Rings of Uniformly Continuous Functions

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Abstract

There is a natural bijective correspondence between the compactifications of a Tychonoff space \( X \), the totally bounded uniformities on \( X \), and the unital \( C^\ast \)-subalgebras of \( C^\ast(X) \) (the algebra of bounded continuous complex valued functions on \( X \)) with what we call the completely regular separation property. The correspondence of compactifications with totally bounded uniformities is well known and can be found in several books on general topology (e.g. [3]). The correspondence with \( C^\ast \)-subalgebras of \( C^\ast(X) \) is more obscure, and we have not found it explicitly stated in the literature. We offer a new proof of this three way correspondence.

Keywords: compactification, \( C^\ast \)-algebra, uniform space, metric gauge

A pseudometric \( \rho \) on a set \( X \) is a function that satisfies the axioms for a metric, except we allow the possibility that \( \rho(x, y) = 0 \) when \( x \neq y \). A uniform space is a set \( X \) together with a family \( \mathcal{G} \) of pseudometrics on \( X \) that satisfy

1. if \( \rho_1 \) and \( \rho_2 \) are in \( \mathcal{G} \), then so is \( \rho_1 \vee \rho_2 \) (the maximum of \( \rho_1 \) and \( \rho_2 \)).

2. If \( d \) is a pseudometric on \( X \) such that, for every \( \epsilon \), there exists \( \rho \in \mathcal{G} \) and \( \delta > 0 \) for which \( \rho(x, y) < \delta \) implies \( d(x, y) < \epsilon \), then \( d \in \mathcal{G} \).

Let us refer to \( \rho_1 \vee \rho_2 \) as the join of \( \rho_1 \) and \( \rho_2 \), and we refer to a pseudometric \( d \) that satisfies the hypothesis in the second condition as one that lies below the family \( \mathcal{G} \). With this language, \( \mathcal{G} \) forms a uniformity when it is closed under joins and contains all pseudometrics that lie below it. (We have taken this definition from [1] chapter IX, and it is further developed in [2].) The set \( \mathcal{G} \)
will be referred to as the *gauge* for the uniformity on $X$, and because of the first condition in our definition above, the collection of open balls obtained from the elements of $G$ form a base for the topology on $X$.

A uniform space $(X, G)$ is *totally bounded* when $(X, \rho)$ is a totally bounded pseudo-metric space for every $\rho \in G$. With a Tychonoff space $X$ given, we will say that a gauge $G$ is *compatible* if the topology generated by its collection of open balls coincides with the topology on $X$. We will let $\mathcal{T}$ denote the collection of all totally bounded compatible gauges on $X$, partially ordered by set inclusion.

With $\mathcal{A} \subseteq C^*(X)$ a unital $C^*$-subalgebra, we will say that $\mathcal{A}$ has the *completely regular separation property* if points lying outside of closed subsets of $X$ can be separated with a continuous function from $\mathcal{A}$. In case $X$ is locally compact, these $C^*$-subalgebras are precisely the unital ones that contain the continuous functions with compact support. We will let $\mathfrak{A}$ denote the collection of all such $C^*$-subalgebras of $C^*(X)$, partially ordered by set inclusion.

Let $\mathcal{C}$ denote the set of all (equivalence classes of) compactifications of $X$, where two compactifications are related $c_1 \leq c_2$ when there exists a continuous function $f : c_2 X \to c_1 X$ allowing the embedding of $X$ into $c_1 X$ to factor through its embedding into $c_2 X$ (see page 138 of [3]). The equivalence classes in $\mathcal{C}$ are those relative to *topologically equivalent* compactifications, i.e. two compactifications $c_1$ and $c_2$ for which $c_1 \leq c_2$ and $c_2 \leq c_1$.

Our strategy is to define three functions mapping between these three sets, show that each preserves order, and that any threefold composition is the identity. We will then have proved that each set is order isomorphic to the other two. Towards this end, define

$$\alpha : \mathfrak{A} \to \mathcal{C}$$

by letting $\alpha_{\mathfrak{A}}$ be the maximal ideal space of $\mathfrak{A}$, with $X$ embedded via the point evaluation homomorphisms. The reader may verify that this is indeed an embedding (because of the completely regular separation property) and that the point evaluations are dense in $\alpha_{\mathfrak{A}}$ (we are using the weak$^*$ topology on $\alpha_{\mathfrak{A}}$).

We then define

$$\beta : \mathcal{C} \to \mathcal{T}$$

by letting $\beta_c$ be the uniformity $X$ inherits as a subspace of $cX$: since $cX$ is compact, it has a unique uniformity, and if $G$ denotes the corresponding gauge, then we are restricting the pseudo-metrics of $G$ to $X$ to obtain $\beta_c$. Again by the compactness of $cX$, we see that the result is a totally bounded gauge uniformity $\beta_c$ on $X$.

Finally, we define

$$\gamma : \mathcal{T} \to \mathfrak{A}$$
by letting \( \gamma_G \) be the set of all bounded uniformly continuous (complex valued) functions on \( X \). It is easily verified that this is a unital \( C^* \)-subalgebra of \( C^*(X) \) (note that the product of uniformly continuous functions need not be uniformly continuous, but products of uniformly continuous bounded functions are uniformly continuous). If \( x \in X \) and \( F \) is a closed set not containing \( x \), then there must be a pseudo-metric \( \rho \in G \) and an \( \epsilon > 0 \) so that the corresponding \( \rho \)-ball of radius \( \epsilon \) about \( x \) is contained in the complement of \( F \) (since \( G \) is compatible). Then the function \( f(y) = \rho(y, F) \) is uniformly continuous and separates \( x \) from \( F \), so that \( \gamma_G \) has the completely regular separation property.

**Theorem 1** Each of \( \alpha, \beta, \) and \( \gamma \) is an order isomorphism.

**Proof.** Assume that \( A \in \mathfrak{A} \). If \( f \) is a continuous function on \( \alpha_A \), then \( f \) is uniformly continuous, and so is its restriction to \( X \), so \( f \in (\gamma \circ \beta \circ \alpha)_A \). Conversely, any uniformly continuous function on \( X \) will extend to \( \alpha_A \), so

\[
\mathfrak{A} = (\gamma \circ \beta \circ \alpha)_A.
\]

If \( c \in \mathfrak{C} \), then, as in the previous paragraph, the bounded uniformly continuous functions on \( (X, \beta_c) \) are exactly the functions that extend continuously to \( cX \), so \( C(cX) \) and \( (\gamma \circ \beta)_c \) are isomorphic \( C^* \)-algebras, and the corresponding homeomorphism of \( cX \) with the maximal ideal space of \( (\gamma \circ \beta)_c \) takes the evaluation functionals to evaluation functionals. It follows that \( (\alpha \circ \gamma \circ \beta)_c \) and \( c \) are topologically equivalent compactifications.

Assume now that \( \mathcal{G} \in \mathfrak{T} \). Since \( (\alpha \circ \gamma)_\mathcal{G} \) is compact, any topologically generating family of pseudometrics will generate the unique uniformity of \( (\alpha \circ \gamma)_\mathcal{G} \), and in particular the family

\[
\mathcal{S} = \{ \rho_f : f \in C((\alpha \circ \gamma)_\mathcal{G}) \}
\]

generates the uniformity on \( (\alpha \circ \gamma)_\mathcal{G} \), where

\[
\rho_f(x, y) = |f(x) - f(y)|.
\]

It is clear that \( (\beta \circ \alpha \circ \gamma)_\mathcal{G} \subseteq \mathcal{G} \), so it suffices to prove that \( \mathcal{G} \) is generated by the restrictions of the elements of \( \mathcal{S} \) to \( X \). With this goal in mind, assume that \( \rho \in \mathcal{G} \) and \( \epsilon > 0 \) are given. Being totally bounded, there exists \( x_1, \ldots, x_n \in X \) for which the \( \rho \)-balls of radius \( \epsilon/3 \) about these points, which we denote \( B_\rho(x_i, \epsilon/3) \), cover \( X \). For each \( i \in \{1, \ldots, n\} \) let \( f_i(y) = \rho(y, x_i) \), so \( f_i \) is \( \mathcal{G} \)-uniformly continuous, and if

\[
\rho_i(x, y) = |f_i(x) - f_i(y)| = |\rho(x, x_i) - \rho(y, x_i)|,
\]

then \( \rho_i \in \mathcal{S} \). Let \( \delta = \epsilon/3 \), and let \( x, y \in X \) be given so that \( \forall \rho_i(x, y) < \delta \), i.e.

\[
|\rho(x, x_i) - \rho(y, x_i)| < \epsilon/3
\]
for each $i$. Choose $k$ so that $x \in B_{\rho}(x_k, \epsilon/3)$, which, combined with the inequality above, gives
\[
\rho(y,x_k) < \epsilon/3 + \rho(x,x_k) < 2\epsilon/3,
\]
and
\[
\rho(x,y) \leq \rho(x,x_k) + \rho(y,x_k) < \epsilon.
\]
It follows that $\rho$ lies below $\mathcal{S}$, so it belongs to the gauge generated by $\mathcal{S}$ and
\[
(\beta \circ \alpha \circ \gamma)_\mathcal{G} = \mathcal{G}.
\]
We have proven that each of $\alpha, \beta,$ and $\gamma$ is a bijection, and using the same techniques, the reader will verify that each map preserves order, which completes our proof.

\[\square\]

References


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