subseteq <a^4> \subseteq \ldots$. Hence we have $<a> = <a^2> = <a^3> = \ldots = <a^{k-1}>$.

By both cases we have $<a> = <a^2> = <a^3> = \ldots = <a^{k-1}> = \ldots \ldots (1)$

Now we prove that $O(a) = k$ is a prime number. Suppose not, $k$ is not a prime. Without loss of generality assume that $k = pq$, for some prime $p$ and $q$. Since $p, q | k$ and $<a>$ is a cyclic group, there exists two element $a^l, a^m \in <a>$ such that $<a^l> = p$ and $<a^m> = q$. Which is a contradiction to (1). Hence $k$ is a prime number.

Since $a$ is arbitrary, every element of $G$ is of prime order. Now we have to prove this prime order is unique. Suppose that let $a, b \in G$ such that $O(a) = p_1, O(b) = p_2$ and $O(a*b) = p_3$, where $p_1, p_2$ and $p_3$ are distinct primes. By assumption $a$ is adjacent to $b$. Therefore $<a> \subseteq <a*b>$ or $<b> \subseteq <a*b>$. Suppose $<a> \subseteq <a*b>$ which implies that $p_1 = p_3$. Therefore $<a> = <a*b>$. Therefore $a*b = a^l$ which implies $b = a^{l-1}$. Therefore $O(b) = p_1$. Hence $p_1 = p_2 = p_3$. Similarly we can prove the other case $<b> \subseteq <a*b>$. Therefore every element of $G$ has a unique prime order. Hence $G$ is a group order $p^n$, for some prime $p$.

Claim: $O(G) = p$

Suppose not $O(G) = p^n$ for some positive integer $n > 1$. Since every element of $G$ has an order $p$, $G$ has at least two distinct subgroup of order $p$. Let $a, b \in G$ such that $O(a) = O(b) = p$ and $<a> \cap <b> = \{e\}$, where $e$ is an identity element of $G$. Clearly $a^{-1} \neq b$. Therefore by hypothesis $a$ and $b$ are adjacent in $\Gamma_{OP}(G)$. Therefore $<a> \subseteq <a*b>$ or $<b> \subseteq <a*b>$. Without loss of generality assume that $<a> \subseteq <a*b>$. $a = (a*b)^r$ which implies that $b^r = a^{1-r}$. Therefore $b^r \in <a>$ which is a contradiction. Therefore $O(G) = p$. Hence $G$ is a group of prime order.

Conversely, assume that $G$ is a group of prime order $n$. Clearly $G$ is a cyclic group of order $n$ with generator $a$. By Proposition 2.3, $deg_{\Gamma_{OP}(G)}(e) = n - 1$. Since $G = <a>$, $<a> = <a^2> = \ldots = <a^{n-1}>$. Therefore $deg_{\Gamma_{OP}(G)}(a^i) = n - 2$ for $i = 1, 2, \ldots, n - 1$. From this we get the degree sum $= (n - 1) + (n - 1)(n - 2) = (n - 1)^2$. Hence $q = \frac{(n-1)^2}{2}$.

**Theorem 2.9.** Let $G$ be a group of order $p$. $\Gamma_{OP}(G) \cong K_{1, 2, 2, \ldots, ktimes}$, where $k = \frac{p-1}{2}$ if and only if $p$ is prime.
Proof. Let $G$ be a group of order $p$. Assume that $\Gamma_{OP}(G) \cong K_{1,2,\ldots,k\times}$, where $k = \frac{p-1}{2}$. Clearly the number of edges of the graph $\Gamma_{OP}(G)$ is $\frac{(p-1)^2}{2}$. Therefore by Theorem 2.8, $p$ is a prime number.

Conversely, assume that $p$ is prime. Therefore $G$ is a cyclic group of order $p$ and also every element other than $e$ of $G$ is a generator of $G$. Therefore for every $a \in G - e$ is not adjacent to $a^{-1}$ only. Therefore we can be partition the vertex set of $\Gamma_{OP}(G)$ into $k + 1$ set, where $k = \frac{p-1}{2}$ such that the identity element $e$ belongs into single partition and the remaining $k$ sets, each set contains the pair elements $a$ and $a^{-1}$. Clearly each partition is an independent set and every element of one partition is adjacent to every element of other partition. Hence $\Gamma_{OP}(G) \cong K_{1,2,\ldots,k\times}$, where $k = \frac{p-1}{2}$. 

**References**


Received: August 1, 2014