Abstract

In this paper, we introduce the notion of Boolean like semi ring modules (modules over Boolean like semi rings) and study some basic properties. Also, we establish some significant differences between left and right Boolean like semi ring modules. Further, we introduce the notion of fractions of Boolean like semi ring modules and study its properties.

Mathematics Subject classification: 16Y30, 16Y60

Keywords: Boolean like semi rings, Boolean like semi ring modules, fractions

Introduction

The concept of modules over near rings has been studied by many authors in many ways. Their study was left near modules for right near rings or right near modules for left near rings. But Gary Ross [2] has introduced the notion of left near modules for left near rings in which he introduced a new line of
research which diversified the theory of near ring modules. Imitating the line of thought of Gary Ross, we introduce the notion of Boolean like semi ring modules (left module over left near ring) and also right module over left near ring.

The paper consists of three sections. In section one, we establish basic definitions and results concerning Boolean like semi rings. Especially, we have shown that right and left modules over Boolean like semi rings are structurally different and can’t be dually symmetric. This we substantiate in examples 1.5 and 1.6.

In section two, we introduce the notions of annihilators and congruence relations in Boolean like semi ring modules and study certain basic properties.

Finally in section three, we introduce the notion of fractions of Boolean like semi ring modules and prove that $S^{-1}M$ is Boolean like semi ring module (see theorem 3.3). Moreover we have studied certain properties between sub modules of M and $S^{-1}M$.

We refer the reader to [1, 3, 4, 5] for further definitions and results. We begin with the following:

1 Basic Properties

Where no confusion can arise, we hereafter shall refer a Boolean like semi ring module M simply as an R module.

**Lemma 1.1.** Every Boolean like semi ring is zero symmetric.

**Proof.** Let $r \in R$ be any element and 0 be the additive identity element of R. Clearly $r0 = 0$ follows from the left distributive property of ”.” over ”+” on R. Then, $0r = (r0)r = (r0)(r + 0 + r0) = r0 = 0 \square$

Next, we state the following results whose proofs are straightforward.

**Lemma 1.2.** Let M be a right R module, then the following basic results will follow from the definition of an R module and the above lemma.

1. $m0_R = 0_M$
2. $0_Mr = (m0_R)r = m(0_Rr) = m0_R = 0_M$
3. Every R-ideal of an R module M is a sub module of M.

**Definition 1.3.** An R module M is called of characteristic 2 if $m + m = 0 \forall m \in M$

**Lemma 1.4.** For $m \in M$, $mR = \{mr/r \in R\} = N$ is a sub module of M with characteristic 2.
Proof. Trivial

To have a better insight into the properties of left R modules, let us have the following two examples;

Example 1.5. Let \( R = \{0, a, b, c\} \) and define '+' and '.' by the following first two tables, and take \( M = \{0, 1, 2, 3\} \) and define '+' on \( M \) to be addition modulo four and a map \( \mu : R \times M \longrightarrow M \) by the last table;

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Thus we obtain a left module \( M \) over the Boolean like semi ring \( R \) and we observe that for every non zero \( m \) in \( M \), \( 0m \neq 0 \). Also, the characteristic of \( M \) as well as that of the sub set \( N = rM \) is not equal to 2 since \( 1+1 = 3 + 3 = 2 \) and \( a3 + a3 = 2 = a1 + a1 \). As a result, it is not always possible to get a sub structure of the kind \( N = rM \) of \( M \) with characteristic 2. Moreover, this example illustrates that the structure of modules is distinct from the structure of semi modules (modules over semi rings) because \((a + b)m = am + bm\) does not hold true in this case and not all semi modules are near modules.

Example 1.6. Let \( R = \{0, a, b, c\} \) and define '+' and '.' as in example 2.1, and define '+' on \( M \) to be addition modulo four and a map \( \mu : R \times M \longrightarrow M \) by the following table;

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In this example, see that every element is zero symmetric.

2 Annihilators and Congruence relations

2.1 Annihilators

In this, we obtain certain results that follow from annihilating sets for a module \( M \) over \( R \).
Definition 2.1. Let $M$ a right $R$ module. The annihilating set of $M$, denoted by $\text{A}(M)$, is defined as; $\text{A}(M) = \{ r \in R | mr = 0 \forall m \in M \}$

Similarly, the annihilating set of an element $m$ of an $R$ module $M$ is defined as, $\text{A}(m) = \{ r \in R | mr = 0 , m \in M \}$

Remark 2.2. Note that $\text{A}(0) = R$, which is not the case in general for a near module. This result is true for a Boolean like semi ring module because of the zero symmetric property of Boolean like semi rings which in turn emanates from the axiom $ab = ab(a + b + ab) \forall a, b \in R$.

Moreover, we have the following,

Lemma 2.3. Let $M$ be a right $R$ module. Then the set $\text{A}(M)$ is a right ideal of $R$.

Proof. Clearly $\text{A}(M)$ contains 0 and hence it is non empty. Let $m \in M$ and $x, y \in \text{A}(M)$ hence $mx + my = 0$. Since $M$ is an $R$ module we have $m(x + y) = 0$ which implies $x + y \in \text{A}(M)$. Hence $\text{A}(M)$ is a sub group of $R$. Next, let $m \in M, x \in \text{A}(M)$ and $r, s \in R$, then $m[(r + x)s + rs] = m(r + x)s + m(rs) = (mr + mx)s + m(rs) = m(rs) + m(rs) = m(rs + rs) = m0 = 0$ hence $(r + x)s + rs \in \text{A}(M)$. Thus $\text{A}(M)$ is a right ideal of $R$.

Remark 2.4.

1. If a Boolean like semi ring $R$ satisfies the condition $abc = bac$ which is called the left weak commutative property, then $\text{A}(M)$ becomes a left ideal and hence an ideal of $R$.

Proof. Let $x \in \text{A}(M)$ and $r \in R$. Then $m(rx) = m[rx(r + x + rx)] = m(rxr) + m(rxx) + m(rrx) = m(xr) + m(xrx) + m(xrrx) = (mx)rr + (mx)rx + (mx)rrx = 0rr + 0rx + 0rrx = 0$ hence $rx \in \text{A}(M)$.

2. An $R$ module $M$ is called unitary (right) if $R$ is unitary with the right unity element 1 and $m1 = m, \forall m \in M$. If $M$ is a unitary $R$ module and $R$ is a weak commutative, then $\text{A}(M)$ is also a left ideal and hence an ideal of $R$.

Proof. Let $x \in \text{A}(M)$ and $r \in R$, then $m(rx) = (m1)rx = m1rx = m1xr = m(xr) = (mx)r = 0r = 0$. Hence $rx \in \text{A}(M)$.

3. If $M$ is a left $R$ module, $\text{Ann}(M)$ may be empty. Also, let alone an ideal, it may not have a subgroup structure.

Example 2.5. In example 1.1 above $\text{A}(M) = \emptyset$.
2.2 Congruence relation

Let \( \sim \) be a relation on the left \( R \) module \( M \) with respect to the sub module \( N \) of \( M \). Define \( \sim \) on \( M \) by \( m_1 \sim m_2 \) if and only if \( m_1 - m_2 \in N \). Then, \( \sim \) is an equivalence relation.

**Definition 2.6.** Let \( M \) be an \( R \) module. A congruence relation \( \theta \) on \( M \) is an equivalence relation on \( M \) satisfying; \( (m, m') \in \theta \Rightarrow (m + n, m' + n) \in \theta \) and \( (rm, rm') \in \theta \) \( \forall m, m', n \in M \) and \( \forall r \in R \).

**Lemma 2.7.** Let \( \theta \) be an equivalence relation on a unitary \( R \) module \( M \) and \( \Theta = \{(m, m') \in MXM \text{ such that } (m, m') \in \theta \} \). If \( \theta \) is a congruence relation on \( M \), then \( \Theta \) is an \( R \) sub module of \( MXM \) and the maps; \( p_1 \) and \( p_2 : \Theta \rightarrow M \) defined by \( p_1(m,m') = m \) and \( p_2(m,m') = m' \) are homomorphisms of \( R \) sub modules.

**Proof.** Clearly \( MXM \) is a module over \( R \). Since \( \theta \) is an equivalence relation, \( \Theta \) contains the pair \((0,0)\) and hence \( \Theta \) is non empty. And let \( x = (m_1, m_2) \) and \( y = (m_3, m_4) \) be elements of \( \Theta \). Then by the congruence relation, \( (m_1 + m_3, m_2 + m_4) \in \Theta \) and \( (m_1 + m_3, m_2 + m_4) \in \Theta \Rightarrow (m_1 + m_3, m_2 + m_4) = x + y \in \Theta \) since \( \Theta \) is a congruence relation. Hence \( \Theta \) is a subgroup. Next, let \( r \in R, x = (m_1, m_2) \in \Theta \). Since \( \Theta \) is a congruence relation, \( (rm_1, rm_2) \in \Theta \Rightarrow rx \in \Theta \)

Clearly, \( p_1, p_2 \) are well defined maps. To show \( p_1, p_2 \) are homomorphisms; let \( x = (m_1, m_2), y = (m_3, m_4) \in \Theta \). Then, \( p_1(x + y) = p_1(m_1 + m_3, m_2 + m_4) = m_1 + m_3 = p_1(m_1, m_2) + p_1((m_3, m_4) \text{ And for } r \in R, x = (m_1, m_2) \in \Theta, \text{ we have } p_1(rx) = p_1((rm_1, rm_2))) = rm_1 = rp_1((m_1, m_2)) = rp_1(x). \) Hence \( p_1 \) is a module homomorphism. Similarly , \( p_2 \) is also a module homomorphism. \( \square \)

We know that there is a one to one correspondence between a congruence relation on a set \( M \) due to a sub module \( N \) and partitions \( \wp \) of \( M \). In this direction, we have the following result.

**Theorem 2.8.** Let \( M \) be an \( R \) module. Let \( \sim \) be an equivalence relation on \( M \) and \( M/ \sim \) be the associated partition on \( M \). Then,

a). If \( \sim \) is a congruence relation on \( M \) then \( M/ \sim \) is an \( R \) module by \( \overline{m + m'} = \overline{m} + \overline{m'} \) and \( \overline{mr} = \overline{mr} \)

b). If \( M/ \sim \) is an \( R \) module, and the canonical map, \( f : M \rightarrow M/ \sim \) with \( m \mapsto \overline{m} \) is a homomorphism of \( R \) modules, then \( \sim \) is a congruence relation.

**Proof.** To prove the theorem, we first show that \(+\) and \(\cdot\) are well defined.

Let \( \overline{m_1} = \overline{m_1'} \) and \( \overline{m_2} = \overline{m_2} \), then \( \overline{m_1} + \overline{m_2} = \overline{m_1 + m_2} = \overline{m_1} + \overline{m_2} = \overline{m_1 + m_2'} = \overline{m_1} + \overline{m_2} \). And, let \( \overline{m_1} = \overline{m_1'} \) and \( \overline{r_1} = \overline{r_2} \). Then, \( \overline{m_1 r_1} = \overline{m_1} \overline{r_1} = m_1 r_1 \) [since \( \sim \) is a congruence relation], \( \overline{m_1} \overline{r_2} = \overline{m_1} r_2 \). Thus both operations
are well defined.  
Clearly, $M/\sim$ is an Abelian group. Now, $\overline{m(r+s)} = m(r+s) = mr+ms = m\overline{r} + \overline{ms} = \overline{mr} + \overline{ms}$ and $\overline{m(rs)} = m(rs) = (mr)s = (\overline{mr})s$. Hence $M/\sim$ is an $R$ module.

To prove the second part, let $m_1 \sim m_2$, $m \in M$. $\Rightarrow \overline{m_1} = \overline{m_2} \Rightarrow f(m_1) = f(m_2)$. Then, $\overline{m_1} + m = f(m_1 + m) = f(m_1) + f(m) = f(m_2) + f(m) = f(m_2 + m) = \overline{m_2} + m$ and $\overline{m_1}r = f(m_1r) = f(m_1)r = \overline{m_1}r = \overline{m_2}r = \overline{m_2}r$ 

\[ \square \]

3 Modules of Fractions

We begin with the following,

**Theorem 3.1.** Let $S$ be a multiplicative subset of a weak commutative Boolean like semi ring $R$. Define a relation $\sim$ on $S \times M$ by: $(s_1, m_1) \sim (s_2, m_2) \iff t[s_2m_1] = t[s_1m_2]$ for some $t \in S$. Then $\sim$ is an equivalence relation.

**Proof.** Clearly $\sim$ is reflexive and symmetric. To prove $\sim$ is transitive, let $(s_1, m_1) \sim (s_2, m_2)$ and $(s_2, m_2) \sim (s_3, m_3)$.

$\Rightarrow t_1[s_2m_1] = t_1[s_1m_2]$ and $t_2[s_3m_2] = t_2[s_2m_3]$ for some $t_1, t_2 \in S$. Claim: There exists $t$ in $S$ such that $t[s_3m_1] = t[s_1m_3]$. Choose $t = t_1^2t_2s_2 \in S$ Then, $\overline{t(s_3m_1)} = (t_1^2t_2s_2)(s_3m_1) = (t_1t_2s_2)(s_2s_3m_1) = t_1(t_2s_2)(s_2s_3m_1) = (t_1^2s_2)(s_1s_2m_2) = t_1(t_2s_2)(s_1s_2m_2) = t_1(t_2s_2)(s_1s_2m_2) = t(s_1m_3)$ 

We denote the equivalence class containing $(s,m)$ by $\overline{m_s}$ and the set of all equivalence classes by $S^{-1}M$. The following are easy consequences of the above theorem and we have stated all without proof.

**Lemma 3.2.** Let $M$ be a left Boolean like semi ring module, then

\[ \begin{align*}
& a. \quad \frac{m}{s} = \frac{s'}{s} \quad \text{for all } m \in M, s', s \in S \\
& b. \quad \frac{m}{s} = \frac{s'm}{ss'} \quad \forall m \in M, s', s \in S \\
& c. \quad \frac{sm}{s} = \frac{s'm}{ss'} \quad \forall m \in M, s', s \in S
\end{align*} \]

**Theorem 3.3.** Let $S$ be a multiplicative set in a weak commutative Boolean like semi ring $R$ and $M$ be a left $R$-module. Define the operations $'+'$: $S^{-1}MXS^{-1}M \to S^{-1}M$ and $'.' : S^{-1}RXS^{-1}M \to S^{-1}M$ by; $\frac{m_1}{s_1} + \frac{m_2}{s_2} = \frac{s_2m_1 + s_1m_2}{s_1s_2}$; $\frac{r}{s} \cdot \frac{m}{t} = \frac{rm}{st}$ Then $S^{-1}M$ is an $S^{-1}R$ module.

**Proof.** we prove in two steps as;

First we prove $'+'$ is well defined. Let $\frac{m_1}{s_1} = \frac{m_1'}{s_1'}$ and $\frac{m_2}{s_2} = \frac{m_2'}{s_2'}$

$\Rightarrow t_1(s_1m_1) = t_1(s_1m_1')$ and $t_2(s_2m_2) = t_2(s_2m_2')$ for some $t_1, t_2 \in S$. choose $t = t_1^2t_2 \in S$ then, $t(s_1m_1)[s_2m_1 + s_1m_2] = (t_1^2t_2)(s_1m_1)(s_2m_1 + s_1m_2) = (t_1^2t_2)(s_1m_1)(s_2m_1 + s_1m_2) = \frac{s_2m_1 + s_1m_2}{s_1s_2} 

\[ \square \]
\[(t_1t_2)(t_1's_1')(s_2's_2)m_1 + t_1^2(t_2's_2')(s_2's_1)m_2 = t_1(t_2's_2s_2)t_1(s_1'm_1) + t_1^2(s_1's_1)t_2(s_2'm_2) =
(t_1^2t_2)(s_2'm_1) + (t_1^2t_2)(s_1's_1)(s_1'm_2) = t(s_1s_2)[s_2'm_1 + s_1'm_2].
\]

With the given definition of ‘+’, we have \(\frac{m_1}{s} + \frac{m_2}{s} = \frac{m_1 + m_2}{s}\)

Clearly \((S^{-1}M, +)\) is an Abelian group. Next, To show ‘.’ is well defined, Let \(\frac{r_1}{s_1} = \frac{r_2}{s_2}\) and \(\frac{m_1}{s_1} = \frac{m_2}{s_2}\) for some \(r_1, r_2 \in R\), \(m, m' \in M\) and \(s_1, s_2, s' \in S\)

\[\Rightarrow \exists t_1, t_2 \in S \ni t_1(s_2r_1) = t_1(s_1r_2)\] and \(t_2(s'm') = t_2(sm')\)

Now choose \(t = t_2t_2' \in S\) then, \(t[(s_2's')(r_1)m] = (t_2t_2')(s_2's')(r_1)m = t_1(t_2's')t_1(s_2r_1)m = t_1(t_2's')(t_1(s_1r_2))m = t_1(t_1(s_1r_2))t_2(s'm') = t_1t_2(s_1r_2)(sm') = (t_2t_2')(s_1s)(r_2m) = t[(s_1s)(r_2m)].\)

Hence ‘.’ is well defined. Next;

**Remark 3.4**) let \(\frac{r}{s} \in S^{-1}R\) and \(\frac{m_1}{s_1}, \frac{m_2}{s_2} \in S^{-1}M,\)

\[\frac{r_1}{s_1} \frac{r_2}{s_2} = \frac{r_1r_2}{s_1s_2} = \frac{r_1r_2}{s_1s_2}, \frac{r_1m_1}{s_1s_1} + \frac{r_2m_2}{s_2s_2} = \frac{r_1m_1 + r_2m_2}{s_1s_2}\]

b) for any \(\frac{r_1}{s_1}, \frac{r_2}{s_2} \in S^{-1}R\) and \(\frac{m}{s} \in S^{-1}M, \frac{r_1r_2}{s_1s_2} = \frac{r_1r_2m}{s_1s_2s_2} = \frac{r_1}{s_1} \frac{r_2}{s_2}.\) Thus \(S^{-1}M\) is a module over the Boolean like semi ring \(S^{-1}R.\)

**Notation:** We use the following notations: \(S^{-1}M := M_s, S^{-1}R := R_s\) and \(\frac{r}{s} := r, \frac{m}{s} := m_s\)

**Lemma 3.5.** Let \(N\) be a sub module of a Boolean like semi ring module \(M.\)

Then,

1. \(N_s\) is a sub module of \(M_s,\)
2. \(m_s \in N_s\) if and only if \(tm \in N\) for some \(t \in S\)
3. If \(N \cap S \neq \emptyset\) then, \(N_s = M_s\)

**Proof.** The proofs are straight forward. \(\square\)

**Lemma 3.6.** Let \(M\) and \(N\) be modules over a Boolean like semi ring. then

1. \(M \cap N\) is a sub module and \((M \cap N)_s = M_s \cap N_s\)
2. \((A(M))_s = A(M_s)\)

**Proof.**

1. Let \(u_s \in (M \cap N)_s \Leftrightarrow tu \in M \cap N\) for some \(t \in S. \Leftrightarrow tu \in M, tu \in N \Leftrightarrow u_s \in M_s, u_s \in N_s \Leftrightarrow u_s \in (N_s \cap M_s)\)

2. (⊇) Let \(r_s \in (A(M))_s \Leftrightarrow (tr)m = 0 \forall m \in M \Rightarrow t(rm) = 0 \Rightarrow (rm)_s = 0_s \Rightarrow r_s m_s = 0_s \Rightarrow r_s \in A(M)\)

(⊇) Let \(r_s \in A(M_s) \Rightarrow r_s m_s = 0_s \Rightarrow (rm)_s = 0_s \Rightarrow t(rm) = 0\) for some \(t \in S. \Rightarrow (tr)m = 0 \Rightarrow tr \in A(M) \Rightarrow (tr)_s \in (A(M))_s \Rightarrow r_s \in (A(M))_s. \square\)
Lemma 3.7. Let $M$ be a module over a Boolean like semi ring $R$ and $N$ be a sub module of $M$. Then, $(N : M)_s = N_s : M_s$.

Proof. Suppose $r_s \in (N : M)_s \iff tr \in (N : M) \iff (tr)M \subseteq N \iff (tr)_sM_s \subseteq N_s \iff r_s \in (N_s : M_s)$

References


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