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A Note on Continuities of the Poset of Turing Degrees[1]

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Abstract

In this note, continuities of the poset of Turing degrees are considered. Main results are: (1) The poset $\mathcal{D}$ of Turing degrees is an algebraic (and continuous) poset with a least element; (2) The poset $\mathcal{D}$ is not strongly continuous; (3) The directed completion of $\mathcal{D}$ is just the ideal completion of $\mathcal{D}$, and thus is an algebraic lattice; (4) $\mathcal{D}$ can be embedded into an algebraic lattice as an embedded base.

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1 Introduction

In mathematical logic, there is a branch of recursively (or computably) enumerable sets and degrees. For the poset $\mathcal{D}$ of Turing degrees and its subposet $\mathcal{R}$ of all r.e. degrees, there are many detailed studies on its logic aspects and

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order properties. The famous Sack's Density Theorem reflects the complexity of $\mathcal{D}$ in order aspects. The Lachlan-Lerman Theorem (see [4, Theorem IX-2.1] shows that any countable atomless Boolean algebra B can be embedded into $\mathcal{R}$, preserving sups, infs and least elements of $\mathcal{R}$.

Different with discusses before, this note concerns continuities of the poset $\mathcal{D}$ and densely embedding $\mathcal{D}$ to other special lattices. We will see that $\mathcal{D}$ is continuous and algebraic but not strongly continuous. We also see that though $\mathcal{D}$ itself is not a lattice, $\mathcal{D}$ can be densely embedded into some algebraic lattices.

2 Preliminaries

We recall some notions and basic results. Most of them come from [2] and [3].

Let $P = (P, \leq)$ be a poset. Then $(P, \geq)$ is also a poset and called the opposed poset (or dual poset) of $(P, \leq)$, denoted by P^{op}. In a poset P, a nonempty subset D of P is *directed* if x, $y \in D$ implies there exists $z \in D$ with $x \leq z$ and $y \leq z$. A nonempty subset F of P is *filtered* if it is directed in the dual poset P^{op}. A *dcpo* means a poset in which any directed subset has a supremum. An *ideal* means a lower set which is directed. A *filter* of P means an ideal of P^{op}. A *principal ideal* (*filter*) is a set of the form $\downarrow x = \{y \in P : y \leq x\}$ ($\uparrow y = \{x \in P : y \leq x\}$). The notation $\sup_x A$ denotes the supremum of A in the principal ideal $\downarrow x$.

Intuitively, we say that state x approximates state y if any computation of y yields the information of x at some finite stage. One of the important insights of the theory of "continuous posets" that has emerged in the last forty years is the following mathematical formalization.

Definition 2.1. (see [3]) Let P be a poset, x, $y \in P$. We say that x *approximates* y, written $x \ll y$, if whenever D is directed with $\sup D \geq y$, then $x \leq d$ for some $d \in D$. We use $\Downarrow x$ to denote the set $\{a \in P : a \ll x\}$. If for every element $x \in P$, the set $\Downarrow x := \{a \in P : a \ll x\}$ is directed and $\sup \Downarrow x = x$, then P is called a *continuous poset.* A *continuous poset* which is also a dcpo (resp., complete lattice) is called a *continuous domain* or briefly a *domain* (resp. *continuous lattice*). A poset P is said to be *algebraic* if every element of P is the directed supremum of compact elements.

A subset A of a poset P is *Scott-closed* if $\downarrow A = A$ and for any directed set $D \subseteq A$, $\sup D \in A$ if $\sup D$ exists. All the Scott-closed sets of P in the order of set-inclusion is denoted by $\sigma^*(P)$. It is easy to check that $\sigma^*(P)$ is a complete lattice, called the Scott-closed set lattice of P. The complements of the Scott-closed sets form a topology, called the *Scott topology*, denoted by $\sigma(P)$.

In a topological space, a set A is said to be irreducible if for any pair of closed sets F_1 and F_2 with $A \subseteq F_1 \cup F_2$, one always has that $A \subseteq F_1$ or $A \subseteq F_2$.

Definition 2.2. (see [6]) Let P be a poset and let $c(P)(\subseteq \sigma^*(P))$ be the set of all irreducible Scott-closed sets of P. Then in the set-inclusion order, $c(P)$ forms a dcpo, called the directed completion of P.

It is known (see [6]) that if P is a(n) continuous (resp: algebraic) poset, then $c(P)$ is a(n) continuous (resp: algebraic) dcpo.

Definition 2.3. (see [8]) Let P be a poset and $x, y \in P$. We write $x \ll_l y$ and say that x universally approximates y if, for any directed set D and any upper bound z of D such that $y \leq \sup_z D$, there is $d \in D$ such that $x \leq d$. The subscript l means x approximates y locally and in the large. The set $\{y \in P | x \ll_l y\}$ will be denoted $\Uparrow_l x$ and $\{y \in P | y \ll_l x\}$ denoted $\Downarrow_l x$.

Definition 2.4. (see [8]) Let P be a poset. If for all $x \in P$, $\Downarrow_l x$ is directed and $\sup \Downarrow_l x = x$, then we say that P is a strongly continuous poset, or shortly, an SC-poset.

Definition 2.5. (cf. [5]) The Scott topology on a poset P is called *lower hereditary* if, for every Scott-closed subset A, the relative Scott topology on A agrees with the Scott topology of the poset A.

Lemma 2.6. (see [5]) *Let P be a poset. The following statements are equivalent:*

(1) *P has a lower hereditary Scott topology;*

(2) *For all $x \in P$, the inclusion map from the poset $\downarrow x$ into P is Scott-continuous;*

(3) *For $z \in P$ and directed $D \subseteq \downarrow z$, $x = \sup_z D$ implies $x = \sup_P D$.*

It is known that every strongly continuous poset is a continuous one but not vice versa,. More precisely, we have

Lemma 2.7. (see [8]) *Let P be a poset. Then P is an SC-poset iff P is a continuous poset and has a lower hereditary Scott topology*

Definition 2.8. (see [2]) (a) A subset A of naturals is Turing reducible to the set B if A is B-recursive (equivalently, if the characteristic function c_A is B-computable). This is written $A \leq_T B$.

(b) The sets A, B are Turing equivalent if $A \leq_T B$ and $B \leq_T A$. We write this $A \equiv_T B$.

(c) Let A be a set of naturals. The equivalence class $d_T(A) = \{B : B \equiv_T A\}$ is called the Turing degree of A.

(d) The poset of all the Turing degrees with the partial ordering induced by the relation $\leq_T$ is denoted by $\mathcal{D}$, and letters $\alpha, \beta, \gamma, \ldots,$ are used for Turing degrees.

It is known [2, Theorem 9-5.10] that, $\mathcal{D}$ is a sup-semilattice but a lattice. And it is know that a sup-semilattice which is also a dcpo must be not a complete lattice. By this observation, we can infer that the poset $\mathcal{D}$ of Turing degrees is not a dcpo.

3 Main Results

In this section, we will give our main results on the poset $\mathcal{D}$ of Turing degrees.

Firstly, it is know form Wikipedia Encyclopedia under the item "Turing degrees" that for each degree α, the set of degrees below α is at most countable, and there is no infinite, strictly increasing sequence of degrees that has a least upper bound. So, we have the following

Proposition 3.1. (see [9]) *In the poset $\mathcal{D}$ of Turing degrees, every ideal either has a largest element or has no least upper bound.*

Proof. In terms of the above facts, it is a routing work by the contrary argument. □

Theorem 3.2. *In the poset $\mathcal{D}$ of Turing degrees, every element is compact and $\mathcal{D}$ is an algebraic poset. Especially, $\mathcal{D}$ is a continuous poset.*

Proof. By Proposition 3.1, if D in $\mathcal{D}$ is a directed subset with a supremum, than D has a largest element. Thus, every element in $\mathcal{D}$ is compact. This trivially follows that $\mathcal{D}$ is an algebraic poset. □

Definition 3.3. Let M be a proper ideal on a poset P. If there is a filter F s. t. M is maximal among the ideals which do not intersect F (for an ideal I on P, $I \cap F = \emptyset$ and $I \supseteq M$ imply $I = M$), then we say M is a maximal ideal relative to filter F on poset P, or roughly, a maximal ideal relative to filters.

Lemma 3.4. *Let P be a poset, I an ideal and F a filter of P with $I \cap F = \emptyset$. Then there always exists a maximal ideal M relative to filter F on P s.t. $M \cap F = \emptyset$ and $M \supseteq I$.*

Proof. Define $\mathcal{A} = \{J : J \text{ be an ideal of } P,\ J \cap F = \emptyset \text{ and } J \supseteq I\}$. By the assumption, we have $I \in \mathcal{A} \neq \emptyset$ and $\mathcal{A}$ is a poset in the set inclusion relation " $\subseteq$ ". Let $\mathcal{B}$ be a linear subset of $\mathcal{A}$. Let $K = \bigcup_{J \in \beta} J$. Claim that $K \in \mathcal{A}$. It is clear that K is a lower set. For $x, y \in K$, there are $J_1, J_2 \in \mathcal{B}$ s. t. $x \in J_1,\ y \in J_2$. Since $\mathcal{B}$ is a linear subset of $\mathcal{A}$, $J_1 \subseteq J_2$ or $J_2 \subseteq J_1$. Suppose that $J_1 \subseteq J_2$ without losing generality. Then $x \in J_2$ and there is $z \in J_2 \subseteq K$ s.t. $x, y \leq z$. This shows that K is directed and K is an ideal of P. Since $I \subseteq K$ and $K \cap F = \emptyset$, $K \in \mathcal{A}$. By the Zorn's Lemma, in $\mathcal{A}$ there is a maximal element $M \in K$. This M is indeed what we need a maximal ideal relative to filter F. □

Theorem 3.5. *If $\alpha \in \mathcal{D}$ is a limit degree, then there is an ideal I such that $I \cap {\uparrow}\alpha = \emptyset$ and $\sup_\alpha I = \alpha$.*

Proof. Let $P = ({\uparrow}\ \alpha \cup {\downarrow}\ \alpha) \cap \mathcal{D}$. Then P is a subposet of $\mathcal{D}$. By Lemma 3.4, there is in P a maximal ideal I relative to the filter ${\uparrow}\alpha$. It is easy to see that I is also an ideal of $\mathcal{D}$, $I \subseteq {\downarrow}\ \alpha$ and $I \cap {\uparrow}\alpha = \emptyset$. We need to show that $\sup_\alpha I = \alpha$. Firstly, α is an upper bound of I. Secondly, let degree β is another upper bound of I with $\beta \leq \alpha$. Suppose that $\beta < \alpha$. Then it follows from the assumption that there is a degree γ such that $\beta < \gamma < \alpha$. This contradicts to the maximality of I in P relative to the filter ${\uparrow}\alpha$. So, $\beta = \alpha$. This shows that $\sup_\alpha I = \alpha$. □

It is known by the Sack's Density theorem [2, Theorem 9-5.15] that the first jump degree $\mathbf{0}'$ is not a minimal cover of any degree which is strictly below it. This means that $\mathbf{0}'$ is a limit r.e. degree. So, we have the following

Theorem 3.6. *The poset $\mathcal{D}$ is not an SC-poset.*

Proof. For the limit degree $\alpha = \mathbf{0}'$, using Theorem 3.5, we can obtain an ideal I satisfying that $\sup_\alpha I = \alpha$ and $\alpha \notin I$. It follows from Proposition 3.1 that $\sup I$ does not exist. By Lemma 2.6(3), we see that the Scott topology on $\mathcal{D}$ is not lower hereditary. So, by Lemma 2.7, $\mathcal{D}$ is not strongly continuous, as desired. □

Proposition 3.7. *In a poset P with every element being compact, a set $F \subseteq P$ is an irreducible Scott-closed set iff F is an ideal. Particularly, $\mathcal{D}$ is such kind of poset and the irreducible Scott-closed sets of $\mathcal{D}$ are exactly the ideals of $\mathcal{D}$.*

Proof. Since every element of P is compact, any ideal in P either has a largest element or has no least upper bound. So, the Scott-closed sets in P are exactly the lower set of P. By this observation, the if part is thus clearly true. To show the only if part, let F be an irreducible Scott-closed set in P. Then F is a lower set. What we need to show is that F is directed. To this end, suppose there are two elements $x, y \in F$ with no common upper bound in F. Then $F - \uparrow x \neq F$ and $F - \uparrow y \neq F$ are Scott-closed sets and $(F - \uparrow x) \cup (F - \uparrow y) = F - (\uparrow x \cap \uparrow y) = F$, contradicting to the irreducibility of F. So, F is directed and is an ideal, as desired.

It follows from Theorem 3.2 that $\mathcal{D}$ is such kind of posets. □

The ideal completion of a poset P is defined to be $Idl(P) = \{I \subseteq P : I$ is an ideal in $P\}$ in the set-inclusion order. It is well-known that $Idl(P)$ is always an algebraic domain. Since $\mathcal{D}$ is a sup-semilattice with a lest element $\mathbf{0}$, it is easy to check that $Idl\mathcal{D}$ is an algebraic lattice.

Definition 3.8. (See [3] for case of dcpos.) Let P be a poset, $B \subseteq P$. The set B is called a basis for P if $\forall a \in P$, there is a directed set $D_a \subseteq B$ such that $\forall d \in D_a$, $d \ll_P a$ and $\sup_P D_a = a$, where the subscript P means to take relevant operations in the poset P.

It is well-known that a poset P is continuous iff it has a basis. To go further, we have

Definition 3.9. (see [7]) Let B and P be posets. If there is a map $j : B \to P$ satisfying

(1) j preserves existing directed sups,

(2) $j : B \to j(B)$ is an order isomorphism,

(3) $j(B)$ is a basis for P,

then (B, j) is called an embedded basis for P. If $B \subseteq P$ and (B, i) is an embedded basis for P, where i is the inclusion map, then we say also that B is an embedded basis for P.

It is easy to see that if $B \subseteq P$, then B is an embedded basis for P iff B is a basis for P and for every directed set $D \subseteq B$ with existing $\sup_B D$, one has $\sup_B D = \sup_P D$. If P has a basis B, then P is continuous. If B is an embedded basis for P, then B itself is also continuous.

Example 3.10. It is easy to see that the rationales $\mathbb{Q}$ is an embedded basis for the reals $\mathbb{R}$. So, $\mathbb{R}$ and $\mathbb{Q}$ are continuous posets. Actually, it is easy to check that every linear ordered set is a continuous poset.

Proposition 3.11. *If P is a poset satisfying each element is compact, then (P, j) is an embedded basis for $Idl(P)$, where $j : P \to Idl(P)$ is defined for all $x \in P$, $j(x) = \downarrow x \in Idl(P)$.*

Proof. It is easy to check that $j(P)$ is a basis for $Idl(P)$. Since every element in P is compact, it is easy to check that j is Scott-continuous and $j : P \to j(P) \subseteq Idl(P)$ is an order isomorphism. So, by Definition 3.9, the proposition holds. □

By Theorem 3.2 and Propositions 3.7 and 3.11, we immediately have the following

Theorem 3.12. *The directed completion $c(\mathcal{D})$ is just the ideal completion of $Idl(\mathcal{D})$, i.e., $c(\mathcal{D}) = Idl(\mathcal{D})$. And $c(\mathcal{D})$ is an algebraic lattice. Consequently, $\mathcal{D}$ can be embedded into the algebraic lattice as an embedded base. When the Scott topologies are concerned, the embedding $\mathcal{D}$ to its directed completion $c(\mathcal{D}) = Idl(\mathcal{D})$ is a densely embedding.*

The above theorem has also an inference that if one requires a domain which has $\mathcal{D}$ as an embedded base, then the domain must be the algebraic lattice $c(\mathcal{D}) = Idl(\mathcal{D})$ up to an isomorphism.

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