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On Hilbert-Schmidt n-Tuples

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Abstract

In this paper we study the Hilbert-Schmidt Tuples of operators on a Banach space.

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1 Introduction

Let $T_1, T_2, ..., T_n$ be commutative bounded linear operators on a Hilbert space $\mathcal{H}$, and $\mathcal{T} = (T_1, T_2, ..., T_n)$ be an n-Tuple, put

$$\Gamma = \{T_1^{m_1} T_2^{m_2} ... T_n^{m_n} : m_1, m_2, ..., m_n \geq 0\}$$

the semigroup generated by $\mathcal{T}$. For $x \in \mathcal{X}$, the orbit of x under $\mathcal{T}$ is the set $Orb(\mathcal{T}, x) = \{S(x) : S \in \Gamma\}$, that is

$$Orb(\mathcal{T}, x) = \{T_1^{m_1} T_2^{m_2} ... T_n^{m_n}(x) : m_1, m_2, ..., m_n \geq 0\}$$

The vector x is called Hypercyclic vector for $\mathcal{T}$ and n-Tuple $\mathcal{T}$ is called Hypercyclic n-Tuple, if the set $Orb(\mathcal{T}, x)$ is dense in $\mathcal{X}$, that is

$$\overline{Orb(\mathcal{T}, x)} = \overline{\{T_1^{m_1} T_2^{m_2} ... T_2^{m_n}(x) : m_1, m_2, ..., m_n \geq 0\}} = \mathcal{X}$$

Suppose above assumptions, also let $\{a_i\}_{i=1}^{+\infty}$ and $\{b_j\}_{j=1}^{+\infty}$ be orthonormal basises in a Hilbert space $\mathcal{H}$. The Tuple $\mathcal{T}$ is said to be Hilbert-Schmidt Tuple, if we have

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mid (T_1T_2...T_na_i, b_j) \mid^2 < \infty$$

All operators in this paper are commutative operator, reader can see [1–9] for more information.

2 Main Results

Theorem 2.1.[The Hypercyclicity Criterion] Let $\mathcal{B}$ be a separable Banach space and $\mathcal{T} = (T_1, T_2, ..., T_n)$ is an n-tuple of continuous linear mappings on $\mathcal{B}$. If there exist two dense subsets $\mathcal{Y}$ and $\mathcal{Z}$ in $\mathcal{B}$, and strictly increasing sequences $\{m_{j,1}\}$, $\{m_{j,2}\}$, ...,$\{m_{j,n}\}$ such that:
1. $T_1^{m_{j,1}}T_2^{m_{j,2}}...T_n^{m_{j,n}} \to 0$ on $\mathcal{Y}$ as $m_{j,i} \to \infty$ for $i = 1, 2, 3, ..., n$,
2. There exist function $\{S_k : Z \to \mathcal{B}\}$ such that for every $z \in Z$, $S_kz \to 0$, and $T_1^{m_{j,1}}T_2^{m_{j,2}}...T_n^{m_{j,n}}S_kz \to \ z$,
then $\mathcal{T}$ is a Hypercyclic n-tuple.
If the tuple T satisfying the hypothesis of previous theorem then we say that $\mathcal{T}$ satisfying the hypothesis of Hypercyclicity criterion.

Theorem 2.2. Suppose $\mathcal{X}$ be an F-sequence space whit the unconditional basis $\{e_\kappa\}_{\kappa\in\mathcal{N}}$. Let $T_1, T_2, ..., T_n$ are unilateral weighted backward shifts with weight sequence $\{a_{i,1} : i \in \mathcal{N}\}$, $\{a_{i,2} : i \in \mathcal{N}\}$,..., $\{a_{i,n} : i \in \mathcal{N}\}$ and $\mathcal{T} = (T_1, T_2, ..., T_n)$ be an n-tuple of operators $T_1, T_2, ..., T_n$. Then the following assertions are equivalent:
(1). $\mathcal{T}$ is chaotic,
(2). $\mathcal{T}$ is Hypercyclic and has a non-trivial periodic point,
(3). $\mathcal{T}$ has a non-trivial periodic point,
(4). the series $\Sigma_{m=1}^{\infty}(\prod_{k=1}^{m}(a_{k,i})^{-1}e_m)$ convergence in $\mathcal{X}$ for $i = 1, 2, ..., n$.

Proof. Proof of the cases (1) → (2) and (2) → (3) are trivial, so we just proof (3) → (4) and (4) → (1). First we proof (3) → (4), for this, Suppose that $\mathcal{T}$ has a non-trivial periodic point, and $x = \{x_n\} \in \mathcal{X}$ be a non-trivial periodic point for $\mathcal{T}$, that is there are $\mu_1, \mu_2, ..., \mu_n \in \mathcal{N}$ such that, $T_1^{\mu_1}T_2^{\mu_2}...T_n^{\mu_n}(x) = x$. Comparing the entries at positions $i + kM_\lambda$ for $\lambda = 1, 2, 3, ..., n$, $k \in \mathcal{N} \cup \{0\}$, of x and $T_1^{\mu_1}T_2^{\mu_2}...T_n^{\mu_n}(x)$ we find that

$$x_{j+kM_\lambda} = (\prod_{t=1}^{M_\lambda}(a_{j+kN+t}))x_{j+(k+1)}, \lambda = 1, 2, 3, ..., n$$

so that we have,

$$x_{j+kM_\lambda} = (\prod_{t=j+1}^{j+kM_\lambda}(a_t))^{-1}x_j = c_\lambda(\prod_{t=1}^{j+kM_\lambda}(a_t))^{-1}, k \in \mathcal{N} \cup \{0\}, \lambda = 1,2,3,...,n$$

with

$$c_\lambda = (\prod_{t=1}^{j}(m_{j,\lambda}))x_j, \lambda = 1,2,3,...,n$$

Since $\{e_\kappa\}$ is an unconditional basis and $x \in \mathcal{X}$ it follows that

$$\sum_{k=0}^{\infty}(\frac{1}{\prod_{t=1}^{j+kM_\lambda}(m_{j,\lambda})})e_{j+kM_\lambda} = \frac{1}{c_\lambda}\sum_{k=0}^{\infty}x_{j+kM_\lambda}.e_{j+kM_\lambda}, \lambda = 1,2,3,...,n$$

convergence in $\mathcal{X}$. Without loss of generality, assume that $j \geq N$, applying the operators $T_1^{\mu_{1,j}}T_2^{\mu_{2,j}}...T_n^{\mu_{n,j}}(x)$ for $j = 1,2,3,...Q-1$, with $Q = Min\{M_i : i = 1,2,...,n\}$, to this series and note that $T_1^{\mu_{1,j}}T_2^{\mu_{2,j}}...T_n^{\mu_{n,j}}(e_n) = a_j e_{n-1}$ for $n \geq 2$ and $j = 1,2,3,...,n$, we deduce that

$$\sum_{k=0}^{\infty}(\frac{1}{\prod_{t=1}^{j+kM_\lambda-\upsilon_\lambda}(m_{j,\lambda})})e_{j+kM_\lambda-\upsilon_\lambda}, \lambda = 1,2,3,...,n$$

convergence in $\mathcal{X}$ for $\gamma = 0,1,2,...,N-1$. By adding these series, we see that condition (4) holds.

Proof of (4) $\Rightarrow$ (1). It follows from theorem (2.1), that under condition (4) the operator $\mathcal{T}$ is Hypercyclic. Hence it remains to show that $\mathcal{T}$ has a dense set of periodic points. Since $\{e_\kappa\}$ is an unconditional basis, condition (4) with proposition 2.3 implies that for each $j \in \mathcal{N}$ and $M, N \in \mathcal{N}$ the series

$$\psi_1(j, M_\lambda) = \sum_{k=0}^{\infty}(\frac{1}{\prod_{t=1}^{j+kM_\lambda}(m_{k,\lambda})})e_{j+kM_\lambda} = (\prod_{t=1}^{j}m_{k,\lambda}).(\sum_{k=0}^{\infty}\frac{1}{\prod_{t=1}^{j+kM_\lambda}m_{k,\lambda}}e_{j+kM_\lambda})$$

converges for $\lambda = 1,2,3,...,n$ and define n elements in $\mathcal{X}$. Moreover, if $M \geq i$ then $T_1^{m_{j_\lambda,1}}T_2^{m_{j_\lambda,2}}...T_n^{m_{j_\lambda,n}} = \psi_\lambda(j_\lambda, M_\lambda)$ for $\lambda = 1,2,3,...,n$, and

$$T_1^{m_{j_\lambda,1}}T_2^{m_{j_\lambda,2}}...T_n^{m_{j_\lambda,n}} = \psi_\lambda(j_\lambda, M_\lambda)T_1^{M_1}T_2^{M_2}...T_n^{M_n}\psi_\lambda(j_\lambda, M_\lambda) = \omega(j_\lambda, M_\lambda) \quad (1)$$

Also, if $N \geq j_i$ then

$$T_1^{m_{j,1}}T_2^{m_{j,2}}...T_n^{m_{j,n}}\omega((j,i),N) = \omega((j,i),N) \quad (2)$$

for $m_{j,i} \geq N$ and $i = 1,2,...,n$. So that each $\psi(j,N)$ for $j \leq N$ is a periodic point for $\mathcal{T}$. We shall show that $\mathcal{T}$ has a dense set of periodic points. Since $\{e_\kappa\}$ is a basis, it suffices to show that for every element $x \in span\{e_\kappa : \kappa \in \mathcal{N}\}$

there is a periodic point y arbitrarily close to it. For this, let $x = \sum_{j=1}^{m} x_j e_j$ and $\varepsilon > 0$. Without lost of generality, for $\lambda = 1, 2, 3, ..., n$ we can assume that

$$| x_i \prod_{t=1}^{i} a_{t,\lambda} | \leq 1 \quad , \quad i = 1, 2, 3, ..., m_\lambda$$

Since $\{e_n\}$ is an unconditional basis, then condition (4) implies that there are an $M, N \geq m$ such that

$$\| \sum_{n=M_\lambda+1}^{\infty} \varepsilon_{\kappa,1} \frac{1}{\prod_{t=1}^{\kappa} a_{t,\lambda}} e_\kappa \| < \frac{\varepsilon}{m_\lambda}, \lambda = 1, 2, 3, ..., n$$

for every sequences $\{\varepsilon_{\kappa,i}\}$, $i = 1, 2, ..., n$ taking values 0 or 1. By (1) and (2) the elements

$$y_1 = \sum_{i=1}^{m_1} x_i \psi(i, M_\varphi), \varphi = 1, 2, 3, ..., n$$

of $\mathcal{X}$ is a periodic point for $\mathcal{T}$, and we have

$$\begin{aligned} \|y_\lambda - x\| &= \| \sum_{i=1}^{m_\lambda} x_i(\psi(i, M_\lambda) - e_i) \| \\ &= \| \sum_{i=1}^{m_\lambda} (x_i \prod_{t=1}^{i} d_{t,M_\lambda}) (\sum_{k=1}^{\infty} \frac{1}{\prod_{t=1}^{i+kM_\lambda} a_{t,M_\lambda}} e_{i+kM_\lambda}) \| \\ &\leq \sum_{i=1}^{m_\lambda} \|(x_i \prod_{t=1}^{i} d_{t,M_\lambda}) (\sum_{k=1}^{\infty} \frac{1}{\prod_{t=1}^{i+kM_\lambda} a_{t,M_\lambda}} e_{i+kM_\lambda}) \| \\ &\leq \sum_{i=1}^{m_\lambda} \|(\sum_{k=1}^{\infty} \frac{1}{\prod_{t=1}^{i+kM_\lambda} a_{t,M_\lambda}} e_{i+kM_\lambda}) \| \\ &\leq \epsilon \end{aligned}$$

as $\lambda = 1, 2, ..., n$ and by this, the proof is complete.

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