A New Hierarchy Related to Generalized Korteweg-de Vries Equations

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Abstract

It is shown that the compatibility condition derived from a pair of linear equations in two independent variables leads to a class of generalized Korteweg-de Vries equations as well as an associated hierarchy of related nonlinear equations. These kinds of equation can be integrated in the case in which the solutions have a translationally invariant structure. Some solutions are determined and presented as well.

Solitons have appeared as a class of solutions of nonlinear partial differential equations which have been used to model specific nonlinear phenomena such as a weakly anharmonic mass-spring chain, among others. Thus the study of such equations has been of importance as well recently [1]. The Korteweg-de Vries (KdV) and m-KdV equations are two examples of nonlinear equations which exhibit this type of physical behavior. Moreover, if the dependent function in the third derivative is raised to a power which is greater than one, the dispersion mechanism blocks an instantaneous spread of the wave front and a wave of constant span is obtained that propagates with a constant velocity. These types of solutions have been of considerable interest and will continue remain [2-4]. In a recent paper [5], the properties and solutions of a generalized KdV equation of the form

\[ u_t + \alpha (u^n)_x + \beta (u^n)_{xxx} = 0, \] (1)
where $a$ and $b$ are real nonzero constants has been studied. Here we intend to present some observations and results related to the integrability properties of (1) and to develop a related class of generalized KdV equation based on a compatibility condition of a particular linear system.

Let us begin by formulating a result which is related to the existence of a strong symmetry for (1) when $n = 1$ and the constants are set to minus one. Consider the evolution equation

$$u_t = K(x, t, u, u_x, u_{xx}, \cdots)$$  \hspace{1cm} (2)

which is also written $u_t = K(x, t, u)$ or even $u_t = K(u)$. Let $K'(u)$ be the linearized pert of the operator $K(u)$ defined by

$$K'(u)\sigma = K'\sigma = \frac{\partial}{\partial \epsilon}K(u + \epsilon \sigma)|_{\epsilon = 0}. \hspace{1cm} (3)$$

This is the Gateaux derivation of $K$ at $u$ in the direction $\sigma$. For any $\bar{u}(u, \epsilon)$, an orbit of $u$, its tangent vector at $u$ is given by

$$\frac{d\bar{u}}{d\epsilon}|_{\epsilon = 0} = \sigma(x, t, u). \hspace{1cm} (4)$$

This is the vector field corresponding to a one-parameter transformation group which acts on $M$, that is $\bar{u} = g_{\epsilon}u$. Developing $\bar{u}$ and $K(\bar{u})$ in powers of $\epsilon$, we have

$$\bar{u}(u, \epsilon) = u + \epsilon \frac{d\bar{u}}{d\epsilon}|_{\epsilon = 0} + \cdots,$$

$$K(\bar{u}) = K(u) + \epsilon K'(u)\frac{d\bar{u}}{d\epsilon}|_{\epsilon = 0} + \cdots = K(u) + \epsilon K'(u)\sigma + \cdots.$$

Comparing $u_t = K(u)$ with $\bar{u}_t = K(\bar{u})$, we obtain the condition satisfied by $\sigma(u)$,

$$\sigma_t = K'\sigma. \hspace{1cm} (4)$$

In this event, $\sigma(u)$ is called a symmetry of (1). Since $\sigma_t = \frac{\partial \sigma}{\partial t} + \sigma'K$, condition (4) can be represented as follows

$$\frac{\partial \sigma}{\partial t} = K'\sigma - \sigma'K. \hspace{1cm} (5)$$

In particular, when $\sigma$ does not include $t$ explicitly, (5) reduces to the form $K'\sigma - \sigma'K = 0$.

**Definition 1.** The operator $\Phi(x, t, u)$ is called a strong symmetry of the equation (2) if it transforms the symmetries of (2) to symmetries, or in other words, if $\sigma$ is a symmetry of (2), then $\Phi \sigma$ is also a symmetry of (2).

**Theorem 1.** $\Phi(x, t, u)$ is a strong symmetry of (1) if it satisfies

$$\frac{d\Phi}{dt} = K'\Phi - \Phi K'. \hspace{1cm} (6)$$
Proof: Assume that \( \sigma \) is a symmetry of (1), then
\[
\frac{d(\Phi\sigma)}{dt} = \frac{d\Phi}{dt} \sigma + \Phi \frac{d\sigma}{dt} = (K'\Phi - \Phi K')\sigma + \Phi K'\sigma = K'(\Phi\sigma).
\]
Thus \( \Phi\sigma \) is a symmetry of (1) since it satisfies Definition 1. Therefore \( \Phi \) is a strong symmetry of (2).

Consider the class of equations of the form (1)-(2) where \( K \) and the equation have the following form,
\[
K(x, t, u, u_x, u_{xx}, u_{xxx}) = u_{xxx} + mu_x^{m-1}u_x, \quad u_t = u_{xxx} + (u^m)_x. \tag{7}
\]
This is (1) in which \( \alpha \) and \( \beta \) are set equal to minus one. In this case, we can work out the Gateaux derivative
\[
\frac{dK}{d\epsilon} \bigg|_{\epsilon=0} = (D^3 + mu_x^{m-1}D + m(m-1)u_{m-2}u_x)\sigma \equiv K'\sigma.
\]
This implies that
\[
K' = D^3 + mu_x^{m-1}D + m(m-1)u_{m-2}u_x. \tag{8}
\]
It is common to define \( D = \partial_x \) in this context. If \( \Phi \) does not depend on \( t \), \( \Phi \) is a strong symmetry if
\[
\Phi[K] = K'\Phi - \Phi K'. \tag{9}
\]
The following theorem shows that a generalization of a strong symmetry for the classical KdV equation with the same derivative dimension or order does not generalize to (7) when \( m \) is not two.

**Theorem 2.** The operator
\[
\Phi = D^2 + au^{m-1} + bu^{m-2}u_xD^{-1}, \tag{10}
\]
is a strong symmetry of (7) only in the case in which \( m = 2 \), and only in this case.

**Proof:** In this case, \( \Phi \) in (10) does not depend on \( t \) explicitly, so we show that (9) cannot be satisfied when \( m \neq 2 \).

Evaluating the Gateaux derivative of \( \Phi \), the left-hand side of (9) is given by
\[
\Phi'[K] = a(m-1)u^{m-2}K + (b(m-2)u^{m-3}u_xK + bu^{m-2}K_x)D^{-1}
= a(m-1)u^{m-2}(u_{xxx} + mu^{m-1}u_x) + (b(m-2)u^{m-3}u_xu_{xxx} + bm(m-2)u^{2m-4}u_x^2
+bmu^{2m-3}u_{xxx} + bm(m-1)u^{2m-4}u_x^2)D^{-1}. \tag{11}
\]
It would suffice to show that the \( D^{-1} \) term cannot be matched in (9) by calculating both sides of (9) when \( m \) is different from two. Let \( g \) be an arbitrary function, then
\[
K'\Phi g = (D^3 + mu_x^{m-1}D + m(m-1)u_{m-2}u_x) \cdot (D^2 + au^{m-1} + bu^{m-2}u_xD^{-1})
\]
\[ = D^5g + aD^3(u^{m-1}g) + bD^3(u^{m-2}u_xD^{-1}g) \\
+ mu^{m-1}D^3g + amu^{m-2}D(u^{m-1}g) + bD^3(u^{m-2}u_xD^{-1}g) \\
+ mu^{m-1}D^3g + amu^{m-1}D(u^{m-1}g) + bm(m-1)u^{2m-4}u_x^2D^{-1}g. \]

Expanding the last terms in each line and then collecting all \(D^{-1}g\) terms together, we obtain the \(D^{-1}g\) term

\[
\left[ b(m-2)(m-3)(m-4)u^{m-5}u_x^4 + 6b(m-2)(m-3)u^{m-4}u_x^2u_{xx} + 3b(m-2)u^{m-3}u_{xx}^2 \\
+ 4b(m-2)u^{m-3}u_xu_{xxx} + bu^{m-2}u_{xxxx} + bm(m-3)u^{2m-4}u_x^2 \right]D^{-1}g.
\] (12)

Since the second term in (9), \(\Phi K'g\) does not contribute terms of the form \(D^{-1}g\) to the result, comparing (11) and (12), it is clear the coefficients of \(D^{-1}g\) on both sides match only when \(m = 2\).

In fact, when \(m = 2\), these results reduce to

\[
\Phi'[K] = a(u_{xxx} + 2uu_x) + (u_{xxxx} + 2uu_{xx} + 2u_x^2)D^{-1},
\] (13)

and

\[
(K' \cdot \Phi - \Phi \cdot K') = (a + 3b - 2)u_{xxx} + 2auu_x + (3a - 4)u_xD^2 \\
+ 3(a + b - 2)u_{xx}D + b(u_{xxxx} + 2uu_{xx} + 2u_x^2)D^{-1}.
\] (14)

Requiring that \(a + 3b - 2 = a\) and \(3a - 4 = 0\), we obtain \(a = 4/3\) and \(b = 2/3\), hence (13) agrees with (14). Therefore,

\[
\Phi = D^2 + \frac{4}{3}u + \frac{2}{3}u_xD^{-1}
\] (15)

is a strong symmetry for the case in which \(m = 2\).

This of course does not completely resolve the question as to whether there exists a strong symmetry to (7) of some form, with some order of the derivative. If a strong symmetry can be found, it may be possible to write down a Lax pair. There is often a relationship between a strong symmetry and a Lax pair, as the next Theorem demonstrates.

**Theorem 3.** The Lax equations for KdV equation (7) when \(m = 2\) can be written in the form

\[
\Phi \sigma = 4\lambda \sigma, \quad \sigma_t = K' \sigma,
\]

where \(\lambda\) is a parameter, and \(\Phi\) is the strong symmetry (15) of (7) when \(m = 2\).

Consider time-dependent constraints of the form

\[
\psi_{xx}(x, t) = -(\lambda + u(x, t))\psi, \quad \psi_t = A\psi + B\psi_x.
\] (16)

The first equation in (16) can be used on its own as a differential constraint to obtain solutions of the KdV equation [7]. It is known that if \(B\) is chosen so
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that all \( \lambda \) dependence is eliminated from the compatibility condition for (16), the classical KdV equation as well as an associated hierarchy of related KdV equations will result. Here we generalize the first constraint in an obvious way and obtain the corresponding nonlinear generalized KdV system. Dependence of the potential \( u(x, t) \) on time \( t \) causes the dependence of the eigenfunctions \( \psi(x, t) \) on \( t \) in spite of the conservation of the eigenvalues \( \lambda \). Thus, with regard to (16), it can be said that the spectrum of the Schrödinger equation in (16) does not change if the potential \( u(x, t) \) evolves according to the KdV type equation.

The first order system (16) with time dependence can be generalized by supposing that

\[
\psi_{xx} = -(\lambda + qu^m(x, t))\psi, \quad \psi_t = A\psi + B\psi_x, \quad (17)
\]

where \( q \) is a constant and \( \psi = \psi(x, t) \). Differentiating \( \psi_{xx} \) in (17) with respect to \( t \), we obtain

\[
\psi_{xxt} = -(\lambda + qu^m)(A\psi + B\psi_x) - qmu^{m-1}u_t\psi. \quad (18)
\]

Now differentiating \( \psi_t \) with respect to \( x \), we obtain

\[
\psi_{txx} = (A_{xx} - B_x(\lambda + qu^m) - Bqmu^{m-1}u_x)\psi - (A + B_x)(\lambda + qu^m)\psi
+ A_x\psi_x - B(\lambda + qu^m)\psi_x + (A_x + B_{xx})\psi_x. \quad (19)
\]

The compatibility condition requires that we equate (18) to (19), that is \( \psi_{xxt} = \psi_{txx} \), therefore

\[
-qmu^{m-1}u_t\psi = (A_{xx} - B_x(\lambda + qu^m) - Bqmu^{m-1}u_x)\psi - B_x(\lambda + qu^m)\psi + A_x\psi_x + (A_x + B_{xx})\psi_x.
= (A_{xx} - 2B_x(\lambda + qu^m) - qmBu^{m-1}u_x)\psi + (2A_x + B_{xx})\psi_x. \quad (20)
\]

The coefficient of \( \psi_x \) in (20) will vanish provided that we require

\[
A_x = -\frac{1}{2}B_{xx}. \quad (21)
\]

Equating coefficients of \( \psi \) on both sides of (20) and substituting (21), we obtain

\[
qmu^{m-1}u_t = \frac{1}{2}B_{xxx} + 2(\lambda + qu^m)B_x + qmu^{m-1}u_xB. \quad (22)
\]

Now as in the treatment of the classical KdV equation, \( B \) in (22) can be chosen in such a way that \( \lambda \) disappears from (22) entirely. To carry this out, we take \( B \) to have the form of a polynomial in \( \lambda \),

\[
B = b_n + b_{n-1}\lambda + b_{n-2}\lambda^2 + \cdots + b_0\lambda^n.
\]
Substituting \( B \) given by (23) into (22) and equating powers of \( \lambda \) on both sides of the resulting expression for \( \lambda^{n+1}, \lambda^n, \ldots, \lambda \), the following system or hierarchy of equations is obtained

\[
b_{0,x} = 0,
\]

\[
\frac{1}{2} b_{0.xxx} + 2qb_{0,x}u^m + 2b_{1,x} + mqb_0u^{m-1}u_x = 0,
\]

\[
\frac{1}{2} b_{1.xxx} + 2qb_{1,x}u^m + 2b_{2,x} + mqb_1u^{m-1}u_x = 0,
\]

\[
\vdots
\]

\[
\frac{1}{2} b_{n-1.xxx} + 2qb_{n-1,x}u^m + 2b_{n,x} + mqb_{n-1}u^{m-1}u_x = 0.
\]

The terms which do not multiply \( \lambda \) yield the equation

\[
mqu^{m-1}u_t = \frac{1}{2} b_{n.xxx} + 2qb_{n,x}u^m + mqu^{m-1}u_x b_n.
\]

(25)

The first equation of (24) requires that \( b_0 \) be constant. Using this fact in the next equation of the hierarchy (24) implies that \( b_1 \) satisfies

\[
b_{1,x} = -\frac{1}{2} q b_0 (u^m)_x.
\]

This can be integrated to give

\[
b_1 = -\frac{1}{2} q b_0 u^m.
\]

(26)

Putting \( n = 1 \) in (25) gives the equation for \( u_t \) to be

\[
mqu^{m-1}u_t = \frac{1}{2} b_{1.xxx} + 2qu^m b_{1,x} + qb_1 (u^m)_x.
\]

(27)

Replacing \( b_1 \) in (27) by its form in (26), the equation can be expressed entirely in terms of \( u \)

\[
u^{m-1}u_t = -\frac{b_0}{4m}(u^m)_{xxx} - \frac{3qb_0}{2m}u^m(u^m)_x.
\]

Of course, when \( n = 1 \), \( B \) terminates at order \( \lambda \), and so \( A \) and \( B \) are given by

\[
A = -\frac{1}{2} B_x = \frac{1}{4} q b_0 (u^m)_x, \quad B = b_1 + b_0 \lambda = b_0 \lambda - \frac{1}{2} q b_0 u^m.
\]

The results obtained in this regard can be summarized in the form of the following Theorem.
**Theorem 4.** The generalized KdV equation

\[ u_t = - \frac{b_0}{4m} u^{1-m} (u^m)_xxx - \frac{3q}{2m} b_0 u(u^m)_x, \tag{28} \]

is the compatibility condition for the following pair of linear equations

\[ \psi_{xx} = - (\lambda + qu^m) \psi, \quad \psi_t = \frac{1}{4} q b_0 (u^m)_x \psi + b_0 (\frac{q}{2} u^m) \psi_x, \tag{29} \]

where \( q \) and \( b_0 \) are real constants. When \( m = 1, q = 1/3 \) and \( b_0 = 4 \), equation (28) reduces to a form of the classical KdV equation,

\[ u_t + 2uu_x + u_{xxx} = 0. \]

Expanding out the derivatives \((u^m)_x\) and \((u^m)_{xxx}\) in (28), it can be expressed in the equivalent form

\[ u_t + \frac{b_0}{4}(m-1)(m-2)u^{-2}u^3_x + \frac{3b_0}{4}(m-1)u^{-1}u_x u_{xx} + \frac{3}{2} q b_0 u^m u_x + \frac{b_0}{4} u_{xxx} = 0. \tag{30} \]

An equation with a similar structure has been proposed in [8] except in (30) two of the coefficients depend directly on \( m \), and can be made to vanish by setting \( m = 1 \) or \( m = 2 \).

Equations (24) can be written in the form

\[ \frac{\partial b_n}{\partial x} = \frac{1}{4}(-\partial_x^3 - 4qu^m \partial_x - 2q(u^m)_x) b_{n-1} = \frac{1}{4} R b_{n-1}. \tag{31} \]

In (31), \( R \) can be thought of as a recursion operator. In the case in which \( b_0 \) is a constant, this can be integrated to obtain \( b_1 \) and \( b_2 \) and further \( b_n \) as follows

\[ b_1 = -\frac{1}{2} q b_0 u^m, \quad b_2 = \frac{1}{8} q b_0 (3qu^{2m} + (u^m)_{xx}). \]

In (25) we have obtained an infinite sequence of equations such that the first one is the generalized KdV equation. A non-recursive version of (31) can be introduced in terms of a suitably differentiable function \( g \) by taking

\[ g_{xxx} = -2q(u^m)_x g - 4(\lambda u^m + \lambda) g_x, \tag{32} \]

or equivalently,

\[ \lambda g_x = \frac{1}{4} R g. \]

The product for the basis functions after appropriate normalization could serve as a generating function for the \( b_n \). Analogous to the second equation in (17), we suppose the function \( g(x,t) \) satisfies the temporal evolution equation

\[ g_t = C g_x - C_x g. \tag{33} \]
This equation can be put in the form of the conservation law
\[
\frac{\partial}{\partial t} \left( \frac{1}{g} \right) - \frac{\partial}{\partial x} \left( \frac{C}{g} \right) = 0. \tag{34}
\]

**Theorem 5.** The compatibility condition for the equations (32) and (33), namely \((g_{xxx})_t = (g_t)_{xxx}\) is satisfied provided that \(u\) and \(C\) satisfy
\[
qmu^{m-1}u_t = \frac{1}{2}C_{xxx} + 2(qu^m + \lambda)C_x + q(u^m)_xC.
\]
In the case in which \(m = 1\), \(q = 1\) and \(C = -2u + 4\lambda\), the constraint above reduces to the classical KdV equation, \(u_t + 6uu_x + u_{xxx} = 0\).

**Proof:** By direct calculation of derivatives, the integrability condition takes the form
\[
(g_{xxx}) - (g_t)_{xxx} = 2(\frac{1}{2}C_{xxx} + 2(qu^m + \lambda)C_x + q(u^m)_xC - qmu^{m-1}u_t) g_x
\]
\[
+ \left( \frac{1}{2}C_{xxx} + 2(qu^m + \lambda)C_x + q(u^m)_xC - qmu^{m-1}u_t \right)_x g.
\]
Imposing the constraint forces this expression to vanish. Putting \(C = -2u + 4\lambda\) and \(q = m = 1\) in the constraint, the last claim follows.

Equation (28) can be reduced to a quadrature in the case in which the function \(u(x, t)\) has the translation invariant form
\[
u(x, t) = f(x - ct). \tag{35}\]
Substituting \(u\) of this form into (28), the differential equation takes the form
\[
c(f^m)' = \frac{b_0}{4}(f^m)'' + \frac{3}{4}qb_0(f^{2m})'.
\]
Differentiation is now with respect to \(\eta = x - ct\). Integrating this equation once, we obtain
\[
4cf^m = b_0(f^m)'' + 3qb_0(f^{2m}) + c_1. \tag{36}
\]
To make this clearer, set \(g = f^m\) in (36) and then multiply both sides of (36) by \(g'\) to give
\[
2c(g^2)' = \frac{1}{2}b_0((g')^2)' + qb_0(g^3)' + c_1g'.
\]
In this form, the equation can be integrated. Doing so and solving for \((g')^2\), we have
\[
(g')^2 = \frac{4c}{b_0}g^2 - 2qg^3 + C_1g + C_2.
\]
It is now clear that this form of the equation can be separated and integrated to give
\[ \int \frac{\epsilon \, dg}{\sqrt{\frac{4c}{b_0} g^2 - 2qg^3 + C_1 g + C_2}} = \int d\eta + K, \quad \epsilon = \pm 1, \] (37)
where \( K \) is an integration constant.

As an example in which the integral can be calculated in closed form, consider the case in which \( C_1 = C_2 = 0 \). This can be integrated to the end, and to see this, we put \( \mu = 4c/b_0 \) and \( \nu = 2q \) so the integral takes the form
\[ \int \frac{dg}{\sqrt{g^2(\mu - \nu g)}} = x - ct + K. \]
This integral can be evaluated in closed form, and we obtain the result
\[ -\frac{2}{\sqrt{\mu}} \tanh^{-1}\left(\frac{\sqrt{\mu - \nu g}}{\sqrt{\alpha}}\right) = x - ct + K. \]
This equation can be solved for \( g = f^{m-1} \) in terms of \( \eta = x - ct \). We obtain that
\[ u(x, t) = \left(\frac{\mu}{\nu \cosh^2\left(\frac{\sqrt{\mu}}{2}(x - ct + K)\right)}\right)^{\frac{1}{m-1}}. \] (38)
Of course, other classes of solutions will be generated when nonzero values are selected for the constants \( C_1 \) and \( C_2 \) in (37).

It has been shown that a pair of linear equations lead to a form of generalized KdV equation given in Theorem 4 as well as an associated hierarchy of linked equations. A further extension of this work would be to extend system (17) to other linear systems and obtain the related equation from the compatibility condition.

References


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