Generalised Strongly Prime Ideals in Near Rings

Om Prakash

Department of Mathematics
IIT Patna, Patliputra colony, Patna-800 013, India

Kalpana

Department of Mathematics and Statistics
Banasthali Vidyapith, Banasthali, Rajasthan-304022, India

Abstract

In 1984, Groenewald and Potgieter, generalised results for \( f \)-prime ideal in near rings. In this paper, we introduce the notion of strongly \( f \)-prime ideal in near rings and generalise some of basic results of strongly prime ideal of near ring for the same. We show every strongly \( f \)-prime ideal is strongly prime but the converse is not true.

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Keywords: \( f \)-nilpotent element, \( f \)-prime radical, \( f \)-nil ideal.

1 Introduction

Throughout this article, \( N \) denotes a zero symmetric right near ring and \( \mathbb{Z}_+ \) denotes the set of positive integers. For any subset \( X \) of \( N \), \( < X > \) denotes the smallest ideal containing \( X \). For preliminary definitions and results related to near rings, we refer Pilz [3]. Let \( N \) be a near ring. A subset \( H \) of \( N \) is said to be an \( m \)--system if, for every \( h_1, h_2 \in H \), there exists \( h_1^1 \in < h_1 > \) and \( h_2^1 \in < h_2 > \) such that \( h_1^1 h_2^1 \in H \). A subset \( H \) of \( N \) is said to be an \( f \)--system if \( H \) contains an \( m \)--system \( H^* \), called kernel of \( H \), such that for
every \( h \in H \), \( f(h) \cap H^* \neq \phi \). An ideal \( A \) of \( N \) is said to be \( f \)-prime if \( N \setminus A \) is an \( f \)-system. The \( f \)-prime radical of near ring \( N \) is defined as the intersection of all \( f \)-prime ideal of \( N \) and is denoted by \( f^-\text{rad}(N) \). An ideal \( f(a) \) is a uniquely determined ideal by an element \( a \in N \) which satisfies the following:

(i) \( a \in f(a) \) (ii) \( x \in f(a) + A \Rightarrow f(x) \subseteq f(a) + A \), for any ideal \( A \) of \( N \).

For any ideal \( I \) of a near ring \( N \), we extend the map \( f \) on \( N/I \) to the set of all ideals of \( N/I \) such that \( f(x+I) = (f(x)+I)/I, x \in N \). Then, this map satisfies the above mentioned two conditions (i) and (ii) for ideal \( f(a) \).

An element \( a \in N \) is said to be \( f \)-nilpotent if \( f(a) \) is nilpotent. We introduced the concept of 2-\( f \)-primal near ring in [2]. A near ring \( N \) is said to be 2-\( f \)-primal near ring if \( f^-\text{rad}(N) \) equal to the set of all \( f \)-nilpotent element of \( N \). An element \( a \in N \) is said to be a strongly nilpotent if for every sequence \( a_1, a_2, \ldots \) in \( N \) satisfying \( a_1 = a \) and \( a_i = a_{i-1}^*a_{i-1}^{**} \), for some \( a_{i-1}^*, a_{i-1}^{**} \in < a_{i-1} >, \) there exists an integer \( k \) such that \( a_t = 0, \) for \( t \geq k \). An element \( a \) of \( N \) is said to be an \( f \)-strongly nilpotent if every element of \( f(a) \) is strongly nilpotent. An ideal \( A \) of \( N \) is said to be strongly prime if \( A \) is prime and \( N/A \) has no non-zero nil ideals. Observe that every strongly prime is a prime ideal of \( N \).

**Lemma 1.1** \( f^-\text{rad}(N) = \{ x \in N | x \text{ is an } f \text{-strongly nilpotent element} \} \).

**Proof.** See proof of Theorem 2.2 in [4] ■

**Proposition 1.2** Every \( f \)-strongly nilpotent element is an \( f \)-nil element, but converse is not true.

**Proof.** Let \( x \) is an \( f \)-strongly nilpotent element. We show every element of \( f(x) \) is nilpotent. If possible, let \( x \) is not nilpotent \( \Rightarrow x^n \neq 0 \), for any \( n \in \mathbb{Z}_+ \). Let \( S = \{ x, x^2, \ldots \} \). Then \( S \) is an \( f \)-system with kernel \( S \) itself. By using Lemma [1.5 (a)] in [1], we have \( (0) \subseteq P \) such that \( P \cap S = \phi \). This implies, \( x \notin P \) i.e. \( x \notin \cap P = f^-\text{rad}(N) \) = set of all \( f \)-strongly nilpotent elements of \( N \). This implies, \( x \) is not an \( f \)-strongly nilpotent element, which is a contradiction. Hence the result. ■

But the converse is not true.

**Example 1.3** Let \( T \) be an ideal of near ring \( N \), which is nil but not nilpotent. Now, we define \( f(a) = < a, T > \), for all \( a \in N \). Then, for any \( t \in T \), \( f(t) = T \) is not nilpotent. This implies, \( t \) is an \( f \)-nil but not \( f \)-strongly nilpotent.

## 2 Strongly \( f \)-prime ideal

**Definition 2.1** An ideal \( I \) of the near ring \( N \) is said to be strongly \( f \)-prime if \( I \) is an \( f \)-prime ideal and \( N/I \) has no non-zero \( f \)-nil ideal. In another words, \( I \) is a strongly prime ideal if it is an \( f \)-prime and for any other ideal \( J \supset I \), we have \( J/I \) is not an \( f \)-nil.
Example 2.2 Consider the dihedral group $N = \{0, a, 2a, 3a, b, a+b, 2a+b, 3a+b\}$ with addition and multiplication given as in [scheme 84, [3]] defined as in the following table:

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There are 3 proper ideals of $N$, namely $A = \{0, 2a\}$, $A_1 = \{0, 2a, b, 2a+b\}$ and $A_2 = \{0, 2a, a+b, 3a+b\}$. Take $f(a) = \langle x, b \rangle$. Clearly, $A$ is an $f$-prime ideal of $N$ with kernel $A' = \{b, 2a+b\}$. Also, $A_1/A = \{0, 2a, b, 2a+b\}$ is not an $f$-nil ideal, since $b$ is not an $f$-nil element. Similarly, we can check that $A_2/A$ is not an $f$-nil ideal. So, $A$ is an $f$-strongly prime ideal of $N$.

Theorem 2.3 Let $M = \{\alpha, \alpha^2, \ldots\}$, where $\alpha$ is not a nilpotent element of near ring $N$. Then there exists a strongly $f$-prime ideal $P$ of $N$ such that $P \cap M = \phi$.

Proof. Let $M = \{\alpha, \alpha^2, \ldots\}$ and $S = \{I | I\}$ is an ideal of $N$ and $I \cap M = \phi$}. Then $S \neq \phi$, since $(0) \in S$. Now, by using Zorn’s Lemma, $\exists$ a maximal element say $P$, which turns out to be an $f$-prime ideal. Now, we show $N/P$ does not have any non zero $f$-nil ideal. If possible, $I/P$ is a non-zero $f$-nil ideal of $N/P$. Then, $P \subset I$ and so, $I \cap M \neq \phi$. This implies, any $x \in I = \alpha^m$, for some $m \in N$. Also, $x + P \in I/P$ is an $f$-nil element. This implies, $f(x + P) = (f(x) + P)/P$ is a nil ideal. Then, we can write $\alpha^{mn} \in P$, for some $m, n \in N$, which is a contradiction to the fact $P \cap M = \phi$.

Remark 2.4 Every strongly $f$-prime ideal is a strongly prime ideal. But converse is not true.
Example 2.5 From the Example [2.2], we have $A = \{0, 2a\}$ is a strongly $f$-prime ideal. But $A$ is not a strongly prime, since $A$ is not a prime ideal.

Proposition 2.6 In a near ring $N$, if every $f$-prime ideal is a strongly $f$-prime ideal. Then $N$ is a $2.f$-primal near ring.

Proof. For showing $N$ is a $2.f$-primal near ring, we prove $f$-rad$(N)$ contains all the $f$-nilpotent element of $N$. Let $x$ is any $f$-nilpotent element of $N$. Then, $[f(x)]^n = (0)$, for some $n \geq 2$. If $x \notin f$–rad$(N)$. Then there exists an $f$-prime ideal $P$ such that $x \notin P$. Using the assumption, $P$ is a strongly $f$-prime ideal such that $x \notin P$. So, $N/P$ has no non zero $f$-nil ideal and thus $((x) + P)/P$ is not an $f$-nil ideal. This implies, $x + P$ is not an $f$-nil ideal. This implies, $(f(x + P))^n \neq P$, for any $n \in \mathbb{Z}_+$ and thus $(0) = [f(x)]^n$ is not contained in $P$, which is a contradiction. Hence the result. ■

Theorem 2.7 The intersection of a linearly ordered set of strongly $f$-prime ideals of near ring $N$ is a strongly $f$-prime ideal.

Proof. Let $\{P_k|k \in \Lambda\}$ be a family of linearly ordered strongly $f$-prime ideals of near ring $N$ and $P = \bigcap P_k, k \in \Lambda$. Then each $P_k$, being an $f$-prime ideal has its complement $C(P_k)$ as an $f$-system with an $m$-system $P_k^*$ (say). First, we show $P$ is an $f$-prime ideal. Consider $P^* = \cup P_k^*$, union is taken over $\Lambda$. Clearly, $P^*$ is an $m$-system, since for any $\alpha, \beta \in P^*$, there exist $k_1, k_2$ such that $\alpha \in P_{k_1}^*$, and $\beta \in P_{k_2}^*$. Then, by using Proposition 2.92 of [3], we have $\alpha, \beta \in P_{k_1}^*$ (say). Since each $P_{k_1}^*$ is an $m$-system. So, $(\alpha)(\beta) \cap P_{k_1}^* \neq \phi$. This implies, $(\alpha)(\beta) \cap P^* \neq \phi$. Now, $P^* = \cup P_k^* \subseteq C(\cap P_k) = C(P)$. Let $a \in P \Rightarrow a \in P_k$, for some $k$. So, there exists $(P_k)^* \subseteq C(P_k)$ such that $f(a) \cap P_k^* \neq \phi$. This implies, $f(a) \cap P^* \neq \phi$, for all $a \in P$.

Also, we show $N/P$ does not have any non zero $f$-nil ideal, when $N/P_k$ does not have any non-zero $f$-nil ideal, for any $k$. Let $J/P, P \subset J$ is a non zero $f$-nil ideal of $N/P$. Now, $P_k \subset P = \cup P_k \subset J$. So, $J/P_k$ is a non-zero $f$-nil ideal of $N/P$, for any $k$, which is a contradiction. Hence the result. ■

Definition 2.8 An element $a \in N$ is said to be $f$-nil if $f(a)$ is nil ideal.

Definition 2.9 A subset $H$ of a near ring $N$ is said to be $f$-nil if every element of $H$ is $f$-nil.

Lemma 2.10 If $A$ be an $f$-nil ideal of a near ring $N$ and $M/A$ be an $f$-nil subset of quotient near ring $N/A$. Then $M$ is $f$-nil.

Proof. Let $m \in M$ be any element. We show $f(m)$ is nil ideal. Since, for any $m \in M, m + A \in M/A$ is an $f$-nil element in $N/A$. So, $f(m + A) = (f(m) + A)/A$ is a nil ideal in $N/A$. This implies, $\hat{m} + A \in (f(m) + A)/A$ is nilpotent in $N/A$ and thus $(\hat{m} + A)^n = A$, for some $n \in \mathbb{Z}_+ \Rightarrow \hat{m}^n \in A$, which is nil, since $A$ is $f$-nil. $\Rightarrow \hat{m}$ is nilpotent. So, $f(m)$ is nilpotent, for any $m \in M$. Thus $M$ is $f$-nil ideal. Hence the result. ■
Theorem 2.11 Any near ring with unity has a unique maximal $f$-nil ideal, if it exists.

**Proof.** The proof follows by using Zorn’s Lemma on the collection of all $f$-nil ideal of near ring $N$. So, it is required to show the maximal $f$-nil ideal say $M$ in near ring $N$ is unique. For this, it is enough to show that $M$ contains every other $f$-nil ideal $I$. It is clear that $(M + I)/M$ is an $f$-nil subset of $N/M$. Then by using Lemma [2.10], $M + I$ is an $f$-nil. But $M$ being a maximal ideal, implies $M + I = M \Rightarrow I \subseteq M$. Hence the result.

Theorem 2.12 In any near ring $N$, Intersection of all strongly $f$-prime ideal of $N$ is equal to the unique maximal $f$-nil ideal $\eta_f(N)$.

**Proof.** Let $\eta_f(N)$ is not contained in $P$, for some strongly $f$-prime ideal of $N$. Then $\eta_f(N)/P$, is a non-zero $f$-nil ideal of $N/P$. But $P$, being a strongly $f$-prime ideal of near ring $N$, $N/P$ cannot have a non zero nil ideal, which is a contradiction. Hence, $\eta_f(N) \subseteq$ intersection of all strongly $f$-prime ideal of $N$. Conversely, we show the intersection of all strongly $f$-prime ideal of $N$ is an $f$-nil ideal and so is equal to $\eta_f(N)$. Let $x \in N$ is not an $f$-nil element. Then $f(x)$ is not a nil ideal. This implies, $f(x)$ cannot be an nilpotent ideal of $N$. Then there exists a strongly $f$-prime ideal which does not contain $f(x)$. Let $S = \{f(x), [f(x)]^2, \ldots\}$. Then by using Zorn’s Lemma, $\exists$ some ideal $P$ maximal with respect to $P \cap S = \phi$, which is an $f$-prime ideal. If $N/P$ has a non-zero $f$-nil ideal say $M/P$, then $P \subseteq M$ and so, $M \cap S \neq \phi \Rightarrow \alpha^n \in M$, where $\alpha = f(a)$ and $n \in \mathbb{Z}_+ \Rightarrow \alpha^n \in P$, which is a contradiction. Hence the result.

**References**


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