The Spectral Norms of Circulant Matrices Involving \((k,h)\)-Fibonacci and 
\((k,h)\)-Lucas Numbers\(^1\)

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Abstract

This paper is an improving of the work from [6], in which the upper and lower bounds for the spectral norms of the matrices \(A_n = \text{Circ}(F_0^{(k,h)}, F_1^{(k,h)}, \ldots, F_{n-1}^{(k,h)})\) and \(B_n = \text{Circ}(L_0^{(k,h)}, L_1^{(k,h)}, \ldots, L_{n-1}^{(k,h)})\) are established. In this new paper, we compute the spectral norms of these matrices.

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1 Introduction and Preliminaries

A matrix \(A = [a_{ij}] \in M_n\) is called a circulant matrix if it is of the form

\[
a_{ij} = \begin{cases} 
a_{j-i}, & j \geq i \\
a_{n+j-i}, & j < i
\end{cases}
\]

Obviously, the circulant matrix \(A\) is determined by its first row elements \(a_0, a_1, \ldots, a_{n-1}\), thus we denote \(A = \text{Circ}(a_0, a_1, \ldots, a_{n-1})\).

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Let \( k, h \) be any real numbers, and \( k > 0, h \leq -1 \), then the \((k,h)\)-Fibonacci sequence \( \{F_{n}^{(k,h)}\}_{n\in\mathbb{N}} \) and the \((k,h)\)-Lucas sequence \( \{L_{n}^{(k,h)}\}_{n\in\mathbb{N}} \) are defined respectively by the following equations [6]:

\[
F_{n+1}^{(k,h)} = kF_{n}^{(k,h)} - hF_{n-1}^{(k,h)}, \quad F_{0}^{(k,h)} = 0, \quad F_{1}^{(k,h)} = 1
\]

\[
L_{n+1}^{(k,h)} = kL_{n}^{(k,h)} - hL_{n-1}^{(k,h)}, \quad L_{0}^{(k,h)} = 2, \quad L_{1}^{(k,h)} = k
\]

Obviously, when \( k = 1, h = -1 \), these two sequences reduce to the well-known Fibonacci sequence \( \{F_{n}\}_{n\in\mathbb{N}} \) and Lucas sequence \( \{L_{n}\}_{n\in\mathbb{N}} \), respectively.

Recently, there have been several papers on the norms of some special matrices. Akbulak and Bozkurt [1] have found the lower and upper bounds for the spectral norms of Toeplitz matrices \( A = [F_{i-j}]_{i,j=1}^{n} \) and \( B = [L_{i-j}]_{i,j=1}^{n} \). Solak [2,3] has given the upper and lower bounds for the spectral norms of circulant matrices whose entries are Fibonacci and Lucas numbers. Then İpek [4] has investigated an improved estimation for the spectral norms of these matrices. Zhou and Jiang [5] have derived some explicit formulas for the spectral norms of \( g \)-circulant matrices whose the first row entries are Fibonacci number, Lucas number and their powers. Shen and Cen [6] have defined \( n \times n \) circulant matrices such that

\[
\mathcal{A}_{n} = \text{Circ}(F_{0}^{(k,h)}, F_{1}^{(k,h)}, \ldots, F_{n-1}^{(k,h)})
\]

and

\[
\mathcal{B}_{n} = \text{Circ}(L_{0}^{(k,h)}, L_{1}^{(k,h)}, \ldots, L_{n-1}^{(k,h)})
\]

then they have established upper and lower bounds for the spectral norms of these matrices.

The main objective of this study is to compute the spectral norms of the matrices in (1) and (2), then we generalize the main results in [4].

Now we give some preliminaries related to our study. For the \((k,h)\)-Fibonacci sequence \( \{F_{n}^{(k,h)}\}_{n\in\mathbb{N}} \) and the \((k,h)\)-Lucas sequence \( \{L_{n}^{(k,h)}\}_{n\in\mathbb{N}} \), then the following identities holds [6]:

\[
\sum_{i=0}^{n-1} F_{i}^{(k,h)} = \frac{1 - F_{n}^{(k,h)} + hF_{n-1}^{(k,h)}}{1 - k + h},
\]

\[
\sum_{i=0}^{n-1} L_{i}^{(k,h)} = \frac{2 - k + (2h - k)F_{n}^{(k,h)} + h(2 - k)F_{n-1}^{(k,h)}}{1 - k + h}.
\]

For any \( A \in M_{m,n} \). The well-known spectral norm of matrix \( A \) is

\[
\|A\|_{2} = \sqrt{\max_{1 \leq i \leq n} \lambda_i(A^HA)}
\]
where \( \lambda_i(A^H A) \) is eigenvalue of \( A^H A \) and \( A^H \) is conjugate transpose of matrix \( A \).

**Lemma 1** \([7]\) Let \( A = [a_{ij}] \in M_n \) and \( A \) is nonnegative, then its spectral radius \( \rho(A) \) satisfy the following inequality
\[
\min_{1 \leq i \leq n} \sum_{j=1}^{n} a_{ij} \leq \rho(A) \leq \max_{1 \leq i \leq n} \sum_{j=1}^{n} a_{ij}.
\] (4)

## 2 Main Results

**Theorem 1** Let \( A_n = [a_{ij}] \in M_n \) be the matrix in (1), then we have
\[
\|A_n\|_2 = \frac{1 - F_n^{(k,h)} + hF_{n-1}^{(k,h)}}{1 - k + h},
\]
where \( \|A_n\|_2 \) is the spectral norm of the matrix \( A_n \).

**Proof.** Since the circulant matrix \( A_n \) is normal, there exists a unitary matrix \( U \in M_n \) such that \( U^H A_n U = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n) \), where \( \lambda_i \) is eigenvalue of \( A_n \), hence
\[
U^H A_n^H A_n U = \text{diag}(|\lambda_1|^2, |\lambda_2|^2, \cdots, |\lambda_n|^2).
\]

Thus, the spectral norm of \( A_n \) is given by its spectral radius. Also since \( A_n \) is nonnegative, its spectral radius \( \rho(A_n) \) satisfy the following inequality:
\[
\min_{1 \leq i \leq n} \sum_{j=1}^{n} a_{ij} \leq \rho(A_n) \leq \max_{1 \leq i \leq n} \sum_{j=1}^{n} a_{ij},
\]
while
\[
\sum_{j=1}^{n} a_{ij} = \sum_{l=0}^{n-1} F_l^{(k,h)} = \frac{1 - F_n^{(k,h)} + hF_{n-1}^{(k,h)}}{1 - k + h}
\]
for any \( i = 1, 2, \cdots, n \), thus the proof is completed.

If we choose \( k = 1, h = -1 \) in Theorem 1, then we obtain the following result from \([4]\).

**Corollary 1** Let \( A_n = \text{Circ}(F_0, F_1, \cdots, F_{n-1}) \) be circulant matrix, then we have
\[
\|A_n\|_2 = F_{n+1} - 1.
\]

**Theorem 2** Let \( B_n = [b_{ij}] \in M_n \) be the matrix in (2), then we have
\[
\|B_n\|_2 = \frac{2 - k + (2h - k)F_n^{(k,h)} + h(2 - k)F_{n-1}^{(k,h)}}{1 - k + h},
\]
where \( \|B_n\|_2 \) is the spectral norm of the matrix \( B_n \).
Proof. The proof is similar to the proof of Theorem 1. Since the circulant matrix $B_n$ is normal and nonnegative, its spectral norm is the same as its spectral radius $\rho(B_n)$. Applying result obtained in Lemma 1, we have
\[
\min_{1 \leq i \leq n} \sum_{j=1}^{n} b_{ij} \leq \rho(B_n) \leq \max_{1 \leq i \leq n} \sum_{j=1}^{n} b_{ij},
\]
but
\[
\sum_{j=1}^{n} b_{ij} = \sum_{l=0}^{n-1} L_l^{(k,h)} = \frac{2 - k + (2h - k)F_n^{(k,h)} + h(2 - k)F_{n-1}^{(k,h)}}{1 - k + h}
\]
for any $i = 1, 2, \cdots, n$, thus the proof is completed.

In the case $k = 1, h = -1$ from Theorem 2, then we have the following result from [4].

**Corollary 2** Let $B_n = \text{Circ}(L_0, L_1, \cdots, L_{n-1})$ be circulant matrix, then we have
\[
\|B_n\|_2 = F_{n+2} + F_n - 1.
\]

Proof. We select $k = 1, h = -1$ in Theorem 2, then the following is valid
\[
\|B_n\|_2 = 3F_n + F_{n-1} - 1,
\]
Thus, the proof is completed from the following identity
\[
3F_n + F_{n-1} = F_{n+2} + F_n.
\]

References


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