Geometry on

$G^L$-Systems of Homogenous Polynomials

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Abstract

For integers $n \geq 2$, $m \geq 1$, a $G^L$-system $(ES_m^{n+1})$ is such a system consisting of $m$ homogenous polynomials $P_1(x_1, x_2, \ldots, x_{n+1})$, $P_2(x_1, x_2, \ldots, x_{n+1})$, $\ldots$, $P_m(x_1, x_2, \ldots, x_{n+1})$ in $n + 1$ variables with coefficients in $\mathbb{C}$, which inherits a topological graph $G[ES_m^{n+1}]$ in $\mathbb{P}^n \mathbb{C}$. We characterize the geometrical properties of system $(ES_m^{n+1})$ in $\mathbb{P}^n \mathbb{C}$, find its combinatorial invariants under linear transformations, and determine its normalization, i.e., a combinatorial manifold $\tilde{M}$ such that $\phi : \tilde{M} \rightarrow \tilde{S}$ is a homeomorphism by combinatorics, where $\tilde{S} = \bigcup_{i=1}^m S_i$ with $S_i$ a hypersurface determined by $P_i(\overline{x}) = 0$ in $\mathbb{P}^n \mathbb{C}$ for integers $1 \leq i \leq m$. Indeed, such a combinatorial manifold $\tilde{M}$ is nothing else but a manifold of dimensional $n$. Particularly, if $n = 2$, its genus is determined in this paper also.

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§1. Introduction

A polynomial $P(x_1, x_2, \ldots, x_{n+1})$ in $n + 1$ variables with coefficients in $\mathbb{C}$ is homogenous of degree $d \geq 1$ if for any constant $\lambda \neq 0$, there is

$$P(\lambda x_1, \lambda x_2, \ldots, \lambda x_{n+1}) = \lambda^d P(x_1, x_2, \ldots, x_{n+1}).$$

Such a polynomial $P(x_1, x_2, \ldots, x_{n+1})$ determines a hypersurface defined by $P(x_1, x_2, \ldots, x_{n+1}) = 0$ in the projective space $\mathbb{P}^n \mathbb{C}$, a set of complex one-dimensional subspace, i.e., hyperplanes passing through origin of the complex vector space $\mathbb{C}^{n+1}$,
which can be classified into two classes

$$\mathbb{P}^n \mathbb{C} = \{(x_1, x_2, \cdots, x_n, 1) \in \mathbb{C}^{n+1}, \; x_i \in \mathbb{C}, \; 1 \leq i \leq n\} \cup \{(x_1, x_2, \cdots, x_n, 0) \in \mathbb{C}^{n+1}, \; x_i \in \mathbb{C}, \; 1 \leq i \leq n\}.$$  

Geometrically, such a one-dimensional subspace spanned by \((x_1, x_2, \cdots, x_n, 1) \in \mathbb{P}^n \mathbb{C}\) can be identified with that of \((x_1, x_2, \cdots, x_n) \in \mathbb{C}^n\), and that of spanned by \((x_1, x_2, \cdots, x_n, 0) \in \mathbb{P}^n \mathbb{C}\) can be identified with that of the hyperplane at the infinite, i.e., \(\mathbb{P}^n \mathbb{C} = \mathbb{C}^n \cup \mathbb{P}^{n-1} \mathbb{C}\). Whence, we can characterize the behavior of one-dimensional subspace \(C\) by its affine behavior \(P(x_1, x_2, \cdots, x_n, 1) \in \mathbb{C}^n\) and that of \(P(x_1, x_2, \cdots, x_n, 0) \in \mathbb{P}^{n-1} \mathbb{C}\). Particularly, if \(\mathbb{C}\) is replaced by \(\mathbb{R}\), then the behavior of \(C\) is divided into to parts: the reality in \(\mathbb{R}^n\) combined with those of in the infinite.

Let \(\varpi = (x_1, x_2, \cdots, x_{n+1})\), \(\bar{\varpi} = (k_1, k_2, \cdots, k_{n+1})\), \(a_{\varpi} = a_{k_1, k_2, \cdots, k_{n+1}}\), \(\varpi^T = x_1^{k_1} x_2^{k_2} \cdots x_{n+1}^{k_{n+1}}\) and

\[ P(\varpi) = \sum_{k_1+\cdots+k_n=d} a_{\varpi}^{\varpi^T} \]

be a homogenous polynomial of degree \(d\), an algebraic covariant of weight \(p\) is a homogenous polynomial \(C(a_{\varpi}^{\varpi^T})\) such that

\[ C(a_{\varpi}^{\varpi^T}) = \Delta^p C(a_{\varpi}^{\varpi^T}) \]

under an invertible linear transformation \(T\):

\[
\begin{align*}
    x_1 &= \alpha_{11} x_1' + \alpha_{12} x_2' + \cdots + \alpha_{1, n+1} x_{n+1}' \\
    x_2 &= \alpha_{21} x_1' + \alpha_{22} x_2' + \cdots + \alpha_{2, n+1} x_{n+1}' \\
    & \vdots & \vdots & \vdots & \vdots \\
    x_{n+1} &= \alpha_{n1} x_1' + \alpha_{n2} x_2' + \cdots + \alpha_{n, n+1} x_{n+1}'
\end{align*}
\]

i.e., \(T = (\alpha_{ij})_{(n+1) \times (n+1)}\) \((\varpi^T)\), where \(\varpi^T\) denotes the transpose of vector \(\varpi\), \(a_{\bar{\varpi}}^{\varpi^T}\) is the coefficient of \(\varpi^T\) in \(P(x_1', x_2', \cdots, x_{n+1}')\) and \(\Delta = \det(\alpha_{ij})_{(n+1) \times (n+1)}\). Clearly, \(P(\varpi)\) is itself a covariant of weight 0 by definition.

The main interesting of this paper is in the combinatorially geometrical properties of systems consisting of homogenous polynomials, particularly, the case of \(n = 2\). In such a case, the subspace determined by \(P(x, y, z) = 0\) is called an algebraic curve \(C\) in \(\mathbb{P}^2 \mathbb{C}\). A point \((a, b, c)\) of a curve \(C\) in \(\mathbb{P}^2 \mathbb{C}\) defined by a homogenous polynomial \(P(x, y, z)\) is called singular if

\[ \frac{\partial P}{\partial x}(a, b, c) = \frac{\partial P}{\partial y}(a, b, c) = \frac{\partial P}{\partial z}(a, b, c) = 0. \]

Denoted by \(\text{Sing}(P)\) all singular points in the curve \(C\) determined by \(P(x, y, z)\) and it is called non-singular if \(\text{Sing}(P) = \emptyset\). By the Noether’s result, we know that for any curve \(C\) determined by a homogenous polynomial \(P(x, y, z)\) of degree \(d\) in \(\mathbb{P}^2 \mathbb{C}\), there is a compact connected Riemann surface \(S\) such that

\[ h : S - h^{-1}(\text{Sing}(C)) \rightarrow C - \text{Sing}(C) \]
is a homeomorphism with genus

\[ g(S) = \frac{1}{2}(d-1)(d-2) - \sum_{p \in \text{Sing}(C)} \delta(p), \]

where \( \delta(p) \) is a positive integer associated with the singular point \( p \) in \( C \). Particularly, if \( \text{Sing}(C) = \emptyset \), i.e., \( C \) is non-singular then there is a compact connected Riemann surface \( S \) homeomorphism to \( C \) with genus \( \frac{1}{2}(d-1)(d-2) \).

For an integer \( m \geq 1 \), let \( M_1, M_2, \cdots, M_m \) be respectively dimensional \( n_1, n_2, \cdots, n_m \) manifolds in a Euclidean space. A finite combinatorial manifold is defined following.

**Definition 1.1**([10]) For manifolds \( M_1, M_2, \cdots, M_m \) with respectively dimensional \( n_1, n_2, \cdots, n_m \) in a topological space, where \( n \geq \max\{n_1, n_2, \cdots, n_m\}, m \geq 1 \) and \( 0 < n_1 < n_2 < \cdots < n_m \), a finite combinatorial manifold \( \widetilde{M} \) of \( M_1, M_2, \cdots, M_m \) is the union \( \bigcup_{i=1}^{m} M_i \) underlying a topological graph \( G^L[\widetilde{M}] \) following:

\[ V(G^L[\widetilde{M}]) = \{M_1, M_2, \cdots, M_m\}, \]

\[ E(G^L[\widetilde{M}]) = \{(M_i, M_j)|M_i \cap M_j \neq \emptyset, 1 \leq i \neq j \leq m\} \] with a labeling

\[ L : M_i \to L(M_i) = M_i \quad \text{and} \quad L : (M_i, M_j) \to L(M_i, M_j) = M_i \cap M_j \]

for integers \( 1 \leq i \neq j \leq m \).

For example, let \( M_0 = \{(x, y, z)|x^2+y^2+z^2 = 1\} \), \( M_1 = \{(x, y, z)|(x-2)^2+y^2+z^2 = 1\} \), \( M_2 = \{(x, y, z)|(x+2)^2+y^2+z^2 = 1\} \), \( M_3 = \{(x, y, z)|x^2+(y-2)^2+z^2 = 1\} \), \( M_4 = \{(x, y, z)|x^2+y^2+(z+2)^2 = 1\} \), \( M_5 = \{(x, y, z)|x^2+y^2+(z-2)^2 = 1\} \) and \( M_6 = \{(x, y, z)|x^2+y^2+(z+2)(z-2) = 1\} \) be 7 spheres in \( \mathbb{R}^3 \). Notice that \( M_0 \cap M_1 = \{(1,0,0)\} \), \( M_0 \cap M_2 = \{(-1,0,0)\} \), \( M_0 \cap M_3 = \{(0,1,0)\} \), \( M_0 \cap M_4 = \{(0,-1,0)\} \), \( M_0 \cap M_5 = \{(0,0,1)\} \), \( M_0 \cap M_6 = \{(0,0,-1)\} \) and \( M_i \cap M_j = \emptyset \) for \( 1 \leq i \neq j \leq 6 \). Whence, its topological graph is a wheel \( W_6 \) with labels such as those shown in Fig.1.

![Fig.1](image-url)
in $\mathbb{R}^n$. Now let

$$P_1(\mathbf{\tau}), P_2(\mathbf{\tau}), \cdots, P_m(\mathbf{\tau})$$

be $m$ homogeneous polynomials in $n+1$ variables with coefficients in $\mathbb{C}$ and each equation $P_i(\mathbf{\tau}) = 0$ determine a hypersurface $M_i$, $1 \leq i \leq m$ in $\mathbb{R}^{n+1}$, particularly, a curve $C_i$ if $n = 2$. Then how can we characterize the system $(ES_{m+1}^n)$? and how can we determine its geometrical behaviors? In this paper, we characterize the system $(ES_{m+1}^n)$, find its combinatorial invariants and answer these questions, particularly for $n = 2$ by a combinatorial approach and determine such a geometrical space is nothing else but a combinatorial Riemann surface $\tilde{M}$ defined by Definition 1.1. Furthermore, we show such a combinatorial Riemann surface $\tilde{M}$ is homeomorphic to a Riemann surface with genus determined in this paper.


§2. $G^L$-System of Homogeneous Polynomials in $n+1$ Variables

For two curves $P(x, y)$, $Q(x, y)$, particularly two lines in $\mathbb{R}^2$, if they are not intersect, we can always say that $P, Q$ are parallel in terminology. Such a property enables one to classify the bundle of lines in $\mathbb{R}^2$ by that of parallel or not parallel. Notice that by the famous Bézout’s theorem, if $P(x, y, z)$, $Q(x, y, z)$ are two complex homogenous polynomials of degrees $m, n$ and have no common components, then their curves determined by $P(x, y, z)$, $Q(x, y, z)$ have precisely $mn$ intersection points counting multiplicities in $\mathbb{P}^2\mathbb{C}$ ([7]). Thus any two curves, particularly, lines in $\mathbb{P}^2\mathbb{C}$ intersect each other and there are no parallel subspace in projective space ([1]). However, denoted by $I(P, Q)$ the set of intersection points of homogenous polynomials $P(\mathbf{\tau})$ with $Q(\mathbf{\tau})$ in $n+1$ variables. We can also introduce the parallel hypersurfaces with degree $\geq 1$ or not in $\mathbb{P}^n\mathbb{C}$ by distinguishing their intersections at infinite or not following.

Definition 2.1 Let $P(\mathbf{\tau}), Q(\mathbf{\tau})$ be two complex homogenous polynomials of degree $d$ in $n+1$ variables. They are said to be parallel, denoted by $P \parallel Q$ if $d \geq 1$ and there are constants $a, b, \cdots, c$ (not all zero) such that for $\forall \mathbf{\tau} \in I(P, Q)$, $ax_1 + bx_2 + \cdots + cx_{n+1} = 0$, i.e., all intersections of $P(\mathbf{\tau})$ with $Q(\mathbf{\tau})$ appear at a hyperplane on $\mathbb{P}^n\mathbb{C}$, or $d = 1$ with all intersections at the infinite $x_{n+1} = 0$. Otherwise, $P(\mathbf{\tau})$ are not parallel to $Q(\mathbf{\tau})$, denoted by $P \nparallel Q$.

For example, let $P(\frac{x}{z}, \frac{y}{z}, 1)z^m$ and $Q = P(\frac{x}{z}, \frac{y}{z}, 1)z^m + kz^m$ for integers $k, m \geq 1$. It is clear that their only intersection points appear at the infinite line $z = 0$. Thus they are parallel in $\mathbb{P}^2\mathbb{C}$ by Definition 2.1. The parallel property naturally enables one to introduce the underlying graph $G[ES_{m+1}^n]$ of $(ES_{m+1}^n)$ following.

Definition 2.2 Let $P_1(\mathbf{\tau}), P_2(\mathbf{\tau}), \cdots, P_m(\mathbf{\tau})$ be homogenous polynomials in $(ES_{m+1}^n)$. 

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Define a topological graph $G^L[ES_m^{n+1}]$ in $\mathbb{C}^{n+1}$ by

$$V(G^L[ES_m^{n+1}]) = \{P_1(\mathbf{x}), P_2(\mathbf{x}), \ldots, P_m(\mathbf{x})\};$$

$$E(G^L[ES_m^{n+1}]) = \{(P_i(\mathbf{x}), P_j(\mathbf{x}))|P_i \parallel P_j, \ 1 \leq i, j \leq m\}$$

with a labeling

$$L : \ P_i(\mathbf{x}) \rightarrow P_i(\mathbf{x}), \ (P_i(\mathbf{x}), P_j(\mathbf{x})) \rightarrow I(P_i, P_j),$$

where $1 \leq i \neq j \leq m$. Such a system $(ES_m^{n+1})$ is called a $G^L$-system and the topological graph of $G^L[ES_m^{n+1}]$ without labels is called the underlying graph of $G^L$-system $(ES_m^{n+1})$, denoted by $G[ES_m^{n+1}]$.

Notice that the parallel property is symmetric and transitive. We can classify polynomials in $(ES_m^{n+1})$ by this property into classes $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_l$ such that $P_i \parallel P_j$ if $P_i, P_j \in \mathcal{C}_k$ for an integer $1 \leq k \leq l$, where $1 \leq i \neq j \leq m$. Such a classification is parallel maximal if each $\mathcal{C}_i$ is maximal for integers $1 \leq i \leq l$, i.e., for $\forall P \in \{P_k(\mathbf{x}), \ 1 \leq k \leq m\}\setminus \mathcal{C}_i$, there is a polynomial $Q(\mathbf{x}) \in \mathcal{C}_i$ such that $P \parallel Q$. Then we know the following result by definition.

**Theorem 2.3** Let $n \geq 2$ be an integer. For a system $(ES_m^{n+1})$ of homogenous polynomials with a parallel maximal classification $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_l$,

$$G[ES_m^{n+1}] \leq K(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_l)$$

and with equality holds if and only if $P_i \parallel P_j$ and $P_i \nparallel P_j$ implies that $P_i \nparallel P_j$, where $K(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_l)$ denotes a complete $l$-partite graphs.

Conversely, for any subgraph $G \leq K(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_l)$, there are systems $(ES_m^{n+1})$ of homogenous polynomials with a parallel maximal classification $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_l$ such that

$$G \simeq G[ES_m^{n+1}].$$

**Proof** Clearly, for any $P(\mathbf{x}), Q(\mathbf{x}) \in \mathcal{C}_k, \ 1 \leq k \leq l$, $P \parallel Q$ by definition. So $(P(\mathbf{x}), Q(\mathbf{x})) \notin E(G[ES_m^{n+1}])$, which implies that

$$G[ES_m^{n+1}] \leq K(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_l).$$

Notice that the equality

$$G[ES_m^{n+1}] = K(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_l)$$

holds implies that for $\forall P(\mathbf{x}) \in \mathcal{C}_i, Q(\mathbf{x}) \in \mathcal{C}_j$ with $1 \leq i \neq j \leq l$, there must be $P \parallel Q$, i.e., $P \parallel P_k$ if $P_k \parallel Q$. Conversely, if $P_k \parallel P_l$ and $P_l \parallel P_j$, then it is clear that for $\forall P(\mathbf{x}) \in \mathcal{C}_i, Q(\mathbf{x}) \in \mathcal{C}_j$ with $1 \leq i \neq j \leq l$, there must be $(P(\mathbf{x}), Q(\mathbf{x})) \in E(G[ES_m^{n+1}])$. Thus

$$G[ES_m^{n+1}] = K(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_l).$$

Conversely, if $G \leq K(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_l)$, we construct a system $(ES_m^{n+1})$ following:
For \( \forall v \in V(G) \), choose homogenous polynomials \( P_v(\overline{x}) \) and \( P_u(\overline{x}) \) with \( P_u \parallel P_v \) if \((v,u) \in E(G)\), otherwise, \( P_u \nparallel P_v \) if \((v,u) \notin E(G)\).

Then it is clear that \( G \simeq G[ES^{n+1}] \) by definition. \( \square \)

Notice that for three hyperplanes \( P_1(\overline{x}), P_2(\overline{x}), P_3(\overline{x}) \), if \( P_1 \parallel P_2 \) and \( P_2 \parallel P_3 \), then there must be that \( P_1 \parallel P_3 \). By Theorem 2.3, we get the following conclusion.

**Corollary 2.4** Let all polynomials in \( (ES^{n+1}_m) \) be degree 1, i.e., hyperplanes with a parallel maximal classification \( \mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_l \), then

\[
G[ES^{n+1}] = K(\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_l).
\]

For example, let the system \( (ES^3_m) \) be consisted of polynomials \( P_1(x, y, z) = x - z \), \( P_2(x, y, z) = x - 2z \), \( P_3(x, y, z) = x - 3z \) and \( P_4(x, y, z) = y - z \), \( P_5(x, y, z) = y - 2z \), \( P_6(x, y, z) = y - 3z \). Then they are all lines in \( \mathbb{P}^2 \mathbb{R} \) with a figure and topological graph shown in Fig. 2.

Furthermore, we consider the existence of \( G^L \)-systems of homogenous polynomial for a topological graph \( G^L \) with labels following:

**Problem 2.5** Let \( G^L \) be a topological graph with labels \( I(e) \in \mathbb{P}^n \mathbb{C} \) for \( \forall e \in E(G) \). Are there systems \( (ES^{n+1}) \) of homogenous polynomials with \( G^L[ES^{n+1}_m] \simeq G^L \)?

The results following answer this question in case of \( n = 2 \), i.e., algebraic curves.

**Theorem 2.6** Let \( n_1, n_2, \cdots, n_l \) be positive integers and let \( G^L \leq K(n_1, n_2, \cdots, n_l) \) be a topological graph with labels \( I(e) \) for \( \forall e \in E(G) \), where \( I(e) \subset \mathbb{P}^2 \mathbb{C} \) is a finite point set. Then

1. There always exists a system \( (ES^3_m) \) of homogenous polynomials of degree

\[
d \geq -3 + \sqrt{9 + 8|I(e)|}
\]

with a maximal parallel classification \( \mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_l \) such that \( G[ES^3_m] \simeq G \) with \( |\mathcal{C}_i| = n_i \), \( 1 \leq i \leq l \) and \( I(P_u, P_v) \supseteq I(e) \) for \( \forall e = (u,v) \in E(G) \).
Thus if 

\[ \mathbf{e} \] 

with coefficients \( (P_m, P_t) \) respectively find two distinct homogenous polynomials \( P \) which has \( (\mathbf{e}, \mathbf{v}) \) or \( (\mathbf{e}, \mathbf{w}) \) or \( (\mathbf{e}, \mathbf{d}) \) with \( \det(M_3) = 0 \) and \( (\mathbf{e}, \mathbf{w}) \) or \( (\mathbf{e}, \mathbf{d}) \) with \( \det(M_6) = 0 \), where

\[ M_3 = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}, \quad M_6 = \begin{pmatrix} x_1^2 & y_1^2 & z_1^2 \\ x_2^2 & y_2^2 & z_2^2 \\ x_3^2 & y_3^2 & z_3^2 \\ x_2^3 & y_2^3 & z_2^3 \\ x_3^3 & y_3^3 & z_3^3 \\ x_1^4 & y_1^4 & z_1^4 \\ x_2^4 & y_2^4 & z_2^4 \\ x_3^4 & y_3^4 & z_3^4 \\ x_2^5 & y_2^5 & z_2^5 \\ x_3^5 & y_3^5 & z_3^5 \\ x_2^6 & y_2^6 & z_2^6 \\ x_3^6 & y_3^6 & z_3^6 \end{pmatrix}. \]

**Proof** Let \( e \in E(G) \). Notice that a homogenous polynomial of degree \( d \) is in the form

\[ P(x, y, z) = \sum_{i+j+k=d} a_{ijk} x^i y^j z^k, \]

which has \( \frac{(d+1)(d+2)}{2} \) terms. Choose a new point \( p \in \mathbb{P}^2 \setminus I(e) \). If

\[ \frac{(d+1)(d+2)}{2} \geq |I(e) \cup \{p\}| = |I(e)| + 1, \]

i.e.,

\[ d \geq -3 + \frac{9 + 8|I(e)|}{2}, \]

there exists a homogenous polynomial \( P(x, y, z) \) passing through points in \( I(e) \cup \{p\} \) with coefficients \( a_{ijk}, i+j+k = d \) not all equal to zero by solving linear equations

\[ \sum_{i+j+k=d} a_{ijk} x_0^i y_0^j z_0^k = 0, \quad (x_0, y_0, z_0) \in I(e). \]

Thus if \( e = (u, v) \in E(G) \), we can always choose \( p_u \neq p_v \in \mathbb{P}^2 \setminus I(e) \) and respectively find two distinct homogenous polynomials \( P_u(x, y, z), P_v(x, y, z) \) of degrees \( D, d \) passing through \( I(e) \cup \{p_u\}, I(e) \cup \{p_v\} \) such that \( I(P_u, P_v) \supseteq I(e) \).

Assuming \( u \in \mathcal{C}_i, 1 \leq i \leq l \). For \( \forall w \in \mathcal{C}_i, w \neq u \), let \( P_u(x, y, z) = P_u(x, y, z) + \lambda w z^d \), where \( \lambda w \) is a complex number. Then it is clear that all polynomials in \( \mathcal{C}_i, 1 \leq i \leq l \) are parallel.

Now let system \( (ES_m^3) \) be consisted all of homogenous polynomials \( P_v(x, y, z), v \in V(G) \). Then such a system \( (ES_m^3) \) is clearly with a maximal parallel classification \( \mathcal{C}_i, 1 \leq i \leq l \) such that \( |\mathcal{C}_i| = n_i, 1 \leq i \leq l \) and \( I(P_u, P_v) \supseteq I(e) \) for \( \forall e = (u, v) \in E(G) \) by construction. Thus we get the conclusion (1).

For (2), by noticing that

\[ \frac{(D+1)(D+2)}{2} > D^2 \geq Dd, \quad \frac{(d+1)(d+2)}{2} > Dd. \]
if \((D, d) \in \{(1, 1), (1, 2), (2, 2), (3, 3)\}\) and
\[
\frac{(D + 1)(D + 2)}{2} > D^2 \geq Dd, \quad \frac{(d + 1)(d + 2)}{2} = Dd
\]
if \((D, d) = (3, 2)\) or \((3, 1)\). Thus the system of linear equations
\[
\sum_{i+j+k=D} a_{i,j,k}x_i^j y_i^j z_i^k = 0, \quad 1 \leq l \leq Dd
\]
and
\[
\sum_{i+j+k=d} b_{i,j,k}x_i^j y_i^j z_i^k = 0, \quad 1 \leq l \leq Dd
\]
have non-zero solutions \(a_{i,j,k}, b_{i,j,k}\) for \(i+j+k = D\) or \(d\) if \((D, d) \in \{(1, 1), (1, 2), (2, 2), (3, 3)\}\), or \((D, d) = (3, 1)\), \((I(e) = \{(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)\}\) with \(\det(M_3) = 0\), or \((D, d) = (3, 2)\), \((I(e) = \{(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4), (x_5, y_5, z_5), (x_6, y_6, z_6)\}\) with \(\det(M_6) = 0\) by linear algebra. Thus we can find homogenous polynomials \(P_u(x, y, z)\) and \(P_v(x, y, z)\) of degrees \(D, d\) such that \(I(P_u, P_v) = I(e)\). □.

The conclusion (2) in Theorem 2.6 implies the nine points theorem holds in the geometry of algebraic curves. Furthermore, we get the following conclusion.

**Corollary 2.7** Let \(\mathcal{L}\) be a finite point set on projective algebraic curve \(P(x, y, z)\) of degree \(d \leq 3\) with \(|\mathcal{L}| = d^2\). Then there always exists a projective algebraic curve \(Q(x, y, z) \neq P(x, y, z)\) of degree \(d\) passing through points in \(\mathcal{L}\).

Corollary 2.7 partially answers Question 2.5 and enables one to get the following result.

**Theorem 2.8** Let \(G^L\) be a topological graph with labels \(I(e) \subset \mathbb{P}^2\mathbb{C}\) on \(e \in E(G)\). If \(|I(e)| = d^2\) with \(d \leq 3\), then there are systems \((ES^3_m)\) of homogenous polynomials \(P_v(x, y, z)\) of degree \(d\) for \(\forall v \in V(G^L)\) such that \(G^L[ES^3_m] \simeq G^L\).

The inverse of Bézout’s theorem is not true in general. For example, let \(p_1, p_2, \ldots, p_d\) be \(d\) points on a projective algebraic curve \(P(x, y, z)\) of degree \(d\), which are not collinear. Clearly, there are no lines \(Q(x, y, z)\) passing through all points \(p_1, p_2, \ldots, p_d\). But the Bézout’s theorem immediately implies the result following, which is useful for constructing systems \((ES^3_m)\) underlying topological graphs \(G^L[ES^3_m]\).

**Theorem 2.9** Let \(P_1(x, y, z), P_2(x, y, z), \ldots, P_k(x, y, z), Q_1(x, y, z), Q_2(x, y, z), \ldots, Q_l(x, y, z)\) be homogenous polynomials for integers \(k, l \geq 1\). Then there are homogenous polynomials \(P(x, y, z), Q(x, y, z)\) such that
\[
I(P, Q) = \bigcup_{1 \leq i \leq k, 1 \leq j \leq l} I(P_i, P_j).
\]
Proof Clearly, \( I(P_i, P_j) \neq \emptyset \) for integers \( 1 \leq i \leq k, 1 \leq j \leq l \) by Bézout’s theorem. Define
\[
P(x, y, z) = \prod_{i=1}^{k} P_i(x, y, z) \quad \text{and} \quad Q(x, y, z) = \prod_{i=1}^{l} Q_i(x, y, z).
\]

Applying Bézout’s theorem, we then know that
\[
I(P, Q) = \bigcup_{1 \leq i \leq k, 1 \leq j \leq l} I(P_i, P_j).
\]

Then we have a criterion for a topological graph \( G^L \) being that an underlying graph of system \( (ES^3_m) \) following.

**Theorem 2.10** Let \( G^L \) be a topological graph labeled with \( I(e) \) for \( \forall e \in E(G^L) \). Then there is a system \( (ES^3_m) \) of homogenous polynomials such that \( G^L[ES^3_m] \simeq G^L \) if and only if there are homogenous polynomials \( P_{vi}(x, y, z), 1 \leq i \leq \rho(v) \) for \( \forall v \in V(G^L) \) such that
\[
I(e) = I(P_u, P_v) = I(\prod_{i=1}^{\rho(u)} P_{ui}, \prod_{i=1}^{\rho(v)} P_{vi})
\]
for \( e = (u, v) \in E(G^L) \), where \( \rho(v) \) denotes the valency of vertex \( v \) in \( G^L \).

Proof If \( (ES^3_m) \) is a system of homogenous polynomials \( P_v, v \in V(G^L) \) such that
\( G^L[ES^3_m] \simeq G^L \), for \( \forall v \in V(G^L) \) let \( P_{v_1} = P_v \) and \( P_{v_i} = 1 \) for integers \( 2 \leq i \leq \rho(v) \). Then, by definition 2.2 we know that
\[
I(e) = I(P_{u}, P_v) = I(\prod_{i=1}^{\rho(u)} P_{ui}, \prod_{i=1}^{\rho(v)} P_{vi})
\]
for \( \forall e = (u, v) \in E(G^L) \).

Conversely, if there are homogenous polynomials \( P_{vi}(x, y, z), 1 \leq i \leq \rho(v) \) for \( \forall v \in V(G^L) \) such that
\[
I(e) = I(\prod_{i=1}^{\rho(u)} P_{ui}, \prod_{i=1}^{\rho(v)} P_{vi})
\]
for \( e = (u, v) \in E(G^L) \), let
\[
L : v \rightarrow \prod_{i=1}^{\rho(v)} P_{vi}(x, y, z) = P_v(x, y, z)
\]
for \( \forall v \in V(G^L) \). Then the system \( (ES^3_m) \) consisting of homogenous polynomials \( P_{vi}(x, y, z), v \in V(G^L) \) is such a system with the conclusion holds. In fact, the identity mapping \( 1_G : v \in V(G^L) \rightarrow v \in V(G^L) \) is such an isomorphism with
\[
I(e) = I(\prod_{i=1}^{\rho(u)} P_{ui}, \prod_{i=1}^{\rho(v)} P_{vi}) = I(P_u, P_v)
\]
for $e = (u, v) \in E(G^L)$. Thus, $G^L[ES^3_m] \simeq G^L$. 

Choosing each $P_i(x, y, z), 1 \leq i \leq \rho(v)$ in Theorem 2.10 being line, we get a special conclusion following.

**Corollary 2.11** Let $p_{ij}$ be the intersection point of lines $L^j_i$ with $L^j_d$ for integers $1 \leq i \leq D$, $1 \leq j \leq d$ in $\mathbb{P}^2 \mathbb{C}$ and $I = \{p_{ij}, 1 \leq i \leq D, 1 \leq j \leq d\}$. Then there are homogenous polynomials $P(x, y, z)$ of degree $D$ and $Q(x, y, z)$ of degree $d$ such that $I(P, Q) = I$.

§3. **Automorphisms of $G^L$-System of Homogenous Polynomials in $n + 1$ Variables**

Classifying isomorphic systems $(ES^m_{n+1})$ enables one to introduce isomorphic covariants on homogenous polynomials following.

**Definition 3.1** Let $(^1ES^m_{n+1})$ and $(^2ES^m_{n+1})$ be systems respectively consisting of covariants $C_i^1(a_\overline{\pi}, \overline{\pi}), C_i^2(a_\overline{\pi}, \overline{\pi})$ on homogenous polynomials $P_i(\overline{\pi})$ for integers $1 \leq i \leq m$. If there is an invertible linear transformation $T$ such that for all $C_i^1(a_\overline{\pi}, \overline{\pi}) \in (^1ES^m_{n+1})$, there is $C_i^2(a_\overline{\pi}, \overline{\pi}) \in (^2ES^m_{n+1})$ with

$$C_i^2(a^T_\overline{\pi}, \overline{\pi}) = \Delta C_i^1(a_\overline{\pi}, \overline{\pi})$$

holds for integers $1 \leq i \leq m$, where $\Delta$ is a constant, then the system $(^2ES^m_{n+1})$ is said to be linear isomorphic to system $(^1ES^m_{n+1})$, denoted by $(^1ES^m_{n+1}) \overset{T}{\simeq} (^2ES^m_{n+1})$, where $\Delta$ is the determinant of $T$.

Notice that a homogenous polynomial $P(\overline{\pi})$ is itself a covariant of weight 0. Whence, a system $(ES^m_{n+1})$ consisting of homogenous polynomials $P_i(\overline{\pi}), 1 \leq i \leq m$ is itself a covariant system by definition. For example, let the systems $(^1ES_3^3)$ be

$$\begin{cases} 
6x^2 + 19y^2 + 18z^2 + 20x y + 14y z + 32yz \\
10x^2 + 33y^2 + 14z^2 + 37xy + 33yz + 53yz \\
x^1 + 3y^2 + 4z^2 + 4xy + 5xz + 7yz 
\end{cases}$$

and let $(^2ES_3^3)$ be

$$\begin{cases} 
x^2 + y^2 + z^2 \\
2xy + yz \\
xz 
\end{cases}$$

Then $(^1ES_3^3) \overset{T}{\simeq} (^2ES_3^3)$ because there is an invertible linear transformation

$$T: \begin{cases} 
x = x' + y' + z' \\
y = 2x' + 3y' + z' \\
z = x' + 3y' + 4z' 
\end{cases}$$

such that $(^1ES_3^3)$ is transformed to $(^2ES_3^3)$ under the transformation $T$. The following result shows that $G^L[ES^m_{n+1}]$ is an invariant on isomorphic systems $(ES^m_{n+1})$. 

Theorem 3.2 Let \((1^ES_m^{n+1}), (2^ES_m^{n+1})\) be systems consisting of covariants \(C_i^1(a_{\overline{\sigma}}, \overline{\tau})\), \(C_i^2(a_{\overline{\sigma}}, \overline{\tau})\) on homogenous polynomials \(P_i(\overline{\tau})\) for integers \(1 \leq i \leq m\) of weight \(p\), respectively. Then \((1^ES_m^{n+1}) \xrightarrow{T} (2^ES_m^{n+1})\) if and only if \(G^L[1^ES_m^{n+1}] \xrightarrow{T} G^L[2^ES_m^{n+1}]\) and for any integer \(i\), \(1 \leq i \leq m\), \(C_i^2(a_{\overline{\sigma}'}, \overline{\tau}) = \Delta^p C_i^1(a_{\overline{\sigma}}, \overline{\tau})\) holds for a constant \(p\), where \(\Delta\) is the determinant of \(T\).

Proof If \((1^ES_m^{n+1}) \xrightarrow{T} (2^ES_m^{n+1})\), we show \(T\) is an isomorphism between topological graphs \(G^L[1^ES_m^{n+1}]\) and \(G^L[2^ES_m^{n+1}]\). In fact, \(T : V(G^L[1^ES_m^{n+1}]) \rightarrow V(G^L[2^ES_m^{n+1}])\) is \(1 - 1\) and onto by definition. Notice that a hyperplane is transferred to a hyperplane by a linear transformation on \(\mathbb{P}^n\mathbb{C}\). Thus, \(C^T_u \parallel C^T_v\) in \((2^ES_m^{n+1})\) if and only if \(C_u \parallel C_v\) in \((1^ES_m^{n+1})\), which implies that \((C^T_u, C^T_v) \in E(G^L[2^ES_m^{n+1}])\) if and only if \((C_u, C_v) \in E(G^L[1^ES_m^{n+1}])\), i.e., \(G^L[1^ES_m^{n+1}] \simeq G^L[2^ES_m^{n+1}]\). Clearly, \(I(C^T_u, C^T_v) = T(I(C_u, C_v))\) for \(\forall (C_u, C_v) \in E(G^L[1^ES_m^{n+1}])\). Consequently, the linear transformation \(T : V(G^L[1^ES_m^{n+1}]) \rightarrow V(G^L[2^ES_m^{n+1}]), \quad E(G^L[1^ES_m^{n+1}]) \rightarrow E(G^L[2^ES_m^{n+1}])\)

is commutative with that of labeling \(L\), i.e., \(T \circ L = L \circ T\), i.e., \(G^L[1^ES_m^{n+1}] \xrightarrow{T} G^L[2^ES_m^{n+1}]\).

By assumption, \(C_i^1(a_{\overline{\sigma}}, \overline{\tau})\) is a covariant on homogenous polynomials \(P_i(\overline{\tau})\) for integers \(1 \leq i \leq m\). Let \(T : C_i^1(a_{\overline{\sigma}}, \overline{\tau}) \rightarrow C_i^2(a_{\overline{\sigma}'}, \overline{\tau})\). Then \(C_i^2(a_{\overline{\sigma}'}, \overline{\tau}) = C_i^2(a_{\overline{\sigma}}, \overline{\tau}) = \Delta^p C_i^1(a_{\overline{\sigma}}, \overline{\tau})\) for integers \(1 \leq i \leq m\), where \(p\) is a constant.

Notice that
\[
(1^ES_m^{n+1}) = \{C_i^1(a_{\overline{\sigma}}, \overline{\tau}) | v \in V(G^L[1^ES_m^{n+1}])\}, \\
(2^ES_m^{n+1}) = \{C_i^2(a_{\overline{\sigma}'}, \overline{\tau}) | u \in V(G^L[2^ES_m^{n+1}])\}.
\]

If \(G^L[1^ES_m^{n+1}] \xrightarrow{T} G^L[2^ES_m^{n+1}]\) with \(C_i^2(a_{\overline{\sigma}'}, \overline{\tau}) = \Delta^p C_i^1(a_{\overline{\sigma}}, \overline{\tau})\) holds for a constant \(p\), let \(T = [\alpha_{ij}]_{(n+1)\times(n+1)}\). Then \(T\) must be a linear isomorphism between systems \((1^ES_m^{n+1})\) and \((2^ES_m^{n+1})\). In fact, for \(\forall C_i^1(a_{\overline{\sigma}}, \overline{\tau}) \in (1^ES_m^{n+1})\), let \(T : C_i^1(a_{\overline{\sigma}}, \overline{\tau}) \rightarrow C_i^2(a_{\overline{\sigma}'}, \overline{\tau}) \in (2^ES_m^{n+1})\). Then \(C_i^2(a_{\overline{\sigma}'}, \overline{\tau}) = \Delta^p C_i^1(a_{\overline{\sigma}}, \overline{\tau})\). Consequently, \((1^ES_m^{n+1}) \xrightarrow{T} (2^ES_m^{n+1})\) by definition.

\[\square\]

Theorem 3.2 enables one immediately knowing the following result.

Corollary 3.3 Let \((ES_m^{n+1})\) be a system consisting of covariants \(C_i^1(a_{\overline{\sigma}}, \overline{\tau})\) on homogenous polynomials \(P_i(\overline{\tau})\) for integers \(1 \leq i \leq m\) of weight \(p\) and \(T\) be an invertible linear transformation. Then

\[G^L[ES_m^{n+1}] \simeq G^L[T^k(ES_m^{n+1})]\]

for any integer \(k\).

Particularly, let \(p = 0\), i.e., \((ES_m^{n+1})\) consisting of homogenous polynomials \(P_1(\overline{\tau}), P_2(\overline{\tau}), \cdots, P_m(\overline{\tau})\) in Theorem 3.2. Then it also implies the following conclusion.
Corollary 3.4 Let \((1ES_m^{n+1}), (2ES_m^{n+1})\) be systems of homogenous polynomials \(P_i(\overline{\tau}), 1 \leq i \leq m\). Then \((1ES_m^{n+1}) \cong (2ES_m^{n+1})\) if and only if \(G^L[1ES_m^{n+1}] \cong G^L[2ES_m^{n+1}]\) with \(T\) an invertible linear transformation on \(\mathbb{P}^n\mathbb{C}\).

Furthermore, if \((1ES_m^{n+1}) = (2ES_m^{n+1}) = (ES_m^{n+1})\) in Definition 3.1, a linear isomorphism is called an automorphism of \((ES_m^{n+1})\). Clearly, all automorphisms of \((ES_m^{n+1})\) form a group \(\text{Aut}[ES_m^{n+1}]\) under the composition operation, which can be determined following.

Theorem 3.5 Let \((ES_m^{n+1})\) be a covariant system of \(C_i(a_{m, \overline{x}}), 1 \leq i \leq m\) of weight \(p\) on homogenous polynomials \(P_1(\overline{x}), P_2(\overline{x}), \cdots, P_m(\overline{x})\) for an integer \(n \geq 1\). Then

\[
\text{Aut}[ES_m^{n+1}] = \left\{ \text{Aut}G^L; \bigcap_{v \in V(H)} \text{Aut}(C_v) : H^L \leq G^L[ES_m^{n+1}] \right\} \cap \text{PGL}(n),
\]

where \(H^L \leq G^L\), \(\text{Aut}(C_v)\) denote respectively an induced topological subgraph \(H\) of graph \(\mathcal{G}\) with labeling \(L\) on \(G^L\) and automorphism group of covariant \(C_v(a_{m, \overline{x}})\) at vertex \(v\).

Proof Let \((ES_m^{n+1})\) be a covariant system of \(C_i(a_{m, \overline{x}}), 1 \leq i \leq m\) of weight \(p\) on homogenous polynomials \(P_1(\overline{x}), P_2(\overline{x}), \cdots, P_m(\overline{x})\) for an integer \(n \geq 1\) with an invertible linear isomorphism \(T: (ES_m^{n+1}) \to (ES_m^{n+1})\) such that \(C_v^T = C_v\) for all \(v \in V(H)\), where \(H \leq G[ES_m^{n+1}]\). Clearly, if \(H = \emptyset\), then \(T \in \text{Aut}G^L[ES_m^{n+1}]\). Now if \(H \neq \emptyset\), by Theorem 3.2 there must be \(T: G^L[ES_m^{n+1}] \to G^L[ES_m^{n+1}] \to H^L\) with \(T \subseteq \bigcap_{v \in V(H)} \text{Aut}(C_v)\). Thus

\[
\text{Aut}[ES_m^{n+1}] \subseteq \left\{ \text{Aut}G^L; \bigcap_{v \in V(H)} \text{Aut}(C_v) : H^L \leq G^L[ES_m^{n+1}] \right\} \cap \text{PGL}(n).
\]

Conversely, such linear transformations \(\omega\) are indeed automorphisms of system \((ES_m^{n+1})\). In fact, let

\[
\omega: C_v \to C_v, \quad v \in V(H^L), \quad V^L(G[ES_m^{n+1}] - H) \to V^L(G[ES_m^{n+1}] - H)
\]

be an invertible linear transformation. Then, \(T \in \text{Aut}[ES_m^{n+1}]\). Thus

\[
\text{Aut}[ES_m^{n+1}] \supseteq \left\{ \text{Aut}G^L; \bigcap_{v \in V(H)} \text{Aut}(C_v) : H^L \leq G^L[ES_m^{n+1}] \right\} \cap \text{PGL}(n). \quad \square
\]

Particularly, let \(p = 0\). We get the automorphism group of system \((ES_m^{n+1})\) of homogenous polynomials following.

Corollary 3.6 Let \((ES_m^{n+1})\) be a system of homogenous polynomials \(P_1(\overline{x}), P_2(\overline{x}), \cdots, P_m(\overline{x})\) for an integer \(n \geq 1\). Then

\[
\text{Aut}[ES_m^{n+1}] = \left\{ \text{Aut}G^L; \bigcap_{v \in V(H)} \text{Aut}(P_v(\overline{x})) : H^L \leq G^L[ES_m^{n+1}] \right\} \cap \text{PGL}(n).
\]
For example, let \((ES^3_4) = \{(P_1(x, y, z) = a_1x + b_1y + c_1z, 1 \leq i \leq 4)\}\) be 4 distinct homogenous polynomials of degree 1, such as those shown in Fig.3.

\[
\begin{align*}
K^L_4 - \{(P_1, P_4)\} & \\
K^L_{1,3} & \\
x + y + z & x + y + 2z & x + 2y + z & x + 2yc + z & x + 4y + z & x + y + z + 2zx + y + 3z
\end{align*}
\]

Fig.3

According to the ratio of coefficients, \(G[ES^3_4]\) and \(\text{Aut}[ES^3_4]\) are listed in Table 1 following,

<table>
<thead>
<tr>
<th>Case</th>
<th>Ratio</th>
<th>(G[ES^3_4])</th>
<th>(\text{Aut}[ES^3_4])</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(a_1 : b_1 \neq a_2 : b_2 \neq a_3 : b_3 \neq a_4 : b_4)</td>
<td>(K_4)</td>
<td>(S_{(P_1, P_2, P_3, P_4)})</td>
</tr>
<tr>
<td>2</td>
<td>(a_1 : a_2 : a_3 : a_4 = b_1 : b_2 : b_3 : b_4)</td>
<td>(K_4)</td>
<td>(S_{(P_1, P_2, P_3, P_4)})</td>
</tr>
<tr>
<td>3</td>
<td>(a_1 : a_2 : a_3 = b_1 : b_2 : b_3, \neq a_4 : b_4)</td>
<td>(K_{1,3})</td>
<td>(S_{(P_1, P_2, P_3, P_4)})</td>
</tr>
<tr>
<td>4</td>
<td>(a_1 : a_2 = b_1 : b_2, \neq a_3 : b_3, \neq a_4 : b_4)</td>
<td>(C_4, K_4 - {(P_1, P_2)})</td>
<td>(S_{(P_1, P_2, P_3, P_4)})</td>
</tr>
</tbody>
</table>

Table 1

where \(S_{(P_1, P_2, P_3, P_4)}\) denotes the symmetric group on \(P_1, P_2, P_3, P_4\) and the last column is obtained by the four lines lemma in algebraic geometry. Clearly, \(\text{Aut}K^L_4 = \text{Aut}C^L_4 = S_{(P_1, P_2, P_3, P_4)}, \langle(P_1 P_2)(P_3 P_4), (P_1 P_3)\rangle = S_{(P_1, P_2, P_3, P_4)}, \langle S_{P_1, P_2, P_3, (P_1 P_4)\rangle = S_{(P_1, P_2, P_3, P_4)}\). Thus, \(\text{Aut}[ES^3_4] \simeq S_4\), a finite group. It should be noted that \(\text{Aut}[ES^m_4]\) maybe not finite. But if it is indeed finite, a natural inverse question is the following:

**Problem 3.7** Let \(H \leq \text{PGL}(n)\) be a finite group. Is there a finite system \((ES^m_4)\) of homogenous polynomials with \(\text{Aut}[ES^m_4] \simeq H\)?

The answer for this question is positive. Thus

**Theorem 3.8** For any finite subgroup \(H \leq \text{PGL}(n)\), there always exists a finite system \((ES^m_4)\) of homogenous polynomials with \(\text{Aut}[ES^m_4] \simeq H\).

**Proof** Let \(P(\overline{x})\) be a homogenous polynomial in \(n + 1\) variables and system \((ES^m_4) = \{P^h(\overline{x}), \forall h \in H\}\). Clearly, \(P^h(\overline{x}) \neq P^g(\overline{x})\) for \(\forall h, g \in H\) if \(h \neq g\). Notice that \(H\) is finite. Thus \(m < \infty\), i.e., a finite system \((ES^m_4)\). Clearly, \(H \leq \text{Aut}[ES^m_4]\) by definition. If \(\text{Aut}[ES^m_4] \neq H\), let \(\theta \in \text{Aut}[ES^m_4] \setminus H\), then there must be \(P^h(\overline{x}) \in (ES^m_4)\). By the construction of \((ES^m_4)\), there is an element \(h \in H\) such that \(P^h(\overline{x}) = P^h(\overline{x})\), i.e., \(\theta \in H\), a contradiction. \(\square\)

The topological graph \(G^L[ES^m_4]\) of system \((ES^m_4)\) constructed in the proof of Theorem 3.8 is dependent on \(P(\overline{x})\) and group \(H\). For example, let \(P(\overline{x}) = x_1\) and
the invertible matrix for $\forall h \in H$ is $[\alpha^{h}_{ij}]_{(n+1) \times (n+1)}$. Then

$$P^{h}(\overline{x}) = \alpha^{h}_{11}x_1 + \alpha^{h}_{12} + \cdots + \alpha^{h}_{1,n+1}x_{n+1}.$$ 

Thus,

$$\emptyset \neq P^{h}(\overline{x}) \cap P(\overline{x}) \subset \{(0, x_2, \cdots, x_{n+1}), \ x_i \in \mathbb{C}, \ 2 \leq i \leq n + 1\}$$

and $P^{h} \parallel P^{g}$ if and only if

$$\frac{\alpha^{h}_{11}}{\alpha^{g}_{11}} = \frac{\alpha^{h}_{12}}{\alpha^{g}_{12}} = \cdots = \frac{\alpha^{h}_{1n+1}}{\alpha^{g}_{1n+1}} \neq \frac{\alpha^{h}_{1n}}{\alpha^{g}_{1n}}$$

for $h, g \in H$. Consequently,

$$V(\mathrm{G}[ES^{n+1}_{m}]) = \{x^{h}_{1}, \ h \in H\}$$

$$E(\mathrm{G}[ES^{n+1}_{m}]) = \{(x^{h}_{1}, x^{q}_{1}), \ h, g \in H \text{ without } \frac{\alpha^{h}_{11}}{\alpha^{g}_{11}} = \frac{\alpha^{h}_{12}}{\alpha^{g}_{12}} = \cdots = \frac{\alpha^{h}_{1n+1}}{\alpha^{g}_{1n+1}} \neq \frac{\alpha^{h}_{1n}}{\alpha^{g}_{1n}}\}$$

with a labeling

$$L: x^{h}_{1} \to x^{h}_{1} \text{ and } (x^{h}_{1}, x^{q}_{1}) \to x^{h}_{1} \bigcap x^{q}_{1}, \ h, g \in H.$$ 

\section{Topology on $G^{L}$-Systems of Homogeneous Polynomials in $n + 1$ Variables}

Let $\pi: \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^{n}\mathbb{C}$ determined by $\pi(x_1, x_2, \cdots, x_{n+1}) = [x_1, x_2, \cdots, x_{n+1}]$ be the projection from $\mathbb{C}^{n+1} \setminus \{0\}$ to $\mathbb{P}^{n}\mathbb{C}$, where $[x_1, x_2, \cdots, x_{n+1}]$ is the homogenous coordinates for points in $\mathbb{P}^{n}\mathbb{C}$. Clearly, the equation $P(\overline{x}) = 0$ determines a smooth function $x_{n+1} = f(x_1, x_2, \cdots, x_n)$ by the implicit theorem if $P(\overline{x})$ is homogenous. It is easily to verify that the subspace $\Gamma[f]$ defined by

$$\Gamma[f] = \{(x_1, x_2, \cdots, x_{n+1}) \in \mathbb{C}^{n+1}: x_{n+1} = f(x_1, x_2, \cdots, x_n), x_i \in \mathbb{C}, 1 \leq i \leq n\}$$

is a complex $n$-manifold, i.e., a hypersurface in $\mathbb{C}^{n+1}$ and $\pi: \Gamma[f] \in \mathbb{C}^{n+1} \to \pi(\Gamma[f]) \in \mathbb{P}^{n}\mathbb{C}$ is a bijection. Thus $\pi(\Gamma[f])$ is a hypersurface in $\mathbb{P}^{n}\mathbb{C}$.

Generally, let $(ES^{n+1}_{m})$ be a $G^{L}$-system of homogenous polynomials $P(\overline{x}_1), P(\overline{x}_2), \cdots, P(\overline{x}_m)$ in $n + 1$ variables with respectively hypersurfaces $S_i = \pi(\Gamma[f_i])$, where $f_i$ is the implicit function $x_{n+1} = f_i(x_1, x_2, \cdots, x_n)$ determined by $P_i(\overline{x}) = 0$ and let $\widetilde{M} = \bigcup_{i=1}^{m} M_i$ with $M_i = \Gamma[f_i]$ for integers $1 \leq i \leq m$. Clearly, for $\forall p \in \widetilde{M}$, there is an open neighborhood $U(p)$ homeomorphic to $\mathbb{C}^n$, $\widetilde{M}$ is Hausdorff and second countable if $m < \infty$. Thus $\widetilde{M}$ is also an $n$-manifold and $\pi: \widetilde{M} \to \widetilde{S}$ is $1 - 1$ by definition. We get a result following.

\textbf{Theorem 4.1} Let $(ES^{n+1}_{m})$ be a $G^{L}$-system consisting of homogenous polynomials $P(\overline{x}_1), P(\overline{x}_2), \cdots, P(\overline{x}_m)$ in $n + 1$ variables with respectively hypersurfaces $S_i = \pi(\Gamma[f_i])$, where $f_i$ is the implicit function $x_{n+1} = f_i(x_1, x_2, \cdots, x_n)$ determined by $P_i(\overline{x}) = 0$ and let $\widetilde{M} = \bigcup_{i=1}^{m} M_i$ with $M_i = \Gamma[f_i]$ for integers $1 \leq i \leq m$. Clearly, for $\forall p \in \widetilde{M}$, there is an open neighborhood $U(p)$ homeomorphic to $\mathbb{C}^n$, $\widetilde{M}$ is Hausdorff and second countable if $m < \infty$. Thus $\widetilde{M}$ is also an $n$-manifold and $\pi: \widetilde{M} \to \widetilde{S}$ is $1 - 1$ by definition. We get a result following.
Let \( S_1, S_2, \ldots, S_m \) be a combinatorial surface consisting of \( m \) orientable surfaces \( S_1, S_2, \ldots, S_m \) underlying a topological graph \( G^L[S] \). Then

\[
\beta(G(\tilde{S})) = g(\tilde{S}) + \sum_{i=1}^{m} (-1)^{i+1} \left[ g(S_{k_1} \cap \cdots \cap S_{k_i}) - c(S_{k_1} \cap \cdots \cap S_{k_i}) + 1 \right],
\]

where \( g(S_{k_1} \cap S_{k_2} \cap \cdots \cap S_{k_i}) \) and \( c(S_{k_1} \cap S_{k_2} \cap \cdots \cap S_{k_i}) \) are respectively the genus and number of path-connected components in surface \( S_{k_1} \cap S_{k_2} \cap \cdots \cap S_{k_i} \), \( \beta(G(\tilde{S})) \) is the Betti number of topological graph \( G(\tilde{S}) \).

Proof\ The proof is by induction on \( m \). If \( m = 2 \), by the classification theorem of surfaces, we assume \( S_1 \) is a connected sum \( T^2 \# T^2 \# \cdots \# T^2 \) of \( g(S_i) \) toruses for \( 1 \leq i \leq m \). By this geometrical model, the contribution of genus of surfaces \( S_2 \) to \( S_1 \) clearly is

\[
g(\tilde{S}) = g(S_2) - g(S_1 \cap S_2) + c(S_1 \cap S_2) - 1
\]

because these \( c(S_1 \cap S_2) \) path-connected components contribute \( c(S_1 \cap S_2) - 1 \) new tori to \( S_1 \) and \( \beta(G(\tilde{S})) = 0 \) in this case. Whence,

\[
g(S_1 \cup S_2) = g(S_1) + g(S_2) - g(S_1 \cap S_2) + c(S_1 \cap S_2) - 1.
\]
Thus the conclusion holds with \( m = 2 \).

Now assume the conclusion holds with \( m \leq s \). Notice that

\[
\bar{S} = \bigcup_{i=1}^{s+1} S_i = \left( \bigcup_{i=1}^{s} S_i \right) \bigcup S_{s+1}.
\]

Applying the inclusion-exclusion principle and the contribution of new tori by path-connected components in their intersection, the contribution of genus of surface \( S_{s+1} \) to the combinatorial surface \( \left( \bigcup_{i=1}^{s} S_i \right) \) is

\[
g(S_{s+1}) - \sum_{S_{k_1} \cap \cdots \cap S_{k_l} \neq \emptyset} (-1)^i \left[ g \left( S_{s+1} \cap \left( S_{k_1} \cap S_{k_2} \cap \cdots \cap S_{k_l} \right) \right) \right]
- c \left( S_{s+1} \cap \left( S_{k_1} \cap S_{k_2} \cap \cdots \cap S_{k_l} \right) \right) + 1 \] + loops of \( S_{s+1} \) with \( S_1, S_2, \ldots, S_s \).

Whence,

\[
g(\bar{S}) = g(\bar{S} \setminus S_{s+1}) + \text{loops of } S_{s+1} \text{ with } \bar{S} \setminus S_{s+1}
= g(\bar{S} \setminus S_{s+1}) + g(S_{s+1}) - \sum_{S_{k_1} \cap \cdots \cap S_{k_l} \neq \emptyset} (-1)^i \left[ g \left( S_{s+1} \cap \left( S_{k_1} \cap \cdots \cap S_{k_l} \right) \right) \right]
- c \left( S_{s+1} \cap \left( S_{k_1} \cap \cdots \cap S_{k_l} \right) \right) + 1 \] + loops of \( S_{s+1} \) with \( S_1, S_2, \ldots, S_s \)

\[
= \beta(G(\bar{S} \setminus S_{s+1}))
+ \sum_{i=1}^{m} (-1)^{i+1} \sum_{S_{k_1} \cap \cdots \cap S_{k_l} \neq \emptyset} \left[ g \left( S_{k_1} \cap \cdots \cap S_{k_l} \right) - c \left( S_{k_1} \cap \cdots \cap S_{k_l} \right) + 1 \right]
+ g(S_{s+1}) + \sum_{1 \leq k_1, \ldots, k_i \leq s} (-1)^{i+1} \left[ g \left( S_{s+1} \cap \left( S_{k_1} \cap \cdots \cap S_{k_i} \right) \right) \right]
- c \left( S_{s+1} \cap \left( S_{k_1} \cap \cdots \cap S_{k_i} \right) \right) + 1 \] + loops of \( S_{s+1} \) with \( S_1, S_2, \ldots, S_s \)

\[
= \beta(G(\bar{S}))
+ \sum_{i=1}^{s+1} (-1)^{i+1} \sum_{S_{k_1} \cap \cdots \cap S_{k_i} \neq \emptyset} \left[ g \left( S_{k_1} \cap \cdots \cap S_{k_i} \right) - c \left( S_{k_1} \cap \cdots \cap S_{k_i} \right) + 1 \right].
\]

Thus, the conclusion holds also with \( m = s + 1 \). \( \square \)

**Corollary 4.3** Let \( \bar{S} \) be a combinatorial surface consisting of orientable surfaces \( S_1, S_2, \ldots, S_m \) with \( S_{k_1} \cap \cdots \cap S_{k_i} \) path-connected or empty for integers \( 1 \leq k_1, k_2, \ldots, k_i \leq m \). Then

\[
g(\bar{S}) = \beta(G(\bar{S})) + \sum_{i=1}^{m} (-1)^{i+1} \sum_{S_{k_1} \cap \cdots \cap S_{k_i} \neq \emptyset} g \left( S_{k_1} \cap \cdots \cap S_{k_i} \right).
\]
Furthermore, if $S_{k_1} \cap \cdots \cap S_{k_i}$ is simply-connected or empty for any integers $k_1, k_2, \ldots, k_i$, then

$$g(\tilde{S}) = \beta(G(\tilde{S})) + \sum_{i=1}^{m} g(S_i).$$

Proof Notice that for integers $1 \leq k_1, k_2, \ldots, k_i \leq m$, if $S_{k_1} \cap \cdots \cap S_{k_i}$ path-connected, then $c(S_{k_1} \cap \cdots \cap S_{k_i}) = 1$ and if $S_{k_1} \cap \cdots \cap S_{k_i}$ is simply-connected, then $g(S_{k_1} \cap \cdots \cap S_{k_i}) = 0$. According to Theorem 4.2, this conclusion follows. □

Theorem 4.2 enables one to determine the genus of surface $\tilde{S}$ by applying Noether’s formula in algebraic geometry, particularly, the non-singular case. Notice that the Bézout’s theorem claims that the number of intersection points counting multiplicities of two projective curves $C_1, C_2$ in $\mathbb{P}^2\mathbb{C}$ is $|I(P, Q)| = \deg(P)\deg(Q)$ if they are determined by $P(x, y, z)$ and $Q(x, y, z)$ without common component. Thus, if $S_i$ is the normalizations of $C_i$ for $1 \leq i \leq 2$, then surfaces $S_1$ and $S_2$ are tangent each other at $\deg(P)\deg(Q)$ points. Applying Theorem 4.2 and Corollary 4.3, we get the genus of the normalization $\tilde{S}$ of complex curves $C_1, C_2, \ldots, C_m$ following.

**Theorem 4.4** Let $C_1, C_2, \ldots, C_m$ be complex curves determined by homogenous polynomials $P_1(x, y, z), P_2(x, y, z), \ldots, P_m(x, y, z)$ without common component, and let

$$R_{P_i, P_j} = \prod_{k=1}^{\deg(P_i)\deg(P_j)} (e_{k}^{ij} z - b_{k}^{ij} y)^{e_{k}^{ij}}, \quad \omega_{i,j} = \sum_{k=1}^{\deg(P_i)\deg(P_j)} \sum_{e_{k}^{ij} \neq 0} 1$$

be the resultant of $P_i(x, y, z), P_j(x, y, z)$ for $1 \leq i \neq j \leq m$. Then there is an orientable surface $\tilde{S}$ in $\mathbb{R}^3$ of genus

$$g(\tilde{S}) = \beta(G(\tilde{C})) + \sum_{i=1}^{m} \left( \frac{(\deg(P_i) - 1)(\deg(P_i) - 2)}{2} - \sum_{p^i \in \text{Sing}(C_i)} \delta(p^i) \right)$$

$$+ \sum_{1 \leq i \neq j \leq m} (\omega_{i,j} - 1) + \sum_{i \geq 3} \sum_{c_{k_1} \cap \cdots \cap c_{k_i} \neq \emptyset} \left[ c \left( C_{k_1} \cap \cdots \cap C_{k_i} \right) - 1 \right]$$

with a homeomorphism $\varphi : \tilde{S} \rightarrow \tilde{C} = \bigcup_{i=1}^{m} C_i$. Furthermore, if $C_1, C_2, \ldots, C_m$ are non-singular, then

$$g(\tilde{S}) = \beta(G(\tilde{C})) + \sum_{i=1}^{m} \left( \frac{(\deg(P_i) - 1)(\deg(P_i) - 2)}{2} \right)$$

$$+ \sum_{1 \leq i \neq j \leq m} (\omega_{i,j} - 1) + \sum_{i \geq 3} \sum_{c_{k_1} \cap \cdots \cap c_{k_i} \neq \emptyset} \left[ c \left( C_{k_1} \cap \cdots \cap C_{k_i} \right) - 1 \right],$$

where

$$\delta(p^i) = \frac{1}{2} \left( I_{p^i} \left( P_i, \frac{\partial P_i}{\partial y} \right) - \nu_{0}(p^i) + |\pi^{-1}(p^i)| \right)$$
is a positive integer with a ramification index \( \nu_\phi(p^i) \) for \( p^i \in \text{Sing}(C_i) \), \( 1 \leq i \leq m \).

**Proof** This result is an immediately consequence of Theorem 4.2 and Noether’s result. For its genus, let \( S_i \) be the normalization of \( C_i \), i.e., \( \varphi_i : S_i \rightarrow C_i \) in \( \mathbb{R}^3 \) for integers \( 1 \leq i \leq m \) and \( \tilde{S} = \bigcup_{i=1}^{m} S_i \). Define \( \varphi : \tilde{S} \rightarrow \tilde{C} \) by \( \varphi(p) = \varphi_i(p) \) if \( p \in S_i \), \( 1 \leq i \leq m \). This definition is well-defined because if \( p \in \bigcap S_{k_1} \cdots \bigcap S_{k_l} \), then \( \varphi(p) \in \bigcap C_{k_1} \bigcap \cdots \bigcap C_{k_l} \).

Notice that each \( \varphi_i \) is a homeomorphism. Thus \( \varphi \) is also a homeomorphism from \( \tilde{S} \) to \( \tilde{C} \), i.e., a normalization of surface to \( \bigcup_{i=1}^{m} C_i \) with \( G(\tilde{S}) \simeq G(\tilde{C}) \).

Clearly, \( C_{k_1} \bigcap \cdots \bigcap C_{k_l} \) only consists of isolated points. Whence, \( g(S_{k_1} \bigcap \cdots \bigcap S_{k_l}) = 0 \) for any subset \( \{k_1, \cdots, k_l\} \subset \{1, 2, \cdots, m\} \) with \( i \geq 2 \) and

\[
\sum_{1 \leq i \neq j \leq m} \left[ c \left( S_i \bigcap S_j \right) - 1 \right] = \sum_{1 \leq i \neq j \leq m} \left[ c \left( C_i \bigcap C_j \right) - 1 \right] = \sum_{1 \leq i \neq j \leq m} (\omega_{i,j} - 1),
\]

if \( C_{k_1} \bigcap \cdots \bigcap C_{k_l} \neq \emptyset \). Substituting the Noether’s formula into Theorem 4.2, we get that

\[
g(\tilde{S}) = \beta(G(\tilde{S})) + \sum_{i=1}^{m} (-1)^{i+1} \sum_{C_{k_1} \bigcap \cdots \bigcap C_{k_i} \neq \emptyset} \left[ g(S_{k_1} \bigcap \cdots \bigcap S_{k_i}) - c(S_{k_1} \bigcap \cdots \bigcap S_{k_i}) - 1 \right] \]

\[
= \beta(G(\tilde{C})) + \sum_{i=1}^{m} g(C_i) + \sum_{i \geq 2} \sum_{C_{k_1} \bigcap \cdots \bigcap C_{k_i} \neq \emptyset} \left[ c(C_{k_1} \bigcap \cdots \bigcap C_{k_i}) - 1 \right] \]

\[
= \beta(G(\tilde{C})) + \sum_{i=1}^{m} \frac{(\text{deg}(P_i) - 1)(\text{deg}(P_i) - 2)}{2} - \sum_{P^i \in \text{Sing}(C_i)} \delta(p^i) + \sum_{1 \leq i \neq j \leq m} (\omega_{i,j} - 1) + \sum_{i \geq 3} (-1)^{i} \sum_{C_{k_1} \bigcap \cdots \bigcap C_{k_i} \neq \emptyset} \left[ c(C_{k_1} \bigcap \cdots \bigcap C_{k_i}) - 1 \right]. \quad \square
\]

Theorem 4.4 enables us to get consequences following.

**Corollary 4.5** Let \( C_1, C_2, \cdots, C_m \) be complex non-singular curves determined by homogeneous polynomials \( P_1(x, y, z), P_2(x, y, z), \cdots, P_m(x, y, z) \) without common component, any intersection point \( p \in I(P_i, P_j) \) with multiplicity \( e_p \) and

\[
\begin{align*}
\left\{ \begin{array}{l}
P_i(x, y, z) = 0 \\
P_j(x, y, z) = 0, \quad \forall i, j \in \{1, 2, \cdots, m\} \\
P_k(x, y, z) = 0
\end{array} \right.
\end{align*}
\]
has zero-solution only. Then the genus of normalization $\widetilde{S}$ of curves $C_1, C_2, \cdots, C_m$ is
\[
g(\widetilde{S}) = 1 + \frac{1}{2} \sum_{i=1}^{m} \deg(P_i) (\deg(P_i) - 3) + \sum_{1 \leq i \neq j \leq m} \omega_{i,j}.
\]
Particularly, if $e_p^{ij} = 1$ for $\forall p \in I(P_i, P_j)$, $1 \leq i \neq j \leq m$, then
\[
g(\widetilde{S}) = 1 + \frac{1}{2} \sum_{i=1}^{m} \deg(P_i) (\deg(P_i) - 3) + \sum_{1 \leq i \neq j \leq m} \deg(P_i) \deg(P_j).
\]

**Proof** Notice that $G \langle \tilde{C} \rangle \simeq K_m$ with
\[
\beta(K_m) = \frac{m(m+1)}{2} - m + 1 \quad \text{and} \quad C_{k_1} \cap \cdots \cap C_{k_i} = \emptyset, \ i \geq 3
\]
by assumption. Applying Theorem 4.4, we know that
\[
g(\tilde{S}) = \beta(G \langle \tilde{C} \rangle) + \sum_{i=1}^{m} \frac{(\deg(P_i) - 1)(\deg(P_i) - 2)}{2}
\]
\[
+ \sum_{1 \leq i \neq j \leq m} (\omega_{i,j} - 1) + \sum_{i \geq 3} (-1)^i \sum_{\omega_{i,j} \neq 0} \left[ c \left( C_{k_1} \cap \cdots \cap C_{k_i} \right) - 1 \right]
\]
\[
= \beta(K_m) + \sum_{i=1}^{m} \frac{(\deg(P_i) - 1)(\deg(P_i) - 2)}{2} + \sum_{1 \leq i \neq j \leq m} (\omega_{i,j} - 1)
\]
\[
= 1 + \frac{1}{2} \sum_{i=1}^{m} \deg(P_i) (\deg(P_i) - 3) + \sum_{1 \leq i \neq j \leq m} \omega_{i,j}.
\]
Particularly, if $e_p^{ij} = 1$ for $\forall p \in I(P_i, P_j)$, then $\omega_{i,j} = \deg(P_i) \deg(P_j)$ by Bézout’s theorem for integers $1 \leq i \neq j \leq m$, we get that
\[
g(\tilde{S}) = 1 + \frac{1}{2} \sum_{i=1}^{m} \deg(P_i) (\deg(P_i) - 3) + \sum_{1 \leq i \neq j \leq m} \deg(P_i) \deg(P_j). \quad \square
\]

**Corollary 4.6** Let $C_1, C_2, \cdots, C_m$ be complex non-singular curves determined by homogenous polynomials $P_1(x, y, z), P_2(x, y, z), \cdots, P_m(x, y, z)$ without common component and $C_i \cap C_j = \bigcap_{i=1}^{m} C_i$ with $\bigcap_{i=1}^{m} C_i = \kappa > 0$ for integers $1 \leq i \neq j \leq m$. Then the genus of normalization $\tilde{S}$ of curves $C_1, C_2, \cdots, C_m$ is
\[
g(\tilde{S}) = g(\tilde{S}) = (\kappa - 1)(m - 1) + \sum_{i=1}^{m} \frac{(\deg(P_i) - 1)(\deg(P_i) - 2)}{2}.
\]

**Proof** Notice that $G \langle \tilde{S} \rangle \simeq S_{1,m+1}$ with $\beta(S_{1,m+1}) = 0$, $\omega_{i,j} = \kappa$ for integers
\[1 \leq i \neq j \leq m \text{ and}\
\sum_{1 \leq i \neq j \leq m} (\omega_{i,j} - 1) + \sum_{i \geq 3} (-1)^i \sum_{C_{k_1} \cap \cdots \cap C_{k_i} \neq \emptyset} \left[ c \left( C_{k_1} \cap \cdots \cap C_{k_i} \right) - 1 \right]\
= \sum_{i \geq 2} (-1)^i (\kappa - 1) \left( \frac{m}{i} \right) = (\kappa - 1)(m - 1).\]

Applying Theorem 4.4, we get that
\[g(\tilde{S}) = (\kappa - 1)(m - 1) + \sum_{i=1}^{m} \left( \deg(P_i) - 1 \right) \left( \deg(P_i) - 2 \right).\]

For homogenous polynomials with small degrees, Theorem 4.4 also enables one to get interesting conclusions.

**Corollary 4.7** Let \(L_1, L_2, \cdots, L_m\) be distinct lines in \(\mathbb{P}^2\mathbb{C}\) with respective normalizations of spheres \(S_1, S_2, \cdots, S_m\). Then there is a normalization of surface \(\tilde{S}\) of \(L_1, L_2, \cdots, L_m\) with genus \(\beta(\langle \tilde{L} \rangle)\). Particularly, if \(\langle \tilde{L} \rangle\) is a tree, then \(\tilde{S}\) is homeomorphic to a sphere.

**Corollary 4.8** Let \(C_1, C_2, \cdots, C_m\) be non-singular conics determined by homogenous polynomials \(P_1(x, y, z), P_2(x, y, z), \cdots, P_m(x, y, z)\) of degree 2 without common component in \(\mathbb{P}^2\mathbb{C}\) and let \(S_1, S_2, \cdots, S_m\) be respective normalizations of surfaces conics \(C_1, C_2, \cdots, C_m\). Then there is a normalization surface \(\tilde{S}\) in of genus
\[g(\langle \tilde{S} \rangle) = \beta(\langle \tilde{C} \rangle) + \sum_{i \geq 2} (-1)^i \sum_{C_{k_1} \cap \cdots \cap C_{k_i} \neq \emptyset} \left[ c \left( C_{k_1} \cap \cdots \cap C_{k_i} \right) - 1 \right]\
\text{with} \quad c(C_{k_1} \cap \cdots \cap C_{k_i}) \leq 4 \text{ for integers } C_{k_1} \cap \cdots \cap C_{k_i} \neq \emptyset.\]

Particularly, if no 3 conics of \(C_1, C_2, \cdots, C_m\) pass through a common point, then
\[g(\langle \tilde{S} \rangle) = \beta(\langle \tilde{C} \rangle) + \sum_{1 \leq i \neq j \leq m} \omega_{i,j} - \left( \frac{m}{2} \right).\]

§5. Application to Elliptic Differential Equations

As we known, a Dirichlet problem on Laplace equation is to find functions \(u(\mathbf{x})\) in a region \(D \subset \mathbb{R}^n\) with
\[
\left\{ \begin{array}{l}
\Delta u = 0, \quad \mathbf{x} \in D \\
u \mid_{\partial D} = \varphi(\mathbf{x})
\end{array} \right.
\]
holds, where
\[\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.\]
If $D \subset \mathbb{R}^n$ is nothing else but a ball $B^n$ and $\varphi(\bar{x})$ is continuous for $\|\bar{x}\| = a$, the Poisson’s integral formula (\cite{3})

$$u(\bar{\xi}) = \int_{\|\bar{x}\|=a} \frac{1}{2a \sqrt{\pi}} \frac{1}{\Gamma\left(\frac{1}{2}\right)} \frac{a^2 - \|\bar{\xi}\|^2}{\|\bar{x} - \bar{\xi}\|^n} \varphi(\bar{x}) dS$$

provides the solution of Dirichlet problem. Particularly, if $n = 2$, we get

$$u(\rho, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{a^2 - \rho^2}{a^2 + \rho^2 - 2a \rho \cos(\theta - \alpha)} \varphi(\alpha) d\alpha$$
in spherical coordinates $(\rho, \theta)$.

Let $\varphi_1(\bar{x}), \varphi_2(\bar{x}), \ldots, \varphi_m(\bar{x})$ be $m$ respective continuous functions, particularly, $m$ homogenous polynomials for $\|\bar{x}\| = a$. A natural question on the Laplace equation is that whether a system of

$$\begin{align*}
\Delta u &= 0, & \bar{x} \in D \\
u |_{\partial D} &= \varphi_i(\bar{x})
\end{align*} \quad \{1 \leq i \leq m\}$$

is solvable or not?

We consider a special case of $D = B^n$ and denote by $\Phi(\varphi_1, n), \Phi(\varphi_2, n), \ldots, \Phi(\varphi_m, n)$ the hypersurfaces respectively determined by $x_{n+1} = \varphi_i(\bar{x})$ in $\mathbb{R}^{n+1}$ with a topological graph $G\left(\bar{\Phi}\right)$. Then such a question can be also viewed as the Dirichlet problem on the Laplace equation prescribed with a boundary value $G\left(\bar{\Phi}\right)$ with $\bar{\Phi} = \bigcup_{i=1}^m \Phi(\varphi_i, n)$.

Similarly, let $S(\varphi_1, n), S(\varphi_2, n), \ldots, S(\varphi_m, n)$ be hypersurfaces determined by $x_{n+1} = u(\bar{x})$ in $\mathbb{R}^{n+1}$ with a topological graph $G\left(\bar{S}\right)$. By a geometrical view, such a system $(PDES^D_m)$ is solvable if

$$\bigcap_{i=1}^m S(\varphi_i, n) \neq \emptyset.$$ 

Otherwise, non-solvable in classical meaning.

However, $\bar{S}$ and $\bar{\Phi}$ are both $n$-manifold in $\mathbb{R}^n$ by Theorem 4.1 and there is a bijection $\tau : G\left(\bar{\Phi}\right) \rightarrow G\left(\bar{S}\right)$. Thus, it is also meaningful for knowing the solution behaviors of Laplace equation by generalizing solution in classical meaning to $G^L$-solutions of systems $(PDES^D_m)$ following.

**Definition 5.1** A $G^L$-solution of system $(PDES^D_m)$ is such an $n$-manifold $\bar{S}$ consisting of hypersurfaces $S(\varphi_1, n), S(\varphi_2, n), \ldots, S(\varphi_m, n)$ underlying a topological graph $G\left(\bar{S}\right)$ in $\mathbb{R}^{n+1}$.

Then, all of the previous discussions implies an interesting conclusion following.
**Theorem 5.2** Let $\tilde{\Phi}$ be a $G\langle \tilde{\Phi} \rangle$-system consisting of continuous functions $\varphi_1(\mathbf{x}), \varphi_2(\mathbf{x}), \ldots, \varphi_m(\mathbf{x})$ for $\|\mathbf{x}\| = a$. Then the system $(PDES^D_m)$ of the Dirichlet problem on Laplace equation prescribed with boundary value $\tilde{\Phi}$ is $G^L$-solvable.

Particularly, if $n = 2$ with $\bigcap_{i=1}^{m} \varphi_i(\theta) \neq \emptyset$, we get a surface consisting of $m$ surfaces $S(\varphi_1, n), S(\varphi_2, n), \ldots, S(\varphi_m, n)$ in $\mathbb{R}^3$ with genus determined by Theorem 4.2.

**References**


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