On Semi-invariant Submanifolds of a Nearly Sasakian Manifold with a Quarter Symmetric Non-metric Connection

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Abstract

We define a quarter symmetric non-metric connection in a nearly Sasakian manifold and we study semi-invariant submanifolds of a nearly Sasakian manifold endowed with a quarter symmetric non-metric connection. Moreover, we discuss the integrability of distributions on semi-invariant submanifolds.

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1. Introduction

In [1], A. Bejancu and N. Papaghiuc studied Semi-invariant submanifolds in Sasakian manifolds. The notion of nearly Sasakian manifold was introduced by
Blair et al. in [4]. CR-submanifolds of nearly Sasakian manifolds were studied by M. H. Shahid in [7]. In [8] M. Shahid studied properties of semi-invariant submanifolds of a nearly Sasakian manifold. Das et al. studied semi-invariant submanifolds of a nearly Sasakian manifold with a semi-symmetric non-metric connection in [5]. In this paper we study semi-invariant submanifolds of a nearly Sasakian manifold with a quarter symmetric non-metric connection.

Let $\nabla$ be a linear connection in an $n$-dimensional differentiable manifold $M$. The Tensor $T$ and the Curvature tensor $R$ of $\nabla$ are given respectively by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The connection $\nabla$ is symmetric if torsion tensor $T$ vanishes, otherwise it is non-symmetric. The connection $\nabla$ is metric connection if there is a Riemannian metric $g$ in $M$ such that $\nabla g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if it is the Levi-Civita connection.

In [6], S. Golab introduced the idea of quarter symmetric linear connection. A linear connection $\nabla$ is said to be quarter symmetric connection if its torsion tensor $T$ is of the form

$$T(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y$$

In [9], Mobin Ahmad et. al. studied some properties of hypersurfaces of an almost $r-$ paracontact Riemannian manifold with quarter symmetric metric connection.

The Paper is organized as follows: In section 2, we give a brief introduction of nearly Sasakian manifold. In section 3, we show that the induced connection on a semi-invariant submanifolds of a nearly Sasakian manifold with a quarter symmetric non-metric connection is also a quarter symmetric non-metric. In section 4, we established some lemmas on semi-invariant submanifolds and in the last section we discussed the integrability conditions of distributions of semi-invariant submanifolds of a nearly Sasakian manifold with quarter symmetric non-metric connection.

2. Preliminaries

Let $\bar{M}$ be $(2m + 1)$ - dimensional almost contact metric manifold [3] with a metric tensor $g$, a tensor field $\phi$ of type (1,1), a vector field $\xi$, a 1-form $\eta$ which satisfies

$$\phi^2 = -I + \eta \otimes \xi, \phi \xi = 0, \eta \phi = 0, \eta(\xi) = 1 (2.1)$$
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\[ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.2} \]

for any vector fields \( X , Y \) on \( \bar{M} \). If in addition to the condition for an almost contact metric structure we have \( d\eta(X, Y) = g(X, \phi Y) \), the structure is said to be a contact metric structure.

The almost contact metric manifold \( \bar{M} \) is called a nearly Sasakian manifold if it satisfies the condition [4]

\[ (\bar{\nabla}_X \phi)(Y) + (\bar{\nabla}_Y \phi)(X) = \eta(Y)X + \eta(X)Y - 2g(X, Y)\xi \tag{2.3} \]

where \( \bar{\nabla} \) denotes the Riemannian connection with respect to \( g \). If, moreover, \( M \) satisfies

\[ (\bar{\nabla}_X \phi)(Y) = -g(X, Y)\xi + \eta(Y)X, \bar{\nabla}_X \xi = \phi X \tag{2.4} \]

then it is called Sasakian manifold [3]. Thus every Sasakian manifold is nearly-Sasakian. The converse statement fails in general [4].

Definition : [2] An \( n \)-dimensional Riemannian submanifold \( M \) of a nearly Sasakian manifold \( \bar{M} \) is called a semi-invariant submanifold if \( \xi \) is tangent to \( M \) and there exists on \( M \) a pair of orthogonal distribution \( (D, D^\perp) \) such that

1. \( TM = D \oplus D^\perp \oplus \{\xi\} \)
2. the distribution \( D \) is invariant under \( \phi \), that is \( \phi D_x = D_x \), for all \( x \in M \),
3. the distribution \( D^\perp \) is anti-invariant under \( \phi \), that is \( \phi D^\perp_x \subset T^\perp_x M \), for all \( x \in M \), where \( T_x M \) and \( T^\perp_x M \) are the tangent space of \( M \) at \( x \).

The distribution \( D \) (resp. \( D^\perp \) ) is called the horizontal (resp. vertical distribution). A semi-invariant submanifold \( M \) is said to be an invariant (resp. anti-invariant) submanifold if we have \( D^\perp_x = \{0\} \) (resp. \( D_x = \{0\} \) ) for each \( X \in M \). We also call \( M \) proper if neither \( D \) nor \( D^\perp \) is null. It is easy to check that each hypersurface of \( M \) which is tangent to \( \xi \) inherits a structure of semi-invariant submanifold of \( \bar{M} \).

A quarter symmetric non-metric connection \( \bar{\nabla} \) is defined as

\[ \bar{\nabla}_X Y = \bar{\nabla}_X Y + \eta(Y)\phi X \tag{2.5} \]

such that \( (\bar{\nabla}_X g)(Y, Z) = -\eta(Y)g(\phi X, Z) - \eta(Z)g(\phi X, Y) \)

for any \( X, Y \in TM \), where \( \bar{\nabla} \) is induced connection on \( M \).

From (2.4) and (2.5), we have

\[ (\bar{\nabla}_X \phi)(Y) = -g(X, Y)\xi + 2\eta(Y)X - \eta(X)\eta(Y)\xi \tag{2.6} \]

\[ (\bar{\nabla}_X \phi)(Y) + (\bar{\nabla}_Y \phi)(X) = -2g(Y, X)\xi + 2\eta(Y)X \tag{2.7} \]

\[ + 2\eta(X)Y - 2\eta(X)\eta(Y)\xi \]
and
\[ \nabla_X \xi = 2\phi X \], (2.8)

We denote by \( g \) the metric tensor of \( \bar{M} \) as well as that induced on \( M \).

**Theorem 2.1.** The connection induced on semi-invariant submanifolds of a nearly-Sasakian manifold with quarter symmetric non-metric connection is also a quarter symmetric non-metric connection.

**Proof:** Let \( \nabla \) be the induced connection with respect to unit normal \( N \) on semi-invariant submanifolds of Sasakian manifold from quarter symmetric non-metric connection \( \bar{\nabla} \), then
\[ \bar{\nabla}_X Y = \nabla_X Y + m(X, Y), \quad (2.9) \]

where \( m \) is a tensor field of type \((0, 2)\) on semi-invariant submanifold \( M \). If \( \nabla^* \) be the induced connection on semi-invariant submanifolds from Riemannian connection \( \bar{\nabla} \), then
\[ \nabla^*_X Y = \nabla^*_X Y + h(X, Y), \quad (2.10) \]

where \( h \) is a second fundamental tensor satisfying
\[ h(\bar{X}, \bar{Y}) = h(\bar{Y}, \bar{X}) = g(H(\bar{X}), \bar{Y}) \], (2.11)

By the definition of quarter symmetric non-metric connection
\[ \nabla_X Y = \nabla^*_X Y + \eta(Y)\phi X \], (2.12)

Now using above equations, we have
\[ \nabla_X Y + m(X, Y) = \nabla^*_X Y + h(X, Y) + \eta(Y)\phi X \]

Equating tangential and normal components from both the sides, we get
\[ h(X, Y) = m(X, Y) \]

and
\[ \nabla_X Y = \nabla^*_X Y + \eta(Y)\phi X. \]

Thus \( \nabla \) is also a quarter symmetric non-metric connection.

Now, Gauss equation for a semi-invariant submanifolds of a nearly Sasakian manifold with a quarter symmetric non-metric connection is
\[ \bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \], (2.13)
and Weingarten formulas for $M$ is given by

$$\tilde{\nabla}_X N = -A_N X + a\phi X + \nabla^\perp_X N (2.14)$$

where $a = \eta(N)$ is a function on $M$, for $X,Y \in TM, N \in T^\perp M, h$ (resp. $A_N$) is the second fundamental form (resp. tensor) of $M$ in $\tilde{M}$ and $\nabla^\perp$ denotes the operator of the normal connection. Moreover, we have

$$g(h(X,Y), N) = g(A_N X,Y). (2.15)$$

For any vector $X$ tangent to $M$ is given as

$$X = PX + QX + \eta(X)\xi (2.16)$$

where $PX$ and $QX$ belong to the distribution $D$ and $D^\perp$ respectively.

For any vector field $N$ normal to $M$, we put

$$\phi N = BN + CN (2.17)$$

where $BN$ (resp. $CN$) denotes the tangential (resp. normal) component of $\phi N$.

Definition: A semi-invariant submanifold is said to be mixed totally geodesic if $h(X,Z) = 0$ for all $X \in D$ and $Z \in D^\perp$.

The Nijenhuis tensor $N(X,Y)$ for quarter symmetric non-metric connection is defined as

$$N(X,Y) = (\tilde{\nabla}_{\phi X} \phi)(Y) - (\tilde{\nabla}_{\phi Y} \phi)(X) - \phi(\tilde{\nabla}_X \phi)(Y) + \phi(\tilde{\nabla}_Y \phi)(X) (2.18)$$

for any $X,Y \in T\tilde{M}$.

From (2.7), we have

$$(\tilde{\nabla}_{\phi X} \phi)(Y) = -2g(\phi X,Y)\xi + 2\eta(Y)\phi X - (\tilde{\nabla}_Y \phi)\phi X (2.19)$$

Also,

$$(\tilde{\nabla}_Y \phi)\phi X = ((\tilde{\nabla}_Y \eta)(X))\xi + 2\eta(X)\phi Y - \phi(\tilde{\nabla}_Y \phi)X (2.20)$$

Now using (2.20) in (2.19), we have

$$(\tilde{\nabla}_{\phi X} \phi)(Y) = -2g(\phi X,Y)\xi + 2\eta(Y)\phi X - ((\tilde{\nabla}_Y \eta)(X))\xi (2.21)$$

$$-2\eta(X)\phi Y + \phi(\tilde{\nabla}_Y \phi)X$$

By virtue of (2.21) and (2.18), we get

$$N(X,Y) = -4\phi(\tilde{\nabla}_X \phi)Y - 4\phi h(X,\phi Y) - 4h(Y, X) (2.22)$$

$$+8\eta(Y)\phi X - 2g(\phi X,Y)\xi$$

for any $X,Y \in T\tilde{M}$. 
3. Basic Lemmas

Lemma 3.1. Let $M$ be a semi-invariant submanifold of a nearly Sasakian manifold $\bar{M}$ with quarter symmetric non-metric connection, then

$$2(\bar{\nabla}_X \phi)Y = \nabla_X \phi Y - \nabla_Y \phi X + h(X, \phi Y) - h(Y, \phi X) - \phi [X, Y] - 2g(X, Y)\xi$$

Proof: The proof of this lemma is similar to lemma 2.1 in [7].

Similar computations also yields

Lemma 3.2. Let $M$ be a semi-invariant submanifold of a nearly Sasakian manifold with quarter symmetric non-metric connection, then

$$2(\bar{\nabla}_X \phi)Y = -A_{\phi Y}X + \nabla_X^\perp \phi Y - \nabla_Y \phi X - h(Y, \phi X) - \phi [X, Y]$$

for any $X \in D$ and $Y \in D^\perp$.

Lemma 3.3. Let $M$ be a semi-invariant submanifold of a nearly Sasakian manifold $\bar{M}$ with quarter symmetric non-metric connection, then

$$P\nabla_X \phi PY + P\nabla_Y \phi PX - PA_{\phi QY}X - PA_{\phi QX}Y = 2\eta(Y)PX(3.1)$$

$$+2\eta(X)PY + \phi P\nabla_X Y + \phi P\nabla_Y X$$

$$Q\nabla_X \phi PY + Q\nabla_Y \phi PX - QA_{\phi QY}X - QA_{\phi QX}Y = 2\eta(Y)QX(3.2)$$

$$+2\eta(X)QY + 2Bh(X, Y)$$

$$h(X, \phi PY) + h(Y, \phi PX) + \nabla_X^\perp \phi QY + \nabla_Y^\perp \phi QX = 2Ch(X, Y)(3.3)$$

$$+\phi Q\nabla_X Y + \phi Q\nabla_Y X$$

$$\eta(\nabla_X \phi PY + \nabla_Y \phi PX - A_{\phi QY}X - A_{\phi QX}Y) = -2g(X, Y)\xi(3.4)$$

$$+2\eta(X)\eta(Y)\xi,$$

for all $X, Y \in TM$.

Proof: Differentiating (2.16) covariantly and using (2.13) and (2.14), we have

$$(\bar{\nabla}_X \phi)Y + \phi (\nabla_X Y) + \phi h(X, Y) = P\nabla_X (\phi PY) + Q\nabla_X (\phi PY)(3.5)$$

$$+\eta(\nabla_X \phi PY)\xi - PA_{\phi QY}X - QA_{\phi QY}X$$
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\[-\eta(A_{\phi QY}X)\xi + \nabla^\perp_X \phi QY + h(X, \phi PY).\]

Similarly,

\[
(\bar{\nabla}_Y \phi)X + \phi(\nabla_Y X) + \phi h(Y, X) = P\nabla_Y (\phi PX) + Q\nabla_Y (\phi PX) (3.6)
\]

\[+\eta(\nabla_Y \phi PX)\xi - PA_{\phi QX}Y - QA_{\phi QX}Y\]

\[-\eta(\phi QX Y)\xi + \nabla^\perp_Y \phi QX + h(Y, \phi PX)\]

Adding (3.5) and (3.6), we have

\[(\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi)X + \phi(\nabla_X Y + \nabla_Y X) + 2\phi h(Y, X) = P\nabla_X (\phi PY) (3.7)\]

\[+P\nabla_Y (\phi PX) + Q\nabla_Y (\phi PX) - PA_{\phi QY}X + QA_{\phi QY}X - QA_{\phi QY}X\]

\[-Q A_{\phi QX}Y + \nabla^\perp_X \phi QY - PA_{\phi QX}Y + \nabla^\perp_Y \phi QX + h(Y, \phi PX) + h(X, \phi PY)\]

\[+\eta(\nabla_X \phi PY)\xi + \eta(\nabla_Y \phi PX)\xi - \eta(A_{\phi QY}X)\xi - \eta(A_{\phi QY}X)\xi\]

Now using (2.7) and (2.17) in above equation, we get

\[-2g(X, Y)\xi + 2\eta(Y)PX + 2\eta(Y)QX + 2\eta(X)PY + 2\eta(X)QY + \phi P\nabla_X Y (3.8)\]

\[+\phi Q\nabla_X Y + \phi P\nabla_Y X + \phi Q\nabla_Y X + 2Bh(Y, X) + 2Ch(Y, X)\]

\[+2\eta(X)\eta(Y)\xi = P\nabla_X (\phi PY) + P\nabla_Y (\phi PX) + Q\nabla_X (\phi PX)\]

\[-PA_{\phi QY}X + QA_{\phi QY}X - QA_{\phi QY}X\]

\[\nabla^\perp_X \phi QY - PA_{\phi QX}Y + \nabla^\perp_Y \phi QX + h(Y, \phi PX)\]

\[+h(X, \phi PY) + \eta(\nabla_X \phi PY)\xi + \eta(\nabla_Y \phi PX)\xi\]

\[-\eta(A_{\phi QX}Y)\xi - \eta(A_{\phi QY}X)\xi\]

Equations (3.1) to (3.4) follows by comparison of tangential, normal and vertical parts.
Definition: The horizontal distribution $D$ is said to be parallel with respect to the connection $\nabla$ on $M$ if $\nabla_X Y \in D$ for all vector fields $X, Y \in D$.

Proposition 3.4. Let $M$ be a semi-invariant submanifold of a nearly Sasakian manifold $\bar{M}$ with quarter symmetric non-metric connection. If the horizontal distribution $D$ is parallel then $h(X, \phi Y) = h(Y, \phi X)$, for all $X, Y \in D$.

Proof: After similar computations to proposition 2.4 in [7], proposition follows.

Lemma 3.5. Let $M$ be a semi-invariant submanifold of a nearly Sasakian manifold $\bar{M}$ with quarter symmetric non-metric connection, then $M$ is mixed totally geodesic if and only if $A_N X \in D$, for all $X \in D$.

Proof: The proof of the lemma is similar as in lemma 2.5 in [7].

Proposition 3.6. Let $M$ be a mixed totally geodesic semi-invariant submanifold of a nearly Sasakian manifold $\bar{M}$ with quarter symmetric non-metric connection. Then the normal section $N \in \phi D^\perp$ is $D$ parallel if and only if $\nabla_X \phi N \in D$, for all $X \in D$.

Proof: Let $N \in \phi D^\perp$, then from (2.7) $(\bar{\nabla}_X \phi)N = 0$ and by hypothesis we have $h(X, \phi N) = 0$.

$$\nabla_X (\phi N) = (\nabla_X \phi)(N) + \phi(\nabla_X N)$$

$$\nabla_X (\phi N) = \phi(\nabla_X N - A_N X)$$

But $A_N X \in D$ so $\nabla_X N = 0$ if and only if $\nabla_X \phi N \in D$, for all $X \in D$.

4. Integrability conditions for distributions

Theorem 4.1. Let $M$ be a semi-invariant submanifold of a nearly Sasakian manifold $\bar{M}$ with quarter symmetric non-metric connection, then the distribution $D \oplus \langle \xi \rangle$ is integrable if the following conditions are satisfied

$$S(X, Y) \in (D \oplus \langle \xi \rangle)(4.1)$$

$$h(X, \phi Y) = h(\phi X, Y)(4.2)$$

for $X, Y \in D \oplus \langle \xi \rangle$.

Proof: The torsion tensor $S(X, Y)$ of almost contact structure is given by

$$S(X, Y) = N(X, Y) + 2d\eta(X, Y)\xi,$$

where $N(X, Y)$ is Nijenhuis tensor.

$$S(X, Y) = [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + 2d\eta(X, Y)\xi(4.3)$$
Suppose that $D \oplus \langle \xi \rangle$ is integrable so for $X, Y \in D \oplus \langle \xi \rangle$, $N[X, Y] = 0$, then $S(X, Y) = 2d\eta(X, Y)\xi \in D \oplus \langle \xi \rangle$.

From (2.22) we get

$$N(X, Y) = 8\eta(Y)\phi X + 4\eta(\nabla_X Y)\xi - 4Bh(X, \phi Y) - 4Ch(X, \phi Y) \tag{4.4}$$

$$-4h(X, Y) - 4\phi \nabla_X \phi Y - 4\nabla_X Y - 2g(\phi X, Y)\xi$$

From (4.3) and (4.4), we get

$$\phi Q(\nabla_X \phi Y) + Ch(X, \phi Y) + h(X, Y) = 0$$

for all $X, Y \in D$. Replacing $Y$ by $\phi Z$ where $Z \in D$, we have

$$\phi Q(\nabla_{\phi Z} \phi Y) + Ch(\phi Z, \phi Y) + h(Y, \phi Z) = 0.$$ 

Interchanging $X$ and $Z$, we have

$$\phi Q(\nabla_{\phi Y} \phi Z) + Ch(\phi Y, \phi Z) + h(\phi Y, Z) = 0.$$ 

Subtracting above two equations, we have

$$\phi Q[\phi Y, \phi Z] + h(Z, \phi Y) - h(Y, \phi Z) = 0$$

from which the assertion follows.

**Lemma 4.2.** Let $M$ be a semi-invariant submanifold of a nearly-Sasakian manifold $\bar{M}$ with quarter symmetric non-metric connection, then

$$2(\nabla_Y \phi)Z = A_{\phi Y}Z - A_{\phi Z}Y - 2g(Y, Z)\xi + \nabla^\perp_Y \phi Z - \nabla^\perp_Z \phi Y - \phi[Y, Z].$$

**Proof:** Lemma follows after similar computations to lemma 3.2 in [7].

**Proposition 4.3.** Let $M$ be a semi-invariant submanifold of a nearly Sasakian manifold $\bar{M}$ with quarter symmetric non-metric connection, then

$$A_{\phi Y}Z - A_{\phi Z}Y = \frac{1}{3}\phi P[Y, Z] - \frac{2}{3}g(\phi Y, Z)$$

**Proof:** Let $Y, Z \in D^\perp$ and $X \in \chi(M)$ then from (2.14) and (2.16), we have

$$2g(A_{\phi Z}Y, X) = -g(\nabla_Y \phi X, Z) - g(\nabla_X \phi Y, Z) + g((\nabla_Y \phi)X + (\nabla_X \phi)Y, Z)$$

By use of (2.7) and $\eta(Y) = 0$ for $Y \in D^\perp$, we have

$$2g(A_{\phi Z}Y, X) = -g(\phi \nabla_Y Z, X) + g(A_{\phi Y}Z, X) + 2\eta(X)g(Y, Z).$$

Transvecting $X$ from both sides, we get

$$2A_{\phi Z}Y = -\phi \nabla_Y Z + A_{\phi Y}Z + 2g(Y, Z)\xi.$$
Interchanging $Y$ and $Z$,

$$2A_{\phi Y}Z = -\phi\nabla_Z Y + A_{\phi Z}Y + 2g(Z, Y)\xi.$$ 

Subtracting above two equations, we get

$$(A_{\phi Y}Z - A_{\phi Z}Y) = \frac{1}{3}\phi P[Y, Z](4.5),$$

where $[Y, Z]$ is the Lie bracket for $\bar{\nabla}$.

Theorem 4.4. Let $M$ be a semi-invariant submanifold of a nearly Sasakian manifold $\bar{M}$ with quarter symmetric non-metric connection. Then the distribution $D^\perp$ is integrable if and only if

$$A_{\phi Y}Z - A_{\phi Z}Y = \frac{2}{3}g(Y, \phi Z)$$

for all $Y, Z \in D^\perp$.

Proof: The proof is similar to theorem 3.4 in [7].

References


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