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Some Bounds and Exact Results on the Substantial Independence Number of Tensor Product of Two Simple Connected Graphs

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Abstract

Given a graph G, a substantial independent set S is a non empty subset of the vertex set V of the graph G = (V, E) if (i) S is an independent set of G and (ii) every vertex in V\S is adjacent to at most one vertex in S. The substantial independence number β_S (G) is the maximum cardinality of a maximal substantial independent set. In this paper we study the substantial independence number and some bounds for the Tensor product of two simple graphs namely $P_2 \times P_n$, $P_m \times K_{1,3}$, $C_m \times K_{1,3}$, $P_2 \times CP_n$, $P_m \times CP_n$, $K_2 \times G_n$, $K_2 \times H_n$ and $P_m \times P_n$.

Keywords: Substantial independent set, Substantial independence number, tensor product

1. Introduction

Given a graph G, a substantial independent set S is a nonempty subset of the vertex set V of a graph G = (V, E) if (i) S is an independent set of G and (ii) every vertex in V\S is adjacent to at most one vertex in S. A substantial independence set S is set to be maximal if any vertex set properly containing S is not substantial independent set. The substantial independence number $\beta_S(G)$ is the maximum cardinality of a maximal substantial independent set.

Definition 1:1: The Tensor product G$\times$H of two graphs G and H is a graph with vertex set V(G)$\times$V(H) and edge set consisting of those pairs of vertices $(u_1, v_1)(u_2, v_2)$ where $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$. That is $(u_1, v_1)(u_2, v_2) \in E(G \times H)$ whenever $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$.

Example 1.2: Consider the following graph G.

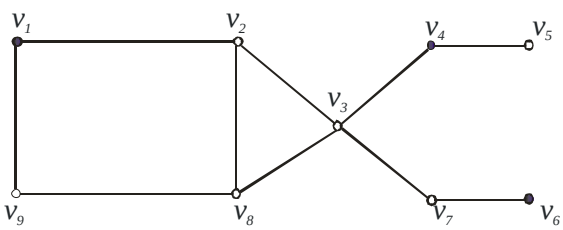

Fig. 1.1

Here {v_1, v_3, v_5, v_6}, {v_2, v_4, v_7, v_8} are maximal independent sets and hence the independence number $\beta(G) = 4$. The sets {v_1, v_4, v_6}, {v_5, v_6, v_9} are substantial independent sets which are maximal with maximum cardinality and hence $\beta_s(G) = 3$. Here {v_2, v_5, v_6, v_9} is a minimal dominating set and hence the dominating number $\gamma(G) = 4$.

Remarks: 1.3

1. For any graph G, $\beta_s \le \gamma \le \beta$ and equality holds for totally disconnected graphs. For bistar and for the graph $P_n o K_1$ we have $\beta_s = \beta$.
2. Let G be a graph with no isolated vertices and if S is a substantial independent set then V/S is a dominating set and hence we have
   $|V - S| \ge \gamma$ that is $n - \beta_s \ge \gamma$ that is $\gamma + \beta_s \le n$ that is $\beta_s \le n - r$
   Equality holds for the graph $P_{\frac{n}{2}} o K_1$ when n is even.

$\gamma\left(P_{\frac{n}{2}} o K_1\right) = \beta_s\left(P_{\frac{n}{2}} o K_1\right) = \frac{n}{2}$ and hence $\gamma + \beta_s = n$.

3. $\beta_s(K_n) = 1 \; n \geq 1$.
4. $\beta_s(K_{m,n}) = 1$ for $m, n \geq 1$
5. $\beta_s(W_n) = 1; n \geq 4$.
6. $\beta_s(Cb_n) = \frac{n}{2}$ where Cbn is a comb with n vertices.
7. $\beta_s(\overline{K}_n) = n$
8. In general for non isolated graphs $1 \leq \beta_s \leq \frac{n}{2}$

Theorem: 1.4: If S is a substantial independent set in a connected graph G=(V,E) and if u,v$\in$S then $d(u,v) \geq 3$.

Proof: Let S be a substantial independent set of G.
Claim: d (u,v)$\geq$3 for every u,v$\in$S.

Suppose $d(u,v) < 3$. Then d (u,v) = 1 or 2. But d(u,v)=1 is not possible, since u and v are independent vertices. If d (u,v)=2 then there is a vertex in V/S which is adjacent to both u and v. This is also not possible since S is a substantial independent set and $u, v \in S$. Hence $d(u,v) \geq 3$ for all $u, v \in S$.

Result: 1.5: Substantial independent is hereditary but not super hereditary hence a substantial independent set is maximal iff it is 1-maximal.

2. Substantial Independent set and the substantial independence number for the tensor product of two simple graphs namely $P_2 \times P_n$, $P_m \times K_{1,3}$, $C_m \times K_{1,3}$, $P_2 \times CP_n$, $P_m \times CP_n$, $K_2 \times G_n$, $K_2 \times H_n$ and $P_m \times P_n$.

Lemma: 2.1 If P_n is a path with n vertices then we have

$$\beta_s(P_2 \times P_n) = \begin{cases} 2\left\lceil \frac{n}{3} \right\rceil & \text{if } n \not\equiv 0 \pmod 3 \\ \frac{2n}{3} & \text{if } n \equiv 0 \pmod 3 \end{cases}$$

Proof: The graph $P_2 \times P_n$ given in figure 2.1 consists of two rows and n columns

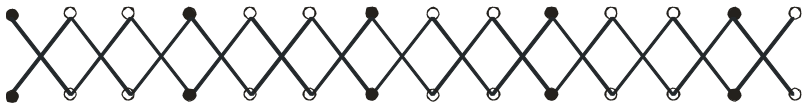

Fig.2.1

Let V ($P_2 \times P_n$) = $\{v_{i,j} : i = 1,2 \text{ and } j = 1,2...n\}$ be the vertex set of $P_2 \times P_n$. Clearly S = $\left\{v_{1,1+3i,} v_{2,1+3i} : i = 0,1,2....\left(\left\lceil \frac{n}{3} \right\rceil - 1\right)\right\}$ is a maximal substantial independent set for $P_2 \times P_n$ with maximum cardinality. Then $\beta_s(P_2 \times P_n) = |S| \geq 2.\left\lceil \frac{n}{3} \right\rceil$.

Since $P_2 \times P_n$ is the union of two paths and $\beta_s(P_n) = \left\lceil \frac{n}{3} \right\rceil$ we have $\beta_s(P_2 \times P_n) \leq 2.\beta_s(P_n) = 2\left\lceil \frac{n}{3} \right\rceil$.

So that we have $\beta_s(P_2 \times P_n) = 2\left\lceil \frac{n}{3} \right\rceil$ Hence $\beta_s(P_2 \times P_n) = \begin{cases} 2\left\lceil \frac{n}{3} \right\rceil \text{ if } n \not\equiv 0 \pmod 3 \\ \frac{2n}{3} \text{ if } n \equiv 0 \pmod 3 \end{cases}$

Note 2.2: When $n \equiv 0$ (mod 3) the maximum cardinality maximal substantial independent set for $P_2 \times P_n$ is a minimum cardinality minimal dominating set. Hence in this case $\beta_s(P_2 \times P_n) = \gamma(P_2 \times P_n)$.

Theorem: 2.3 For the bipartite graph $K_{1,3}$ and the path P_m we have

$$\beta_s(P_m \times K_{1,3}) = \begin{cases} \frac{2m}{3} \text{ if } m \equiv 0 \pmod 3 \\ 2\left\lceil \frac{m}{3} \right\rceil \text{ if } m \not\equiv 0 \pmod 3 \end{cases}$$

Proof: Let V ($P_m \times K_{1,3}$) = $\{v_{i,j} : i = 1,2,...m \text{ and } j = 1,2,3,4\}$

The graph $P_m \times K_{1,3}$ contains m – rows and 4 – columns. We select the vertices $v_{1,1}$ and $v_{1,4}$ from the first row. Then we cannot select any vertices from the second and third row. Next we select the vertices $v_{4,1}$ and $v_{4,4}$ from the fourth row. Proceeding like this we can construct a maximal substantial independent set S. Thus S= $\left\{v_{1+3i,1}, v_{1+3i,4} : i = 0,1,2,...\left(\left\lceil \frac{m}{3} \right\rceil - 1\right)\right\}$. Clearly S is 1 –maximal and hence maximal. If we consider any other maximal substantial independent set S' then

$|S'| \le |S|$. Hence S is a maximum cardinality maximal substantial independent set and $|S| = 2\left\lceil \frac{m}{3} \right\rceil$ and hence $\beta_s(P_m \times K_{1,3}) = 2\left\lceil \frac{m}{3} \right\rceil$ when $m \not\equiv 0 \pmod 3$ and $\beta_s(P_m \times K_{1,3}) = \frac{2m}{3}$ when $m \equiv 0 \pmod 3$.

Corollary: 2.4 We have $\beta_s(P_2 \times K_{1,3}) = 2$

Illustration: 2.5 Consider the graph $P_6 \times K_{1,3}$ in figure 2.2

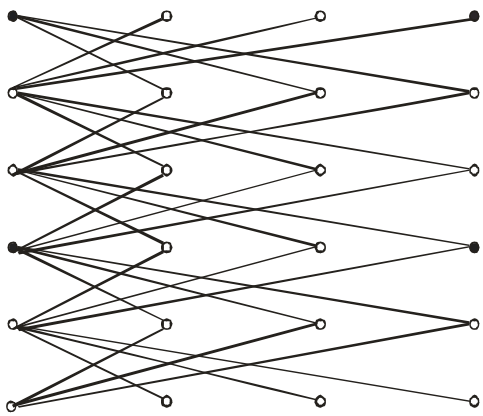

Fig.4.2

Here S = $\{v_{1,1}, v_{1,4}, v_{4,1}, v_{4,4}\}$ is the maximal substantial independent with maximum cardinality and hence $\beta_s(P_6 \times K_{1,3}) = 4$.

Theorem: 2.6 For the bipartite graph $K_{1,3}$ and the cycle C_m we have

$$\beta_s(C_m \times K_{1,3}) = \begin{cases} 2\left\lfloor \frac{m}{3} \right\rfloor \text{ if } m \not\equiv 0 \pmod 3 \\ \frac{2m}{3} \text{ if } m \equiv 0 \pmod 3 \end{cases}$$

Proof: Let $V(C_m \times K_{1,3}) = \{v_{i,j} : i = 1,2...m \text{ and } j = 1,2,3,4\}$. The graph $C_m \times K_{1,3}$ contains m-rows and 4-columns. Also $v_{1,1}$ is adjacent to $v_{m,2}, v_{m,3}$ and $v_{m,4}$ and $v_{m,1}$ is adjacent to $v_{1,2}$, $v_{1,3}$ and $v_{1,4}$. Hence the maximal substantial independent set is of the form.

$$S = \left\{ v_{1+3i,1}, v_{1+3i,4} : j = 0,1,2...\left(\left\lfloor \frac{m}{3} \right\rfloor - 1 \right) \right\}$$

Clearly $S' = S \cup \{v\}$ where $v \in V \setminus S$ is not a substantial independent set. Hence S is 1-maximal and hence maximal.

If S' is any other maximal substantial independent set then $|S'| \le |S|$.

Hence S is a maximal substantial independent set with maximum cardinality.

$$\text{Hence } \beta_s(C_m \times K_{1,3}) = |S| = 2 \times \left\lfloor \frac{m}{3} \right\rfloor = \begin{cases} 2\left\lfloor \frac{m}{3} \right\rfloor \text{ if } m \not\equiv 0 \pmod 3 \\ \frac{2m}{3} \text{ if } m \equiv 0 \pmod 3 \end{cases}$$

Illustration: 2.7 Consider the graph $C_5 \times K_{1,3}$ in figure 2.3

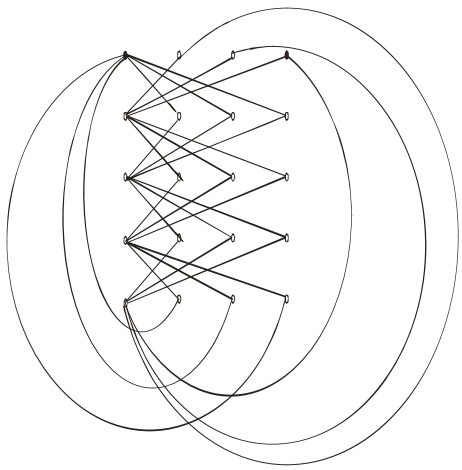

Fig.2.3

Here S = $\{v_{1,1}, v_{1,4}\}$ is the maximal substantial independent with maximum cardinality and hence $\beta_s(C_5 \times K_{1,3}) = 2$

Theorem: 2.8 If CP_n is a caterpillar in which the base vertices are of degree 3 then we have $\beta_s(P_2 \times CP_n) =$ n-2

<u>Proof</u>: Let, $V(P_2 \times CP_n)$ = $\{v_{i,j} : i = 1,2 \text{ and } j = 1,2....n\}$. CP_n is a caterpillar in which the base vertices are of degree 3. The graph $P_2 \times CP_n$ contains 2-rows and n-columns. Let S = $\left\{v_{1,3+2j}, v_{2,3+2j} : j = 0,1,...\left(\frac{n-2}{2} - 1\right)\right\}$

Then clearly S is 1-maximal and hence maximal substantial independent set. It S' is any other substantial independent set than $|S'| \le |S|$ hence $\beta_s(P_2 \times CP_n) = |S| = 2 \times \frac{n-2}{2} = n-2$.

Note: 2.9 The substantial independent number for CP_n is $\beta_s(CP_n) = \frac{n-2}{2}$.

Hence $\beta_s(P_2 \times CP_n) = 2 \times \frac{n-2}{2} = 2.\beta_s(CP_n)$

Illustration: 2.10 Consider the graph $P_2 \times CP_{12}$ in figure 2.4

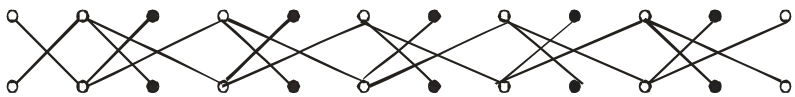

Fig.2.4

Here $\beta_s(P_2 \times CP_{12}) = 10$.

Theorem: 2.11 If CP_n is a caterpillar in which the base vertices are of degree 3

$$\text{then we have } \beta_s(P_m \times CP_n) \ge \begin{cases} \frac{m(n-2)}{4} \text{ if } m \equiv 0 \pmod 4 \\ \frac{(m+1)(n-2)}{4} \text{ if } m \equiv 1 \pmod 4 \\ \left\lceil \frac{m}{4} \right\rceil (n-2) \text{ if } m \equiv 2,3 \pmod 4 \end{cases}$$

Proof: Let $V(P_m \times CP_n) = \{v_{i,j:}, i = 1,2...m \text{ and } j = 1,2...n\}$. The graph $P_m \times CP_n$ contains m-rows and n-columns. To construct a maximal substantial independent set S we select the vertices $v_{1,3}$,$v_{1,3}$ etc. from the first row and $V_{2,3}$,$V_{2,5}$ etc. from

the second row. Then we cannot select any vertices from the third row and fourth row. Next we select the vertices $v_{5,3}, v_{5,5}$ etc. From the fifth row and $v_{6,3}$, $v_{6,5}$ etc. from the sixth row and so on. Then clearly S is 1-maximal and hence maximal substantial independent set. It S' is any other substantial independent set than $|S'| \le |S|$. There are three cases.

Case (i) : $m \equiv 0 \pmod 4$.
In this case the maximal substantial independent set is of the form.

$$S = \left\{ v_{1+4i,3+2j}, v_{2+4i,3+2j} : i = 0,1,\ldots\left(\frac{m}{4}-1\right) \text{ and } j = 0,1,\ldots\left(\frac{n-2}{2}-1\right) \right\}$$

So that $|S| = 2 \times \frac{m}{4} \times \frac{n-2}{2} = \frac{m(n-2)}{4}$. Hence $\beta_s(P_m \times CP_n) \ge \frac{m(n-2)}{4}$

Case (ii) : $m \equiv 1 \pmod 4$

In this case the maximal substantial independent set is of the form.

$$S = \left\{ v_{1+4i,3+2j}, v_{2+4i,3+2j}, v_{m,3+2} : i = 0,1,\ldots\left(\frac{m-1}{4}-1\right) \text{ and } j = 0,1,2,\ldots\left(\frac{n-2}{2}-1\right) \right\}$$

So that $|S| = 2\left(\frac{m-1}{4}\right).\frac{n-2}{2} + \frac{n-2}{2} = \left[2\frac{(m-1)}{4}+1\right]\left(\frac{n-2}{2}\right)$

$= \frac{m+1}{2}.\frac{n-2}{2} = \frac{(m+1)(n-2)}{4}$

Hence $\beta_s(P_m \times CP_n) \ge \frac{(m+1)(n-2)}{4}$

Case (iii) $m \equiv 2,3 \pmod 4$
In this case the maximal substantial independent set S is of the form

$$S = \left\{ v_{1+4i,3+2j}, v_{2+4i,3+2j} : i = 0,1,\ldots\left(\left\lceil\frac{m}{4}\right\rceil-1\right) \text{ and } 0,1,\ldots\left(\frac{n-2}{2}-1\right) \right\}$$

So that $|S| = 2.\left\lceil\frac{m}{4}\right\rceil.\left(\frac{n-2}{2}\right) = \left\lceil\frac{m}{4}\right\rceil(n-2)$. Hence $\beta_s(P_m \times CP_n) \ge \left\lceil\frac{m}{4}\right\rceil(n-2)$

Illustration: 2.12 Consider the graph $P_8 \times CP_{10}$ in figure 2.5

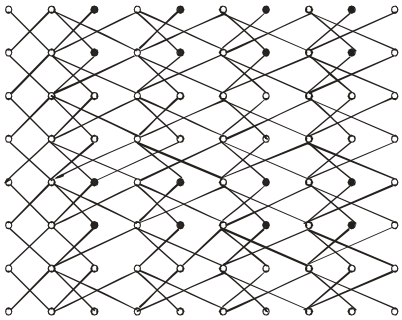

Fig.2.5

Clearly $\beta_s(P_8 \times CP_{10}) = \dfrac{8 \times (10-2)}{4} = 2 \times 8 = 16$

Theorem: 4.13 If G_n is a gear graph then we have $\beta_s(K_2 \times G_n) \geq \begin{cases} n \text{ if n is even} \\ n+1 \text{ if n is odd} \end{cases}$

Proof: G_n is the Gear graph with 2n+1 vertices and 3n edges. Hence the graph $K_2 \times G_m$ consists of 2-rows and 2n+1 columns.

Let $V(K_2 \times G_n) = \{v_{i,j} : i = 1,2 \text{ and } j = 1,2...(2n+1)\}$. To form a maximal substantial independent set we select the vertices $v_{1,3}, v_{1,7}$ etc. from the first row and $v_{2,3}$, $v_{2,7}$ etc. from the second row. Then clearly S is 1-maximal and hence maximal substantial independent set. It S' is any other substantial independent set than $|S'| \leq |S|$. There are two cases.

Case (i) : n is even.
In this case the maximal substantial independent set S=$\left\{ v_{1,3+4j}, v_{2,3+4j} : j = 0,1,...\left(\dfrac{n}{2}-1\right) \right\}$ so that $|S| = 2.\dfrac{n}{2} = n$ and hence $\beta_s(K_2 \times G_n) \geq n$.

Case (ii) n is odd
In this case the maximal substantial independent set S=$\left\{ v_{1,3+4j}, v_{2,3+4j}, v_{1,2n}, v_{2,2n} : j = 0,1,...\left(\dfrac{n-1}{2}-1\right) \right\}$ so that $|S| = 2 \times \left(\dfrac{n-1}{2}\right) + 2$ = $(n-1)+2 = n+1$ and hence $\beta_s(K_2 \times G_n) \geq n+1$

Illustration: 2.14 Consider the graph $K_2 \times G_7$ in figure 2.6

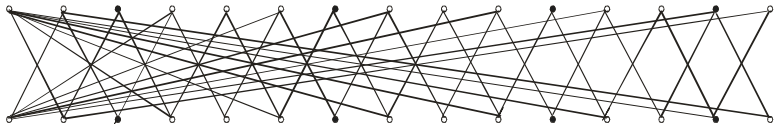

Fig.2.6

Clearly $\beta_s(K_2 \times G_7) = 8$

Illustration: 2.15 Consider the graph $K_2 \times G_8$ in figure 2.7

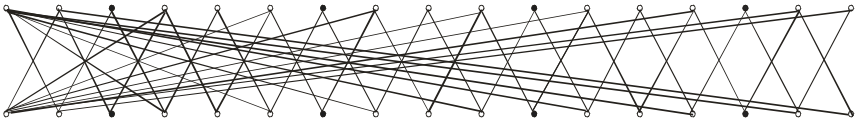

Fig. 2.7

Clearly $\beta_s(K_2 \times G_7) = 8$

Theorem: 2.16 If H_n is a Helm graph we have $\beta_s(K_2 \times H_n) = 2n$

<u>Proof</u>: We know that the Helm graph H_n is a graph with 2n+1 vertices and 3n edges. Let $V(K_2 \times H_n) = \{v_{i,j} : i = 1,2 \text{ and } j = 1,2.....(2n+1)\}$. According to the rule of Tensor product the vertex $v_{1,1}$ is adjacent to $v_{2,2}, v_{2,4}$…and $v_{2,1}$ is adjacent to $v_{1,2}, v_{1,4}$… Also $v_{1,2}$ is adjacent to $v_{2,1}, v_{2,3}$ and $v_{2,14}$ and $v_{2,2}$ is adjacent to $v_{1,1}, v_{1,3}$ and $v_{1,14}$. The vertices $v_{1,3}, v_{1,5}, v_{1,7}, v_{1,9}$ etc. are adjacent to only $v_{2,2}$, $v_{2,4}, v_{1,6}$ etc. respectively.

Similarly the vertices $v_{2,3}, v_{2,5}, v_{2,7}$ etc are only adjacent to $v_{1,2}, v_{1,4}, v_{1,6}$ etc. Hence to form a maximal substantial independent set we select the vertices $v_{1,3}, v_{1,5}$ etc. From the first row and $v_{2,3}, v_{2,5}$ etc. from the second row. Hence S = $\{v_{1,3+2j}, v_{2,3+2j}, : j = 0,1,...(n-1)\}$ is a maximal substantial independent set. Then clearly S is 1-maximal and hence maximal substantial independent set. It S' is any other substantial independent set than $|S'| \le |S|$.

So that $|S|$ =2n and hence $\beta_s(K_2 \times H_n) = |S| = 2n$.

Illustration: 2.17 Consider the graph $K_2 \times H_6$ in figure 2.8

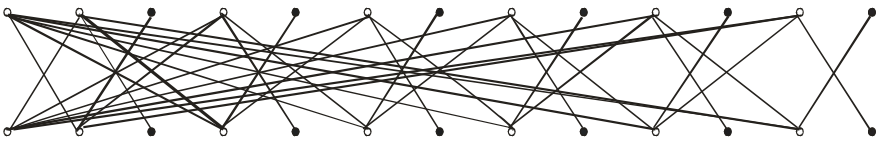

Fig.2.8

Clearly $\beta_s(K_2 \times H_6) = 2 \times 6 = 12$.

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