Some Bounds and Exact Results on the Substantial 

Independence Number of 

Tensor Product of Two Simple Connected Graphs

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Abstract

Given a graph G, a substantial independent set S is a non empty subset of the vertex set V of the graph G = (V, E) if (i) S is an independent set of G and (ii) every vertex in V\S is adjacent to at most one vertex in S. The substantial independence number \( \beta_s(G) \) is the maximum cardinality of a maximal substantial independent set. In this paper we study the substantial independence number and some bounds for the Tensor product of two simple graphs namely \( P_2 \times P_n, \ P_m \times K_{1,3}, \ C_m \times K_{1,3}, \ P_2 \times CP_n, \ P_m \times CP_n, \ K_2 \times G_n, \ K_2 \times H_n \) and \( P_m \times P_n \).

Keywords: Substantial independent set, Substantial independence number, tensor product
1. Introduction

Given a graph G, a substantial independent set S is a nonempty subset of the vertex set V of a graph G = (V, E) if (i) S is an independent set of G and (ii) every vertex in V\S is adjacent to at most one vertex in S. A substantial independence set S is set to be maximal if any vertex set properly containing S is not substantial independent set. The substantial independence number \( \beta_S(G) \) is the maximum cardinality of a maximal substantial independent set.

**Definition 1.1:** The Tensor product G×H of two graphs G and H is a graph with vertex set V(G)×V(H) and edge set consisting of those pairs of vertices \((u_1, v_1)(u_2, v_2)\) where \(u_1u_2 \in E(G)\) and \(v_1v_2 \in E(H)\). That is \((u_1, v_1)(u_2, v_2) \in E(G \times H)\) whenever \(u_1u_2 \in E(G)\) and \(v_1v_2 \in E(H)\).

**Example 1.2:** Consider the following graph G.

![Graph G](image)

Fig. 1.1

Here \(\{v_1, v_3, v_5, v_6\}, \{v_2, v_4, v_7, v_8\}\) are maximal independent sets and hence the independence number \( \beta(G) = 4 \). The sets \(\{v_1, v_4, v_6\}, \{v_5, v_6, v_9\}\) are substantial independent sets which are maximal with maximum cardinality and hence \( \beta_s(G) = 3 \). Here \(\{v_2, v_5, v_6, v_9\}\) is a minimal dominating set and hence the dominating number \( \gamma(G) = 4 \).

**Remarks: 1.3**

1. For any graph G, \( \beta_s \leq \gamma \leq \beta \) and equality holds for totally disconnected graphs. For bistar and for the graph \( P_n o K_1 \) we have \( \beta_s = \beta \).
2. Let G be a graph with no isolated vertices and if S is a substantial independent set then V/S is a dominating set and hence we have \( V - S \geq \gamma \) that is \( n - \beta_s \geq \gamma \) that is \( \gamma + \beta_s \leq n \) that is \( \beta_s \leq n - r \).

Equality holds for the graph \( P_n o K_1 \) when n is even.
Some bounds and exact results on the substantial independence number

\[ \gamma \left( P_n \circ K_1 \right) = \beta_s \left( P_n \circ K_1 \right) = \frac{n}{2} \quad \text{and hence} \quad \gamma + \beta_s = n. \]

3. \( \beta_s (K_n) = 1 \quad n \geq 1. \)

4. \( \beta_s \left( K_{m,n} \right) = 1 \quad \text{for} \quad m, n \geq 1 \)

5. \( \beta_s \left( W_n \right) = 1; n \geq 4. \)

6. \( \beta_s \left( C_{bn} \right) = \frac{n}{2} \) where \( C_{bn} \) is a comb with \( n \) vertices.

7. \( \beta_s \left( \overline{K}_n \right) = n \)

8. In general for non isolated graphs \( 1 \leq \beta_s \leq \frac{n}{2} \)

**Theorem: 1.4:** If \( S \) is a substantial independent set in a connected graph \( G=(V,E) \) and if \( u, v \in S \) then \( d(u,v) \geq 3. \)

**Proof:** Let \( S \) be a substantial independent set of \( G. \)

Claim: \( d(u,v) \geq 3 \) for every \( u, v \in S. \)

Suppose \( d(u,v) < 3. \) Then \( d(u,v) = 1 \) or 2. But \( d(u,v) = 1 \) is not possible, since \( u \) and \( v \) are independent vertices. If \( d(u,v) = 2 \) then there is a vertex in \( V/S \) which is adjacent to both \( u \) and \( v. \) This is also not possible since \( S \) is a substantial independent set and \( u, v \in S. \) Hence \( d(u,v) \geq 3 \) for all \( u, v \in S. \)

**Result: 1.5:** Substantial independent is hereditary but not super hereditary hence a substantial independent set is maximal iff it is 1-maximal.

**2. Substantial Independent set and the substantial independence number for the tensor product of two simple graphs namely \( P_2 \times P_n, P_m \times K_{1,3}, C_m \times K_{1,3}, P_2 \times CP_n, P_m \times CP_n, K_2 \times G_n, K_2 \times H_n, \) and \( P_m \times P_n. \)**

**Lemma: 2.1** If \( P_n \) is a path with \( n \) vertices then we have

\[ \beta_s \left( P_2 \times P_n \right) = \begin{cases} \left\lfloor \frac{n}{3} \right\rfloor & \text{if} \quad n \not\equiv 0 \pmod{3} \\ \frac{2n}{3} & \text{if} \quad n \equiv 0 \pmod{3} \end{cases} \]
Proof: The graph $P_2 \times P_n$ given in figure 2.1 consists of two rows and $n$ columns.

Let $V(P_2 \times P_n) = \{v_{i,j} : i = 1, 2 \text{ and } j = 1, 2, \ldots, n\}$ be the vertex set of $P_2 \times P_n$. Clearly $S = \{v_{i,1+3i}, v_{2,1+3i} : i = 0, 1, 2, \ldots, \left\lceil \frac{n}{3} \right\rceil - 1\}$ is a maximal substantial independent set for $P_2 \times P_n$ with maximum cardinality. Then $\beta_s(P_2 \times P_n) = |S| \geq 2 \left\lceil \frac{n}{3} \right\rceil$.

Since $P_2 \times P_n$ is the union of two paths and $\beta_s(P_n) = \left\lceil \frac{n}{3} \right\rceil$ we have $\beta_s(P_2 \times P_n) \leq 2 \beta_s(P_n) = 2 \left\lceil \frac{n}{3} \right\rceil$.

So that we have $\beta_s(P_2 \times P_n) = 2 \left\lceil \frac{n}{3} \right\rceil$ Hence $\beta_s(P_2 \times P_n) = \begin{cases} 2 \left\lceil \frac{n}{3} \right\rceil & \text{if } n \not\equiv 0 \pmod{3} \\ \frac{2n}{3} & \text{if } n \equiv 0 \pmod{3} \end{cases}$

Note 2.2: When $n \equiv 0 \pmod{3}$ the maximum cardinality maximal substantial independent set for $P_2 \times P_n$ is a minimum cardinality minimal dominating set. Hence in this case $\beta_s(P_2 \times P_n) = \gamma(P_2 \times P_n)$.

Theorem 2.3 For the bipartite graph $K_{1,3}$ and the path $P_m$ we have

$\beta_s(P_m \times K_{1,3}) = \begin{cases} \frac{2m}{3} & \text{if } m \equiv 0 \pmod{3} \\ 2 \left\lceil \frac{m}{3} \right\rceil & \text{if } m \not\equiv 0 \pmod{3} \end{cases}$

Proof: Let $V(P_m \times K_{1,3}) = \{v_{i,j} : i = 1, 2, \ldots, m \text{ and } j = 1, 2, 3, 4\}$
The graph $P_m \times K_{1,3}$ contains $m$ rows and 4 columns. We select the vertices $v_{1,1}$ and $v_{1,4}$ from the first row. Then we cannot select any vertices from the second and third row. Next we select the vertices $v_{4,1}$ and $v_{4,4}$ from the fourth row. Proceeding like this we can construct a maximal substantial independent set $S$. Thus $S = \{v_{i+3i,1}, v_{i+3i,4} : i = 0, 1, 2, \ldots, \left\lceil \frac{m}{3} \right\rceil - 1\}$. Clearly $S$ is 1-maximal and hence maximal. If we consider any other maximal substantial independent set $S'$ then $|S'| \leq |S|$. Hence $S$ is a maximum cardinality maximal substantial independent set and $|S| = 2 \left\lceil \frac{m}{3} \right\rceil$ and hence $\beta_s(P_m \times K_{1,3}) = 2 \left\lceil \frac{m}{3} \right\rceil$ when $m \not\equiv 0 \pmod{3}$ and $\beta_s(P_m \times K_{1,3}) = \frac{2m}{3}$ when $m \equiv 0 \pmod{3}$.

**Corollary: 2.4** We have $\beta_s(P_2 \times K_{1,3}) = 2$

**Illustration: 2.5** Consider the graph $P_6 \times K_{1,3}$ in figure 2.2

Here $S = \{v_{1,1}, v_{1,4}, v_{4,1}, v_{4,4}\}$ is the maximal substantial independent with maximum cardinality and hence $\beta_s(P_6 \times K_{1,3}) = 4$.

**Theorem: 2.6** For the bipartite graph $K_{1,3}$ and the cycle $C_m$ we have

$$\beta_s(C_m \times K_{1,3}) = \begin{cases} 
2 \left\lceil \frac{m}{3} \right\rceil & \text{if } m \not\equiv 0 \pmod{3} \\
\frac{2m}{3} & \text{if } m \equiv 0 \pmod{3}
\end{cases}$$
Proof: Let \( V(C_m \times K_{1,3}) = \{v_{i,j} : i = 1,2...m \text{ and } j = 1,2,3,4 \} \). The graph \( C_m \times K_{1,3} \) contains \( m \)-rows and 4-columns. Also \( v_{1,1} \) is adjacent to \( v_{m,2}, v_{m,3} \) and \( v_{m,4} \) and \( v_{m,1} \) is adjacent to \( v_{1,2}, v_{1,3} \) and \( v_{1,4} \). Hence the maximal substantial independent set is of the form:

\[
S = \left\{ v_{i,j} : j = 0,1,2...\left\lfloor \frac{m}{3} \right\rfloor - 1 \right\}
\]

Clearly \( S' = S \cup \{v\} \) where \( v \in V \setminus S \) is not a substantial independent set. Hence \( S \) is 1-maximal and hence maximal.

If \( S' \) is any other maximal substantial independent set then \( |S'| \leq |S| \).

Hence \( S \) is a maximal substantial independent set with maximum cardinality.

Hence \( \beta_S(C_m \times K_{1,3}) = \left| \left\{ v_{i,j} \right\} \right| = 2 \times \left\lfloor \frac{m}{3} \right\rfloor = 2 \left\lfloor \frac{m}{3} \right\rfloor \) if \( m \not\equiv 0 \pmod{3} \)

\[
= \begin{cases} 
2 \left\lfloor \frac{m}{3} \right\rfloor & \text{if } m \not\equiv 0 \pmod{3} \\
 \frac{2m}{3} & \text{if } m \equiv 0 \pmod{3}
\end{cases}
\]

Illustration: 2.7 Consider the graph \( C_3 \times K_{1,3} \) in figure 2.3

![Fig.2.3](image.png)

Here \( S = \{v_{1,1}, v_{1,4}\} \) is the maximal substantial independent with maximum cardinality and hence \( \beta_S(C_3 \times K_{1,3}) = 2 \).
**Theorem: 2.8** If $CP_n$ is a caterpillar in which the base vertices are of degree 3 then we have $\beta_s(P_2 \times CP_n) = n - 2$

**Proof:** Let, $V(P_2 \times CP_n) = \{v_{ij} : i = 1, 2 \text{ and } j = 1, 2, \ldots, n\}$. $CP_n$ is a caterpillar in which the base vertices are of degree 3. The graph $P_2 \times CP_n$ contains 2-rows and n-columns. Let $S = \{v_{1,3+2j}, v_{2,3+2j} : j = 0, 1, \ldots, \left\lfloor \frac{n-2}{2} \right\rfloor\}$.

Then clearly $S$ is 1-maximal and hence maximal substantial independent set. It $S'$ is any other substantial independent set than $|S'| \leq |S|$ hence $\beta_s(P_2 \times CP_n) = |S| = 2 \times \frac{n-2}{2} = n - 2$.

**Note: 2.9** The substantial independent number for $CP_n$ is $\beta_s(CP_n) = \frac{n-2}{2}$.

Hence $\beta_s(P_2 \times CP_n) = 2 \times \frac{n-2}{2} = 2 \beta_s(CP_n)$

**Illustration: 2.10** Consider the graph $P_2 \times CP_{12}$ in figure 2.4

Here $\beta_s(P_2 \times CP_{12}) = 10$.

**Theorem: 2.11** If $CP_n$ is a caterpillar in which the base vertices are of degree 3

then we have $\beta_s(P_m \times CP_n) \geq \begin{cases} \frac{m(n-2)}{4} & \text{if } m \equiv 0 \pmod{4} \\
\frac{(m+1)(n-2)}{4} & \text{if } m \equiv 1 \pmod{4} \\
\frac{m}{4}(n-2) & \text{if } m \equiv 2,3 \pmod{4} \end{cases}$

**Proof:** Let $V(P_m \times CP_n) = \{v_{ij} : i = 1, 2 \ldots m \text{ and } j = 1, 2 \ldots n\}$. The graph $P_m \times CP_n$ contains m-rows and n-columns. To construct a maximal substantial independent set $S$ we select the vertices $v_{1,3}, v_{1,3}$ etc. from the first row and $V_{2,3}, V_{2,5}$ etc. from
the second row. Then we cannot select any vertices from the third row and fourth row. Next we select the vertices \( v_{5,3}, v_{5,5} \) etc. From the fifth row and \( v_{6,3}, v_{6,5} \) etc. from the sixth row and so on. Then clearly S is 1-maximal and hence maximal substantial independent set. It \( S' \) is any other substantial independent set than \( |S'| \leq |S| \). There are three cases.

Case (i) : \( m \equiv 0 \) (mod 4).

In this case the maximal substantial independent set is of the form.

\[
S = \left\{ v_{1+4i,3+2j}, v_{2+4i,3+2j}, v_{3+4i,3+2j} : i = 0,1,\ldots, \left\lfloor \frac{m}{4} \right\rfloor - 1 \text{ and } j = 0,1,\ldots, \left\lfloor \frac{n-2}{2} \right\rfloor - 1 \right\}
\]

So that \( |S| = 2 \times \frac{m}{4} \times \frac{n-2}{2} = \frac{m(n-2)}{4} \). Hence \( \beta_s(P_m \times CP_n) \geq \frac{m(n-2)}{4} \)

Case (ii) : \( m \equiv 1 \) (mod 4)

In this case the maximal substantial independent set is of the form.

\[
S = \left\{ v_{1+4i,3+2j}, v_{2+4i,3+2j}, v_{3+4i,3+2j} : i = 0,1,\ldots, \left\lfloor \frac{m-1}{4} \right\rfloor - 1 \text{ and } j = 0,1,2,\ldots, \left\lfloor \frac{n-2}{2} \right\rfloor - 1 \right\}
\]

So that \( |S| = 2 \left( \frac{m-1}{4} \right) \frac{n-2}{2} + \frac{n-2}{2} = \left( \frac{m-1}{4} \right) + 1 \left( \frac{n-2}{2} \right) \)

\[
= \frac{m+1}{2} \frac{n-2}{2} = \frac{(m+1)(n-2)}{4}
\]

Hence \( \beta_s(P_m \times CP_n) \geq \frac{(m+1)(n-2)}{4} \)

Case (iii) \( m \equiv 2,3 \) (mod 4)

In this case the maximal substantial independent set S is of the form

\[
S = \left\{ v_{1+4i,3+2j}, v_{2+4i,3+2j}, v_{3+4i,3+2j} : i = 0,1,\ldots, \left\lfloor \frac{m}{4} \right\rfloor - 1 \text{ and } 0,1,\ldots, \left\lfloor \frac{n-2}{2} \right\rfloor - 1 \right\}
\]

So that \( |S| = 2 \left( \frac{m}{4} \right) \left( \frac{n-2}{2} \right) = \left( \frac{m}{4} \right)(n-2) \). Hence \( \beta_s(P_m \times CP_n) \geq \left( \frac{m}{4} \right)(n-2) \)
Illustration: 2.12 Consider the graph $P_8 \times CP_{10}$ in figure 2.5

![Fig.2.5](image)

Clearly $\beta_s(P_8 \times CP_{10}) = \frac{8 \times (10 - 2)}{4} = 2 \times 8 = 16$

Theorem: 4.13 If $G_n$ is a gear graph then we have $\beta_s(K_2 \times G_n) \geq \begin{cases} n & \text{if } n \text{ is even} \\ n + 1 & \text{if } n \text{ is odd} \end{cases}$

Proof: $G_n$ is the Gear graph with $2n+1$ vertices and $3n$ edges. Hence the graph $K_2 \times G_n$ consists of 2-rows and $2n+1$ columns.

Let $V(K_2 \times G_n) = \{v_{i,j} : i = 1,2 \text{ and } j = 1,2\ldots(2n+1)\}$. To form a maximal substantial independent set we select the vertices $v_{1,3}, v_{1,7}$ etc. from the first row and $v_{2,3}, v_{2,7}$ etc. from the second row. Then clearly $S$ is 1-maximal and hence maximal substantial independent set. It $S'$ is any other substantial independent set than $|S'| \leq |S|$. There are two cases.

Case (i) : $n$ is even.
In this case the maximal substantial independent set
$S = \left\{v_{1,3+j}, v_{2,3+j} : j = 0,1,\ldots, \left(\frac{n}{2} - 1\right)\right\}$ so that $|S| = 2 \cdot \frac{n}{2} = n$ and hence $\beta_s(K_2 \times G_n) \geq n$.

Case (ii) $n$ is odd
In this case the maximal substantial independent set
$S = \left\{v_{1,3+j}, v_{2,3+j}, v_{1,2n}, v_{2,2n} : j = 0,1,\ldots, \left(\frac{n-1}{2} - 1\right)\right\}$ so that $|S| = 2 \times (\frac{n-1}{2}) + 2$
$= (n-1) + 2 = n + 1$ and hence $\beta_s(K_2 \times G_n) \geq n + 1$
**Illustration: 2.14** Consider the graph $K_2 \times G_7$ in figure 2.6

Fig. 2.6

Clearly $\beta_s(K_2 \times G_7) = 8$

**Illustration: 2.15** Consider the graph $K_2 \times G_8$ in figure 2.7

Fig. 2.7

Clearly $\beta_s(K_2 \times G_7) = 8$

**Theorem: 2.16** If $H_n$ is a Helm graph we have $\beta_s(K_2 \times H_n) = 2n$

**Proof:** We know that the Helm graph $H_n$ is a graph with $2n+1$ vertices and $3n$ edges. Let $V(K_2 \times H_n) = \{v_{i,j} : i = 1,2 \text{ and } j = 1,2,...,(2n+1)\}$. According to the rule of Tensor product the vertex $v_{1,1}$ is adjacent to $v_{2,2}, v_{2,4},...$ and $v_{2,1}$ is adjacent to $v_{1,2}, v_{1,4},...$ Also $v_{1,2}$ is adjacent to $v_{2,1}, v_{2,3}$ and $v_{2,14}$ and $v_{2,2}$ is adjacent to $v_{1,1}, v_{1,3}$ and $v_{1,14}$. The vertices $v_{1,3}, v_{1,5}, v_{1,7}, v_{1,9}$ etc. are adjacent to only $v_{2,2}, v_{2,4}, v_{1,6}$ etc. respectively.

Similarly the vertices $v_{2,3}, v_{2,5}, v_{2,7}$ etc are only adjacent to $v_{1,2}, v_{1,4}, v_{1,6}$ etc. Hence to form a maximal substantial independent set we select the vertices $v_{1,3}, v_{1,5}$ etc. From the first row and $v_{2,3}, v_{2,5}$ etc. from the second row. Hence $S = \{v_{1,3+2j}, v_{2,3+2j} : j = 0,1,...(n-1)\}$ is a maximal substantial independent set. Then clearly $S$ is 1-maximal and hence maximal substantial independent set. It $S'$ is any other substantial independent set than $|S'| \leq |S|$.

So that $|S| = 2n$ and hence $\beta_s(K_2 \times H_n) = |S| = 2n$.

**Illustration: 2.17** Consider the graph $K_2 \times H_6$ in figure 2.8

Fig. 2.8

Clearly $\beta_s(K_2 \times H_6) = 2 \times 6 = 12$. 
References


Received: August 22, 2014; Published: October 22, 2014