On Hamerstein-Volterra Integral Equation

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Abstract

A numerical method is used to transform the Hamerstein–Volterra integral equation into a system of Hamerstein integral equations. Then a classical method of the degenerate kernel method is applied to solve the system of Hamerstein equations. Also, numerical examples are solved.

Keywords: Integral equation, Hamerstein-Volterra (H-V), degenerate method

Introduction

Many problems of mathematical physics, contact problems in the theory of elasticity and mixed problems of mechanics of continuous media are reduced to mixed type of integral equations, (see [1-4]). Many different methods are used to solve the integral equations analytically, see [5,6,7]. Also, for numerical methods, we refer to [8,9]. This paper is concerned with the problem of finding numerical solution of the Hamerstein-Volterra equation

\[ \varphi(x, t) = \frac{1}{0} k(x, y) \psi(y, \varphi(y, t)) \, dy - \int_{0}^{t} F(t, \tau) \varphi(x, \tau) \, d\tau = f(x, t) \]  

(1.1)
where the kernel \( k(x,y) \) of Hamerstein integral term is considered in space and known continuous function in \( L_2[0,1] \) while the kernel of Volterra term is considered in time and known continuous functions in \( C[0,T] \). The known continuous function \( f(x,t) \) and \( \psi(y,\varphi(y,t)) \) belong to the space \( L_2[0,1] \times C[0,T] \). While \( \varphi(x,t) \) is the unknown function. In order to guarantee the existence of a unique solution we assume through this work the following conditions.

(i) The kernel \( k(x,y) \in C([0,1] \times [0,1]) \), \( x, y \in [0,1] \) satisfies

\[
|k(x,y)| < M, M \text{ is a constant.}
\]

(ii) The positive continuous kernel \( F(t,\tau) \in C[0,T] \) satisfies

\[
|F(t,\tau)| < M_2, M_2 \text{ is a constant.}
\]

(iii) The given function \( f(x,t) \) with its partial derivatives are continuous in the space \( L_2[0,1] \times C[0,T] \) and its norm is defined as

\[
\|f\| = \max_{0 \leq \tau \leq T} \left\{ \int_0^1 \left\{ \int_0^1 f^2(x,\tau) dx \right\}^{1/2} d\tau \right\}
\]

(iv) The given continuous function \( \psi(x,\varphi(x,\tau)) \) satisfies the Lipshitz condition with respect to the unknown argument \( \varphi(x,\tau) \) and its norm is defined as

\[
\|\psi\| = \max_{0 \leq \tau \leq T} \left\{ \int_0^1 \left\{ \int_0^1 \psi^2(y,\varphi(y,\tau)) dy \right\}^{1/2} d\tau \right\}
\]

2. Numerical Method

To represent the H-VIE of Eq. (1.1) to a system of HE. we divide the interval \( [0,T] \), \( 0 \leq t \leq T < \infty \) as \( 0 = t_0 < t_1 < ... < t_N \), where \( t = t_n \), \( n = 0,1,2,...N \). Then using the quadrature formula \( w_j \), \( j = 0,1,...,n \), (see [8,9]) we have

\[
\varphi(x,t_n) - \frac{1}{0} k(x,y)\psi(y,\varphi(y,t_n))dy
\]

\[
- \sum_{j=0}^n w_j F(t_n,t_j) \varphi(x,t_j) + 0(h_n^{p+1}) = f(x,t_n)
\]

(2.1)
where \( h_n = \max_{0 \leq j < n} h_j \), \( h_j = t_{j+1} - t_j \).

The values of the weight formula \( w_j \) and the constant \( p \) depend on the number of derivatives of \( F(t, \tau) \) for all \( \tau \in [0, T] \), for example, if \( F(t, \tau) \in C^3 [0, T] \), then we have \( p = 3, \ p = n \) and \( w_0 = h_0 / 2, \ w_3 = h_3 / 2, \ w_j = h_j, \ j = 1, 2. \)

More information for the characteristic points and the quadrature coefficients are found in [8,9]. Using the following notations

\[
\varphi(x, t_n) = \varphi_n(x), \quad F(t_n, t_j) = F_{n,j} , \quad f(x, t_k) = f_k(x), \quad (2.2)
\]

we can rewrite, after neglecting the error, the formula (2.1) in the form

\[
\mu_n \varphi_n(x) - \int_0^1 k(x, y) \varphi(y, \varphi_n(y)) \, dy = f_n(x) + \sum_{j=0}^{n-1} w_j F_{n,j} \varphi_j(x)
\]

\[
(\mu_n = 1 - w_n F_{nn}) \quad (2.3)
\]

The integral equation (2.3) represents a system of Hamerstein integral equation of the the first kind, when \( \mu_n = 0 \) and for the second kind, for all values of \( n \), \( w_n F_{nn} \neq 1 \).

The solution of the system (2.3), when \( \mu_n \neq 0 \) can be obtained using different methods. In [10], a new collocation method is used to solve (2.3), when \( n = 0 \).

Also a variation of Nystrom method is presented in [11] to obtain the Hamerstein integral equation of the second kind (i.e when \( n = 0 \))

### 3. Degenerate kernel method, see[6]

In this section we will apply the degenerate kernel method for this, assume \( k_l(x, y) \) is an approximation of the kernel \( k(x, y) \), that satisfies the condition

\[
\lim_{l \to \infty} \left\{ \int_0^1 \int_0^1 \left| k(x, y) - k_l(x, y) \right|^2 \, dx \, dy \right\}^{1/2} = 0 \quad as \quad l \to \infty \quad (3.1)
\]

Also, assume

\[
k_l(x, y) = \sum_{i=1}^l B_i(x) C_i(y) \quad (3.2)
\]

where the set of the function \( B_i(x) \) is assumed to be linear.
It is natural to expect that the solution of the following equation associated with the degenerate kernels \( k_n(x,t) \) converges to the exact solution (1.1)

\[
\mu_n \phi_{n_l}(x) - \int_0^1 k_1(x,y) \psi(y,\varphi_{n_l}(y)) \, dy = f_n(x) + \sum_{j=0}^{n_l-1} w_j F_{n_l,j} \varphi_{n_l}(x)
\]  

(3.3)

Using (3.2) in (2.5), we get

\[
\mu_n \phi_{n_l}(x) - \sum_{i=1}^l a_{n_i} B_{n_i}(x) = f_n(x) - \sum_{j=0}^{n_l-1} w_j F_{n_l,j} \varphi_{j_l}(x)
\]  

(3.4)

where

\[
a_{n_i} = \int_0^1 C_i(y) \psi(y,\varphi_{n_i}(y)) \, dy.
\]  

(3.5)

Here \( a_{n_j}'s \) represents the values of the constants that will be determined.

Using (3.3) in (3.5), we get

\[
a_{n_i} = \int_0^1 C_i(y) \psi(y,\varphi_{n_i}(y)) \, dy + \int_0^1 \frac{1}{\mu_n} (f_n(y) + \sum_{m=1}^l a_{n_m} B_{n_m}(y)) - \sum_{j=0}^{n_l-1} w_j F_{n,l,j} \varphi_{j_l}(y)) \, dy
\]  

(3.6)

Define the function

\[
H_{n_0}(a_{n_1},a_{n_2},...a_{n_l}) = \int_0^1 C_i(y) \psi(y,\varphi_{n_i}(y)) - \sum_{m=1}^l a_{n_m} B_{n_m}(y) - \sum_{j=0}^{n_l-1} w_j F_{n,l,j} \varphi_{j_l}(y)) \, dy
\]  

(3.7)

Here, the formula (3.6) represents a nonlinear system of algebraic equation in the form

\[
\begin{bmatrix}
a_{n_1} \\
a_{n_2} \\
. \\
. \\
. \\
a_{n_l}
\end{bmatrix} =
\begin{bmatrix}
H_{n_1}(a_{n_1},a_{n_2},...,a_{n_l}) \\
H_{n_2}(a_{n_1},a_{n_2},...,a_{n_l}) \\
. \\
. \\
. \\
H_{n_l}(a_{n_1},a_{n_2},...,a_{n_l})
\end{bmatrix}
\]  

(3.8)

which can be written in the vector form

\[
a = H(a)
\]  

(3.9)
where $a^T = (a_1, a_2, \ldots, a_l)$ and $H(\alpha)^T = (H_{m_1}(\alpha), H_{m_2}(\alpha), \ldots, H_{m_l}(\alpha))$.

We shall show that the unique solution of Eq. (3.9) corresponds to the unique solution of (3.4) for each $n_l$ under some mild assumptions.

**Theorem 1.** The system of integral equation (3.4) has a unique solution under the following condition

$$\left\{ \int_0^1 k_n(x,y) \, dx \, dy \right\}^{1/2} \leq M_1 \quad (3.10)$$

To prove the theorem, we use the relation

$$|k_n(x,y)| \leq |k(x,y) - k_n(x,y)| + |k(x,y)|$$

Then with the aid of Eq. (3.1), we neglect the small constant $\varepsilon$ where

$$|k(x,y) - k_n(x,y)| < \varepsilon$$

Assume, for $n_l > N$, the integral operator

$$T_{n_l} \varphi(x) = \int_0^1 k_l(x,y) \psi(y, \varphi_n(y)) \, dy$$

and

$$\tilde{T}_{n_l} \varphi(x) = T_{n_l} \varphi(x) + \sum_{j=0}^n w_j F_{n,j} \varphi_{n,j}(x). \quad (3.11)$$

It is straightforward to verify that $\|T_{n_l} \varphi\| \leq M_1 A \|\varphi\|$ for all $\varphi \in L_2 \{0,1\}$. Hence $T_{n_l}$ is a bounded nonlinear operator for all values of $0 \leq n \leq N$, $l$ is finite number. Also, we have

$$\|\tilde{T}_{n_l} \varphi^{(1)} - \tilde{T}_{n_l} \varphi^{(2)}\| = \left| \int_0^1 \int_0^1 k_l^2(x,y) \, dx \, dy \left[ \left| \varphi(y, \varphi_n^{(1)}(y)) - \varphi(y, \varphi_n^{(2)}(y)) \right|^2 \, dy \right]^{1/2} \right|$$

Using the conditions (i) and (ii), we get

$$\|T_{n_l} \varphi^{(1)} - T_{n_l} \varphi^{(2)}\| \leq \text{const.} \|\varphi^{(1)} - \varphi^{(2)}\| \quad (3.13)$$

i.e $\varphi^{(1)} = \varphi^{(2)}$

hence the proof is completed.

**Theorem 2.** For the degenerate kernel $k_l(x,y)$, let
\[ M = C \left\{ \sum_{i=0}^{l-1} \left| B_i(x) \right|^2 dx \right\} \left\{ \sum_{i=0}^{l-1} \left| C_i(x) \right|^2 dx \right\}^{1/2} \quad (M < 1) \] (3.14)

where, \( C \) is the constant of Lipschitz for the second argument of the function \( \psi(x, \varphi(x, t)) \). Then the nonlinear algebraic equation (3.9) has a unique solution

\[ \alpha^*_n = \left( \alpha^*_{n_1}, \alpha^*_{n_2}, \ldots, \alpha^*_{n_l} \right) \] (3.15)

and

\[ \varphi_{n_l} = f_n(x) + \sum_{i=1}^{l} \alpha^*_i B_i(x) + \sum_{j=0}^{l} w_j F_j n_l \varphi_{n_j} \] (3.16)

is the unique solution of (3.4).

Proof of this theorem depends on the definition of the discrete \( l_2 \) norm by

\[ \| \alpha_n \|_2 = \left\{ \sum_{j=1}^{l} \alpha_{n,j}^2 \right\}^{1/2} \quad \text{for} \quad \alpha = \left( \alpha_{n_1}, \alpha_{n_2}, \ldots, \alpha_{n_l} \right)^T \in l_2(l). \]

For \( \alpha^{(1)} = \left( \alpha^{(1)}_{n_1}, \alpha^{(1)}_{n_2}, \ldots, \alpha^{(1)}_{n_l} \right) \) and \( \alpha^{(2)} = \left( \alpha^{(2)}_{n_1}, \alpha^{(2)}_{n_2}, \ldots, \alpha^{(2)}_{n_l} \right) \)

we have

\[ \| H(\alpha^{(1)}) - H(\alpha^{(2)}) \| \leq B \left\{ \sum_{i=1}^{l} \left| B_i(x) \right|^2 dx \right\}^{1/2} \left\{ \sum_{i=1}^{n} \left| C_i(x) \right|^2 dx \right\}^{1/2} \| \alpha^{(1)} - \alpha^{(2)} \|_2 \]

Consequently \( F \) is a contraction operator in \( l(l), M < 1 \).

It is not difficult for the reader to prove the following theorem

**Theorem 3.** For

\[ \left\| k(x, y) - k_n(x, y) \right\| = \left\{ \int_0^1 \int_0^1 \left| k(x, y) - k_n(x, y) \right|^2 \ dx \ dy \right\}^{1/2} \]

we have

\[ \| \varphi - \varphi_n \| \leq \text{const.} \| k - k_n \| \| \varphi \| . \]

3. **Numerical results**

**Example 1:**

In this example, we consider the following integral equation

\[ \varphi(x, t) + \int_0^1 x^2 y \sqrt{\varphi(y, t)} \ dy + \int_0^t \varphi(x, \tau) \ d \tau = x^2 \left[t + \frac{\sqrt{t}}{3} + \frac{t^3}{3} \right], \]
the exact solution is \( \varphi(x, t) = x^2 t \), and the error is calculated in the table.

Example 2:
In this example, we consider the following integral equation
\[
\varphi(x, t) + \int_0^x y \varphi^2(y, t) \, dy + \int_0^t \varphi(x, \tau) \, d\tau = x^2 \left[ t + \frac{t^3}{3} \right] + \frac{x^2}{6},
\]
Also, the exact solution of the previous integral equation is \( \varphi(x, t) = x^2 t \). Using the degenerate kernel method, we obtained the error

<table>
<thead>
<tr>
<th>t</th>
<th>E(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.1</td>
<td>(0.2e^{-x^2} - 0.5e^{-x^2} )</td>
</tr>
<tr>
<td>0.2</td>
<td>(0.3e^{-x^2} - 0.2e^{-x^2} )</td>
</tr>
<tr>
<td>0.3</td>
<td>(0.5e^{-x^2} - 0.4e^{-x^2} )</td>
</tr>
<tr>
<td>0.4</td>
<td>(0.6e^{-x^2} - 0.7e^{-x^2} )</td>
</tr>
<tr>
<td>0.5</td>
<td>(0.8e^{-x^2} - 0.1e^{-x^2} )</td>
</tr>
<tr>
<td>0.6</td>
<td>(0.9e^{-x^2} - 0.1e^{-x^2} )</td>
</tr>
<tr>
<td>0.7</td>
<td>(1e^{-x^2} - 0.2e^{-x^2} )</td>
</tr>
<tr>
<td>0.8</td>
<td>(1e^{-2x^2} - 0.2e^{-4x} )</td>
</tr>
<tr>
<td>0.9</td>
<td>(1e^{-2x^2} - 0.2e^{-4x} )</td>
</tr>
<tr>
<td>1</td>
<td>(1e^{-2x^2} - 0.2e^{-4x} )</td>
</tr>
</tbody>
</table>

At \( h = 0.01, t = 1 \) we have \( E(x) = 0.1e^{-x^2} - 0.2e^{-x^2} \)
So when $h$ decreases, the error is also decreased.

At $h=0.1$, we have

<table>
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<th>$T$</th>
<th>$E(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.1</td>
<td>$1e^{-1.0}$</td>
</tr>
<tr>
<td>0.2</td>
<td>$2e^{-2.0}$</td>
</tr>
<tr>
<td>0.3</td>
<td>$4e^{-3.0}$</td>
</tr>
<tr>
<td>0.4</td>
<td>$5e^{-4.0}$</td>
</tr>
<tr>
<td>0.5</td>
<td>$6e^{-5.0}$</td>
</tr>
<tr>
<td>0.6</td>
<td>$7e^{-6.0}$</td>
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<tr>
<td>0.7</td>
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</tr>
<tr>
<td>0.8</td>
<td>$9e^{-8.0}$</td>
</tr>
<tr>
<td>0.9</td>
<td>$1e^{-9.0}$</td>
</tr>
<tr>
<td>1</td>
<td>$1e^{-10.0}$</td>
</tr>
</tbody>
</table>

At $h = 0.01, t = 1$ we have $E(x) = .1e^{-x^2}$.

References


Hamerstein-Volterra integral equation


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