The Euler’s Function and
the Distribution of Certain Numbers

Rafael Jakimczuk

División Matemática, Universidad Nacional de Luján
Buenos Aires, Argentina

Copyright © 2014 Rafael Jakimczuk. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

In this note we show that the arithmetical function \( \varphi(n)/n^2 \) is related with the distribution of certain positive integers, where \( \varphi(n) \) is the Euler’s function.

Mathematics Subject Classification: 11A99, 11B99

Keywords: Euler’s function, distribution of certain numbers, density

1 Introduction and Preliminary Results

In this note \( \lfloor x \rfloor \) denotes the integer-part function. Note that

\[ 0 \leq x - \lfloor x \rfloor < 1. \tag{1} \]

We shall need the following theorem, it is sometimes called either the principle of inclusion-exclusion or the principle of cross-classification. We now enunciate the principle.

**Theorem 1.1** Let \( S \) be a set of \( N \) distinct elements, and let \( S_1, \ldots, S_r \) be arbitrary subsets of \( S \) containing \( N_1, \ldots, N_r \) elements, respectively. For \( 1 \leq i < j < \ldots < l \leq r \), let \( S_{ij\ldots l} \) be the intersection of \( S_i, S_j, \ldots, S_l \) and let...
N_{ij...l} be the number of elements of S_{ij...l}. Then the number K of elements of S not in any of S_{1},...,S_{r} is

\[ K = N - \sum_{1 \leq i \leq r} N_{i} + \sum_{1 \leq i < j \leq r} N_{ij} - \sum_{1 \leq i < j < k \leq r} N_{ijk} + \cdots + (-1)^{r} N_{12...r}. \]  

(2)

Proof. See, for example, either [2, page 233] or [3, page 84].

If n \geq 1 the Euler’s function \( \varphi(n) \) is defined to be the number of positive integers not exceeding n which are relatively prime to n. The following formula is well-known

\[ \varphi(n) = n \prod_{p|n} \left( 1 - \frac{1}{p} \right) \]  

(3)

where, the product run on the different prime divisors p of n.

In this note we show that the arithmetical function \( \varphi(n)/n^2 \) is related with the distribution of certain positive integers.

2 Main Result

Let us consider the numbers of the form \( p_{1}^{h_{1}} \cdots p_{k}^{h_{k}}C \), where \( p_{1},...,p_{k} \) are \( k \geq 1 \) different primes fixed, \( h_{1},...,h_{k} \) are \( k \geq 1 \) positive integers fixed, the positive integer C is variable and \( p_{1}^{h_{1}} \cdots p_{k}^{h_{k}} \) are relatively prime. The set of these numbers is denoted \( S_{p_{1}^{h_{1}}...p_{k}^{h_{k}}} \) and the number of these numbers not exceeding \( x \) is denoted \( A_{p_{1}^{h_{1}}...p_{k}^{h_{k}}}(x) \).

**Theorem 2.1** The numbers in the set \( S_{p_{1}^{h_{1}}...p_{k}^{h_{k}}} \) have positive density \( D_{p_{1}^{h_{1}}...p_{k}^{h_{k}}} \), where

\[ D_{p_{1}^{h_{1}}...p_{k}^{h_{k}}} = \frac{(p_{1} - 1) \cdots (p_{k} - 1)}{p_{1}^{h_{1}+1} \cdots p_{k}^{h_{k}+1}} = \frac{\varphi(p_{1}^{h_{1}} \cdots p_{k}^{h_{k}})}{(p_{1}^{h_{1}} \cdots p_{k}^{h_{k}})^{2}} \]  

(4)

Proof. Equations (2), (1) and (3) give

\[
A_{p_{1}^{h_{1}}...p_{k}^{h_{k}}}(x) = [\frac{x}{p_{1}^{h_{1}} \cdots p_{k}^{h_{k}}}] - [\frac{x}{p_{1}^{h_{1}+1} \cdots p_{k}^{h_{k}}}] - \cdots - [\frac{x}{p_{1}^{h_{1}} \cdots p_{k}^{h_{k}+1}}] + \cdots
\]

\[
= \frac{x}{p_{1}^{h_{1}} \cdots p_{k}^{h_{k}}} - \frac{x}{p_{1}^{h_{1}+1} \cdots p_{k}^{h_{k}}} - \cdots - \frac{x}{p_{1}^{h_{1}} \cdots p_{k}^{h_{k}+1}} + \cdots + O(1)
\]

\[
= \frac{1}{p_{1}^{h_{1}} \cdots p_{k}^{h_{k}}} \left( 1 - \frac{1}{p_{1}} \right) \cdots \left( 1 - \frac{1}{p_{k}} \right) x + O(1)
\]
\[ = \frac{(p_1 - 1) \cdots (p_k - 1)}{p_1^{h_1+1} \cdots p_k^{h_k+1}} x + O(1) \]
\[ = \frac{\varphi \left( p_1^{h_1} \cdots p_k^{h_k} \right)}{\left( p_1^{h_1} \cdots p_k^{h_k} \right)^2} x + O(1) \]  
Equation (5) gives
\[ D_{p_1^{h_1} \cdots p_k^{h_k}} = \lim_{x \to \infty} \frac{A_{p_1^{h_1} \cdots p_k^{h_k}}(x)}{x} = \frac{(p_1 - 1) \cdots (p_k - 1)}{p_1^{h_1+1} \cdots p_k^{h_k+1}} = \frac{\varphi \left( p_1^{h_1} \cdots p_k^{h_k} \right)}{\left( p_1^{h_1} \cdots p_k^{h_k} \right)^2} \]
That is, equation (4). The theorem is proved.

The set of the multiples of \( p_1 \cdots p_k \) we denote \( E_{p_1^{h_1} \cdots p_k^{h_k}} \), the density of this set is clearly \( \frac{1}{p_1 \cdots p_k} \). Note that the union of the infinite disjoint sets \( S_{p_1^{h_1} \cdots p_k^{h_k}} \) is the set \( E_{p_1^{h_1} \cdots p_k^{h_k}} \). That is
\[ \bigcup_{(h_1, \ldots, h_k)} S_{p_1^{h_1} \cdots p_k^{h_k}} = E_{p_1^{h_1} \cdots p_k^{h_k}} \]
where the symbol
\[ \bigcup_{(h_1, \ldots, h_k)} \]
denotes that the union run on all possible vectors \((h_1, \ldots, h_k)\).

In the next theorem we prove that the sum of the infinity densities of the infinite disjoint sets \( S_{p_1^{h_1} \cdots p_k^{h_k}} \) is the density of the union of these sets, namely, the density of the set \( E_{p_1^{h_1} \cdots p_k^{h_k}} \).

**Theorem 2.2** The following formula holds
\[ \sum_{(h_1, \ldots, h_k)} D_{p_1^{h_1} \cdots p_k^{h_k}} = \frac{1}{p_1 \cdots p_k} \]  
Proof. We have (see equation (4))
\[ \sum_{(h_1, \ldots, h_k)} D_{p_1^{h_1} \cdots p_k^{h_k}} = \sum_{(h_1, \ldots, h_k)} \frac{(p_1 - 1) \cdots (p_k - 1)}{p_1^{h_1+1} \cdots p_k^{h_k+1}} \]
\[ = (p_1 - 1) \cdots (p_k - 1) \left( \frac{1}{p_1^2} + \frac{1}{p_1^3} + \cdots \right) \cdots \left( \frac{1}{p_k^2} + \frac{1}{p_k^3} + \cdots \right) \]
\[ = (p_1 - 1) \cdots (p_k - 1) \frac{1}{p_1^2} \frac{1}{p_1} \cdots \frac{1}{p_k^2} \frac{1}{p_k} = \frac{1}{p_1 \cdots p_k} \]
where we have use the well-known series
\[ \frac{1}{1 - x} = \sum_{m=0}^{\infty} x^m \quad |x| < 1 \]
The theorem is proved.

Note that if we put \( n = p_1^{h_1} \cdots p_k^{h_k} \), then the numbers of the form \( p_1^{h_1} \cdots p_k^{h_k} C = nC \) have density (see equation (4))

\[
D_n = \frac{\varphi(n)}{n^2} \quad (n = 1, 2, 3, \ldots)
\] (7)

Besides, we have the following theorem.

**Theorem 2.3** The following formula holds

\[
\sum_{n \leq x} D_n = \sum_{n \leq x} \frac{\varphi(n)}{n^2} = \frac{6}{\pi^2} \log x + \frac{6}{\pi^2} \gamma - B + O \left( \frac{\log x}{x} \right)
\]

where \( \gamma \) is the Euler’s constant,

\[ B = \sum_{m=1}^{\infty} \frac{\mu(m) \log m}{m^2} \]

and \( \mu(m) \) is the Mobius’s function.

Proof. See [1, chapter 3].

**ACKNOWLEDGEMENTS.** The author is very grateful to Universidad Nacional de Luján.

**References**


Received: September 15, 2014; Published: November 12, 2014