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# A Semi-symmetric Recurrent Metric Connection in a Generalised Co-symplectic Manifold

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**Abstract**

In the present paper, we have studied the properties of genralised co-symplectic manifold and quasi-Sasakian manifolds with respect to the the semi symmetric recurrent Metric Connection. We have also studied the killing condition and first class condition for genralised co-symplectic manifold.

**Keywords:** Semi-symmetric recurrent metric connection, Generalised co-symplectic manifold, Generalised quasi-Sasakian manifold.

## 1. Introduction

In 1924, Friedmann and Schouten [3] introduced the idea of semi-symmetric linear connection on a differentiable manifold. In 1930 H. A. Hayden [4] defined a semi-symmetric metric connection on a Riemannian manifold and this was further developed by K. Yano [8]. In 1976 Sharfuddin and Hussian [7], defined a semi-symmetric metric connection in an an almost contact manifold. In 1992 Agashe and Chafle [1], introduced a semi symmetric non-metric connection on a Riemannian manifold. Recurrent Metric Connection was introduced and studied by Y. Liang [5] in 1994. In 2001 UC. De and J. Sengupta [2] investigated the curvature tensor of an almost contact metric manifold that admit a type of semi-symmetric metric connection and studied the properties of curvature tensor, conformal curvature tensor and projective curvature tensor. In this paper we have studied Generalised co-symplectic manifold with special semi-symmetric recurrent metric connection and discuss its exientance in Generalised quasi-Sasakian manifold. In section 3 we establish the relation between the Riemannian connection and special semi-symmetric recurrent metric connection on Generalised co-symplectic manifold and in section 4 we establish the relation between the Riemannian connection and special semi-symmetric recurrent metric connection on quasi-Sasakian manifolds.

## 2. Preliminaries

An $n$ dimensional differentiable manifold $M_n$ is an almost contact manifold if it admits a tensor field $\phi$ of type (1, 1), a vector field $\xi$ and a 1−form $\eta$ satisfying for arbitrary vector field $X$

$$\overline{\overline{X}} = \eta(X)\xi - X \tag{2.1}$$

$$\overline{\xi} = 0 \tag{2.2}$$

where $\overline{X} = \phi X$

from equations (2.1) and (2.2) we have

$$\eta(\overline{X}) = 0 \tag{2.3}$$

and $\eta(\xi) = 1$ (2.4)

An almost contact manifold $M_n$ in which a Riemannian metric tensor $g$ of type (0, 2) satisfying

$$g(\overline{X}, \overline{Y}) = g(X, Y) - \eta(X)\eta(Y)) \tag{2.5}$$

$$g(X, \xi) = \eta(X), \tag{2.6}$$

for arbitrary vector fields $X$ and $Y$ , is called an almost contact metric manifold.
Let us put

$$F(X, Y) = g(\overline{X}, Y) \tag{2.7}$$

then we have

$$F(\overline{X}, \overline{Y}) = F(X, Y) \tag{2.8}$$

$$F(X, Y) = g(\overline{X}, Y) = - g(X, \overline{Y}) = - F(Y, X) \tag{2.9}$$

An almost contact metric manifold satisfying

$$(D_X F)(Y,Z) = \eta(Y)(D_X\eta)(\overline{Z}) - \eta(Z)(D_X\eta)(\overline{Y}) \tag{2.10}$$

$$\begin{aligned}&(D_X F)(Y,Z) + (D_Y F)(Z,X) + (D_Z F)(X,Y) = \\ &\eta(Y)(D_X\eta)(\overline{Z}) - \eta(Z)(D_X\eta)(\overline{Y}) + \eta(Z)(D_Y\eta)(\overline{X}) - \\ &\eta(X)(D_Y\eta)(\overline{Z}) + \eta(X)(D_Z\eta)(\overline{Y}) - \eta(Y)(D_Z\eta)(\overline{X}) = 0\end{aligned} \tag{2.11}$$

for arbitrary vector fields *X*, *Y* , *Z*; are respectively called generalised cosymplectic and generalised quasi-Sasakian manifolds [6].
If on any almost contact manifold, $\xi$ satisfies

$$(D_X\eta)(\overline{Y}) = -\ (D_{\overline{X}}\eta)(Y) = (D_Y\ \eta)(\overline{X}) \tag{2.12}$$

$$(D_X\eta)(Y) = (D_{\overline{X}}\eta)(\overline{Y}) = -(D_Y\ \eta)(X) \tag{2.13}$$

and

$$(D_\xi\phi) = 0, \tag{2.14}$$

then ξ is said to be of the first class and the manifold is said to be of the first class [8].
If on an almost contact metric manifold *U* satisfies

$$(D_X\eta)(\overline{Y}) = \ (D_{\overline{X}}\eta)(Y) = (D_Y\ \eta)(\overline{X}) \Leftrightarrow \tag{2.15}$$

$$(D_X\eta)(Y) = \ -(D_{\overline{X}}\eta)(\overline{Y}) = -(D_Y\ \eta)(X) \tag{2.16}$$

and

$$(D_\xi\phi) = 0, \tag{2.17}$$

then $\xi$ is said to be of the second class and the manifold is said to be second class [8].
The Nijenhuis tensor in generalised co-symplectic manifold is given by

$$N(X, Y) = (D_{\overline{X}}\phi)(Y) - (D_{\overline{Y}}\ \phi)(X) - \overline{(D_X\phi)(Y)} + \overline{(D_Y\ \phi)(X)} \tag{2.18}$$

$$N(X, Y, Z) = (D_{\overline{X}}F)(Y, Z) - (D_{\overline{Y}}F)(X, Z) + (D_X F)(Y, \overline{Z}) - (D_Y F)(X, \overline{Z}). \tag{2.19}$$

## 3. Semi-symmetric Recurrent metric connection

Let *D* be a Riemannian connection, then an affine connection **B** defined by

$$B_X Y = D_X Y - \eta(X,Y) \tag{3.1}$$

whose torsion tensor *S* of *B* is given by

$$S(X,Y) = \eta(Y)X - \eta(X)Y \tag{3.2}$$

and metric tensor *g* satisfies

$$(B_X g)(Y,Z) = 2\eta(X)g(Y,Z) \tag{3.3}$$

for arbitrary vector fields *X*, *Y* , *Z*; then B is called a semi-symmetric metric connection .
If we put

$$B_X Y = D_X Y + P(X,Y) \tag{3.4}$$

where *P* is a tensor field of type (1, 2), then we have

$$P(X, Y) = -\eta(X)Y \tag{3.5}$$

$$P(X, Y,Z) = -\eta(X)g(Y,Z), \tag{3.6}$$

$$S(X, Y,Z) = \eta(Y)g(X,Z) - \eta(X)g(Y,Z) \tag{3.7}$$

and
$$(B_X\eta)Y = (D_X\eta)Y + \eta(X)\eta(Y) \tag{3.8}$$

where
$$P(X, Y,Z) \stackrel{\text{def}}{=} g(P(X, Y), Z)$$
$$S(X, Y,Z) \stackrel{\text{def}}{=} g(S(X, Y), Z).$$

from equations (2.3),(2.4),(3.2) and (3.5) the following results are obvious.

(3.9)
$$\begin{aligned}
&(i) P(\overline{X},Y) = P(\overline{Y},X) = P(\overline{X},\overline{Y}) = 0\\
&(ii) S(\overline{X},\overline{Y}) = 0\\
&(iii) S(\overline{X},Y) = -P(Y,\overline{X})\\
&(iv) S(X,\overline{Y}) = P(X,\overline{Y})\\
&(v) S(\overline{X},\xi) = -P(\xi,\overline{X})\\
&(vi) S(\xi,\overline{Y}) = P(\xi,\overline{Y})\\
&(vii) S(\overline{X},\xi) + S(\xi,\overline{Y}) = P(\xi,\overline{Y}) - P(\xi,\overline{X})\\
&(viii) S(X,Y) = -S(Y,X)\\
&(ix) P(X,Y) = P(Y,X) \text{ iff } \eta(X)Y = \eta(Y)X
\end{aligned}$$

(3.10)
$$\begin{aligned}
&(i) P(\overline{\overline{X}},Y) = P(\overline{\overline{Y}},X) = P(\overline{\overline{X}},\overline{\overline{Y}}) = 0\\
&(ii) S(\overline{\overline{X}},Y) = S(\overline{\overline{Y}},X) = S(\overline{\overline{X}},\overline{\overline{Y}}) = 0\\
&(iii) S(\overline{\overline{X}},\xi) = -S(\xi,\overline{\overline{X}}) = -P(\xi,\overline{X})\\
&(iv) P(X,\overline{\overline{Y}}) - P(Y,\overline{\overline{X}}) = P(X,Y) - P(Y,X)\\
&(v) S(\overline{\overline{X}},Y) = -P(Y,\overline{\overline{X}}) = -S(Y,\overline{\overline{X}})\\
&(vi) S(X,\overline{\overline{Y}}) = P(X,\overline{\overline{Y}}) = -S(\overline{\overline{Y}},X)
\end{aligned}$$

(3.11)
$$\begin{aligned}
&(i) S(X,Y,Z) = P(X,Y,Z) - P(Y,X,Z)\\
&(ii) P(\overline{\overline{X}},Y,Z) = P(\overline{\overline{X}},\overline{\overline{Y}},Z) = P(\overline{\overline{X}},\overline{\overline{Y}},\overline{\overline{Z}}) = 0\\
&(iii) P(X,\overline{\overline{Y}},Z) = P(X,Y,\overline{\overline{Z}}) = S(X,\overline{\overline{Y}},Z)\\
&(iv) S(\overline{\overline{X}},\overline{\overline{Y}},Z) = S(\overline{\overline{X}},\overline{\overline{Y}},\overline{\overline{Z}}) = 0\\
&(v) S(X,Y,\overline{\overline{Z}}) = S(X,\overline{\overline{Y}},Z) - S(\overline{\overline{X}},Y,Z)
\end{aligned}$$

**Theorem 3.1** An almost contact metric manifold with semi-symmetric recurrent metric connection B satisfies the relation

$$S(\overline{X},Y,\overline{Z})+S(Y,\overline{Z},\overline{X})=P(Y,Z,X)-P(Y,X,Z) \tag{3.12}$$

**Proof** Barring *X* and *Z* in equation (3.7) and using equations (2.3) and (2.5) we have

$$S(\overline{X},Y,\overline{Z})=\eta(Y)g(X,Z)-\eta(X)\eta(Y)\eta(Z) \tag{3.13}$$

again from equation (3.7) and using equations (2.3) and (2.5)

$$S(Y,\overline{Z},\overline{X})=-\eta(Y)g(Z,X)+\eta(X)\eta(Y)\eta(Z) \tag{3.14}$$

In consequence of equations (3.13), (3.14) and (3.6) we obtain (3.12).

**Theorem 3.2** A generalised co-symplectic manifold with semi-symmetric non-metric connection B satisfies the relations

$$\begin{aligned}&(i)(B_X F)(Y,\overline{Z})=\eta(Y)(B_X\eta)\overline{\overline{Z}}-2P(X,\overline{Y},\overline{Z})\\&(ii)(B_X F)(\overline{Y},Z)=-\eta(Z)(B_X\eta)\overline{\overline{Y}}-2P(X,\overline{\overline{Y}},Z)\end{aligned} \tag{3.15}$$

**Proof :** We have

$$B_X\left(F(Y,Z)\right)=(B_X F)(Y,Z)+F(B_X Y,Z)+F(Y,B_X Z) \tag{3.16}$$

$$(B_X F)(Y,Z)=B_X\left(F(Y,Z)\right)-F(B_X Y,Z)-F(Y,B_X Z) \tag{3.17}$$

using equation (3.1) in above equation we obtain

$$(B_X F)(Y,Z)=(D_X F)(Y,Z)+2\eta(X)F(Y,Z) \tag{3.18}$$

Barring Z in (3.18) we obtain

$$(B_X F)(Y,\overline{Z})=(D_X F)(Y,\overline{Z})+2\eta(X)F(Y,\overline{Z}) \tag{3.19}$$

In consequence of equations (2.10) , (2.3) ,(2.7)and (3.6) in equation (3.19) result (3.15)(i) follows.

Barring Y in equation (3.18) and in consequence of equations (2.10) , (2.3) ,(2.7)and (3.6) result (3.14)(ii) follows.

**Theorem 3.3:** An gernalised co-sympletic manifold $M_n$ admits semi-symmetric non- metric recurrent connection **B** , is such that $B_X F=0$, then F is killing iff

$$\eta(X)F(Y,Z)+\eta(Y)F(X,Z)=0$$

**Proof:** from equation (3.18) we have

$$(D_X F)(Y,Z)=(B_X F)(Y,Z)-2\eta(X)F(Y,Z) \tag{3.20}$$

similarly, $(D_Y F)(X,Z)=(B_Y F)(X,Z)-2\eta(Y)F(X,Z)$ (3.21)

Adding equations (3.20) and (3.21) we get

$$(D_X F)(Y,Z)+(D_Y F)(X,Z)=-2\eta(X)F(Y,Z)-2\eta(Y)F(X,Z)$$

which proves the statement

**Theorem 3.4** If ξ is killing on generalised co-symplectic manifold with semi-symmetric recurrent metric connection B, then

$$N^*(X,Y,Z) = (B_X F)(Y,\overline{Z}) + (B_Y F)(Z,\overline{X}) + (B_{\overline{Z}} F)(X,Y) - 2\eta(Z) D_{\overline{X}}\eta(\overline{Y}) \quad (3.22)$$

**Proof:** The nijenhuis tensor of a gernalised co-sympletic manifold with respect to semi symmetric recurrent metric connection B is given by

$$N^*(X,Y,Z) = (B_{\overline{X}} F)(Y,Z) - (B_{\overline{Y}} F)(X,Z) + (B_X F)(Y,\overline{Z}) - (B_Y F)(X,\overline{Z}) \quad (3.23)$$

from equation (3.23) we have

$$\begin{aligned} N^*(X,Y,Z) - (B_X F)(Y,\overline{Z}) - (B_Y F)(Z,\overline{X}) - (B_{\overline{Z}} F)(X,Y) &= (B_{\overline{X}} F)(Y,Z) - (B_{\overline{Y}} F)(X,Z) - (B_{\overline{Z}} F)(X,Y) \\ &= \eta(Y)\left[(D_{\overline{X}}\eta)\overline{Z} + (D_{\overline{Z}}\eta)\overline{X}\right] - \eta(X)\left[(D_{\overline{Y}}\eta)\overline{Z} + (D_{\overline{Z}}\eta)\overline{Y}\right] \\ &\quad - \eta(Z)\left[(D_{\overline{X}}\eta)\overline{Y} + (D_{\overline{Y}}\eta)\overline{X}\right] \end{aligned} \quad (3.24)$$

Since $\xi$ is killing then putting $(D_X\eta)(Y) + (D_Y\eta)(X) = 0$ in (3.24) equation,we obtain (3.22).

**Theorem 3.5** If the generalised co-symplectic manifold is of first class with respect to the Riemannian connection D, then it is also first class with respect to the semi-symmetric recurrent metric connection B.

**Proof :**Barring *X* and *Y* in equation (3.8) respectively and then using equation (2.3) we find

$$(D_{\overline{X}}\eta)Y = (B_{\overline{X}}\eta)Y \quad (3.26)$$

and

$$(D_X\eta)\overline{Y} = (B_X\eta)\overline{Y} \quad (3.27)$$

Adding equations (3.26) and (3.27), we obtain

$$(D_{\overline{X}}\eta)Y + (D_X\eta)\overline{Y} = (B_{\overline{X}}\eta)Y + (B_X\eta)\overline{Y} \quad (3.28)$$

In view ofequations (2.12) and (3.28), we get

$$(B_{\overline{X}}\eta)Y = -(B_X\eta)\overline{Y} \quad (3.29)$$

Again in similar way, we have

$$(B_X\eta)\overline{Y} = -(B_Y\eta)\overline{X} \quad (3.30)$$

From equations (3.29) and (3.30), we find

$$(B_Y\eta)\overline{X} = (B_X\eta)\overline{Y} = -(B_{\overline{X}}\eta)(Y) \quad (3.31)$$

Similarly we can prove

$$(B_X\eta)(Y) = (B_{\overline{X}}\eta)\overline{Y} = -(B_Y\eta)X \quad (3.32)$$

Taking covariant derivative of $\phi Y = \overline{Y}$ with respect to *B* and using equations (2.2), (2.3) and (3.1), we get

$$(B_X\phi)Y + \phi(D_X Y) = D_X\overline{Y} \quad (3.33)$$

Now using $(D_X\phi)(Y) + \phi(D_X Y) = D_X\overline{Y}$ in equation (3.33), we get

$$(B_X\phi)Y = (D_X\phi)Y \quad (3.34)$$

Replacing $X$ by $\xi$ in (3.34) and using (2.17) we get

$$(B_\xi \phi)Y = 0$$

Hence the theorem.

**Theorem 3.6** A generalised co-symplectic manifold equipped with semi-symmetric recurrentmetric connection B is completely integrable.

**Proof**: using (3.18) and (2.10) In equation (3.23), we have

$$\begin{aligned} N^*(X,Y,Z) &= \eta(Y)\left[(D_{\overline{X}}\eta)\overline{Z} + (D_{\overline{X}}\eta)\overline{\overline{Z}}\right] - \eta(X)\left[(D_Y\eta)\overline{Z} + (D_{\overline{Y}}\eta)\overline{\overline{Z}}\right] \\ &- \eta(Z)\left[(D_{\overline{X}}\eta)\overline{Y} - (D_{\overline{Y}}\eta)\overline{X}\right] + 2\eta(X)F(Y,\overline{Z}) + 2\eta(Y)F(X,\overline{Z}) \end{aligned} \quad (3.35)$$

barring X,Y and Z in equation (3.35) and using equation (2.3) we get

$$N^*(\overline{X},\overline{Y},\overline{Z}) = 0$$

Hence the theorem.

**Theorem 3.7** If $\phi$ is killing, then on generalised co-symplectic manifold with semi-symmetric recurrent metric connection B, we have

$$(B_X\eta)\overline{Z} + 2F(X,Z) = 0 \quad (3.36)$$

**Proof:** Since $\phi$ is killing, therefore

$$(B_X F)(Y,Z) + (B_Y F)(X,Z) = 0 \quad (3.37)$$

In consequence of equations (2.10), and (3.18) equation (3.37)becomes

$$\eta(Y)\left[(D_X\eta)\overline{Z} + 2F(X,Z)\right] + \eta(X)\left[(D_Y\eta)\overline{Z} + 2F(Y,Z)\right] - \eta(Z)\left[(D_X\eta)\overline{Y} + (D_Y\eta)\overline{X}\right] = 0 \quad (3.38)$$

using equation (3.8) in above equation

$$\eta(Y)\left[(B_X\eta)\overline{Z} + 2F(X,Z)\right] + \eta(X)\left[(B_Y\eta)\overline{Z} + 2F(Y,Z)\right] - \eta(Z)\left[(B_X\eta)\overline{Y} + (D_Y\eta)\overline{X}\right] = 0 \quad (3.39)$$

putting $\xi$ for $Y$ and using equations (2.2), (2.3) and (2.7), we obtain

$$(B_X\eta)\overline{Z} + 2F(X,Z) + \eta(X)(B_\xi\eta)\overline{Z} - \eta(Z)(B_\xi\eta)\overline{X} = 0 \quad (3.40)$$

Again putting $\xi$ for $X$ and using equations (2.2), (2.3) and (2.7) we get

$$(B_\xi\eta)\overline{Z} = 0$$

From (3.39) and (3.40), we get the result.

## 4. Semi-symmetric recurrent metric connection on quasi-Sasakian manifold

**Theorem 4.1** A quasi-Sasakian manifold is normal if and only if

$$(B_X F)(Y,Z) = \eta(Y)(B_Z\eta(\overline{X}) + \eta(Z)[(B_{\overline{X}}\eta)(Y) - 2\eta(X)g(\overline{X},Z) \quad (4.1)$$

where B being semi-symmetric recurrent metric connection.

**Proof:** The necessary and sufficient condition that a quasi-Sasakian manifold to be normal is[3]

$$(D_X F)(Y, Z) = \eta(Y)(D_Z \eta)(\bar{X}) + \eta(Z)(D_{\bar{X}} \eta)(Y) \qquad (4.2)$$

Using equation (4.2) in equation (3.18) we get (4.1)

**Theorem 4.2** Let D be the Riemannian connection and B be a semisymmetricnon-metric connection. Then an almost contact metric manifoldis a generalised quasi-Sasakian manifold of the first kind if

$$(B_X F)(Y, Z) + (B_Y F)(Z, X) + (B_Z F)(X, Y) - 2[\eta(X)\_F(Y, Z) + \eta(Y)F(Z, X) + \eta(Z)F(X, Y)] = 0$$

**Proof** From equation (3.18), we have

$$(D_X F)(Y, Z) = (B_X F)(Y, Z) - 2\eta(X)F(Y, Z) \qquad (4.3)$$

Taking covariant derivative of $\eta(\bar{Z}) = 0$ with respect to *D* and using equation (3.8), we obtain

$$(D_X \eta)(\bar{Z}) = (B_X \eta)(\bar{Z}) \qquad (4.4)$$

Using equations (3.31), (4.3) and (4.4) in equation (2.11), we get the required result.

**Theorem 4.3** A generalised co-symplectic manifold is quasi-Sasakian manifold if

$$(B_X F)(\xi, \ ) = (B_Z F)(\xi, X) \qquad (4.5)$$

where B being a semi-symmetric recurrent metric connection.

**Proof:** From equation (3.18), we have

$$\begin{aligned}&(D_X F)(Y, Z) + (D_Y F)(Z, X) + (D_Z F)(X, Y)\\ &= (B_X F)(Y, Z) + (B_Y F)(Z, X) + (B_Z F)(X, Y) - \\ &2\eta(X)F(Y, Z) - 2\eta(Y)F(Z, X) - 2\eta(Z)F(X, Y)\end{aligned} \qquad (4.6)$$

Using equation (2.10) in the above expression, we find

$$\begin{aligned}&(D_X F)(Y, Z) + (D_Y F)(Z, X) + (D_Z F)(X, Y)\\ &= \eta(Y)[(B_X F)(\xi, Z) - (B_Z F)(\xi, X)]\\ &+ \eta(Z)[(B_Y F)(\xi, X) - (B_X F)(\xi, Y)] +\\ &\eta(X)[(B_Z F)(\xi, Y) - (B_Y F)(\xi, Z)]\end{aligned} \qquad (4.7)$$

Since manifold is quasi-Sasakian, therefore

$$(D_X F)(Y, Z) + (D_Y F)(Z, X) + (D_Z F)(X, Y) = 0 \qquad (4.8)$$

From equations(4.7) and (4.8), we obtain (4.5).

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