Case Study of Markov Chains Ray-Knight Compactification

HaiXia Du and YanLing Pan

Department of Mathematics and Statistics
Zhengzhou Normal University Zhengzhou, 450044, China

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Abstract

It gives Ray-Knight compactifications of specific examples to illustrate the complexity.

Keywords: Ray - Knight compactification; Transfer function; Resolvent

1. Introduction

Structural problems of Markov chain suffered for decades, is not completely resolved, the main reason is the general Markov chain has only locally strong Markov property. Reference [1] using Ray-Knight compactification method, construct strong Markov process corresponding to the transfer function. Reference [2] using Ray - Knight of the Markov chain method, proves that the Markov chain the existence of the local time of. Reference [3], [4], [5] using Markov chain Ray-Knight compactification method, solves the structural problem of bilateral birth- death process. Reference [6] using Markov chain Ray-Knight compactification method, solves the structural problem of birth- death process. Reference [7], [8] illustrates the relationship of Martin entrance boundary and Ray-Knight compactification of minimal Q-processes. Thus, Ray-Knight compactification is the bridge to solve the problem of Markov chain structure. In this article, we will through specific examples to illustrate the Ray - Knight com-
pactification of Markov chain, at the same time, it shows that Ray-Knight compactification complexity.

2. Preliminaries

Let $E=\{1,2,\ldots\}$, we agree on the topology of $E$ is discrete topology, then $E$ is a locally compact and has a countable topological space, and the function of $E$ is a continuous function. Called the state of the elements in $E$.

All bounded functions on $E$ denoted as $M$, with bounded nonnegative functions denoted as $M^+$, $M$ and $M^+$ topology is the topology of uniform convergence. Sometimes we take $M$ and $M^+$ elements as infinite dimensional column vector. The family of functions $P(t) = \left(p_{ij}(t)\right)_{i,j \in E}, t \geq 0$ on $[0, \infty)$ is called the transfer function on $E$, if

$$p_{ij}(t) \geq 0, \ldots (1)$$

$$\sum_{k=1}^{\infty} p_{ik}(t) \leq 1, \ldots (2)$$

$$p_{ij}(t+s) = \sum_{k=1}^{\infty} p_{ik}(t)p_{kj}(s), \ldots (3)$$

If there is $\lim_{t \to 0} p_{ij}(t) = p_{ij}(0) = \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$, then called $P(t)(t \geq 0)$ as a standard.

Said $P(t)(t \geq 0)$ is honest (or not interrupted), if established for $i \in E$ all equal sign in (2).

Known that if $P(t)(t \geq 0)$ transfer function is the standard, then there is a limit:

$$\lim_{t \to 0} \frac{p_{ij}(t)-\delta_{ij}}{t} = q_{ij}, i, j \in E.$$

and $0 \leq q_{ij} < \infty, 0 \leq q_{ii} = -q_{ij} \leq \infty, \sum_{k \neq i} q_{ik} \leq q_{i}$.

Matrix $Q = \left(q_{ij}(t)\right)_{i,j \in E}$ is called $P(t)(t \geq 0)$ of density matrix on $E$.

Let $R_{ij}(\lambda) = \int_{0}^{\infty} e^{-\lambda t} p_{ij}(t) dt, i, j \in E, \lambda > 0$. $R_{ij}(\lambda)$ is called the resolvent of
$P(t)(t \geq 0) R_{ij}(\lambda)$ is the resolvent of the transfer function of $P(t)(t \geq 0)$, if and only if the following conditions (6) – (9) holds

$$\lambda \sum_{k \in E} R_{ik}(\lambda) \leq 1 \quad (6)$$

$$R_j(\lambda) - R_j(\mu) + (\lambda - \mu) \sum_{k \in E} R_{ik}(\lambda) R_j(\mu) = 0 \quad (7)$$

$$\lim_{\lambda \to \infty} \lambda R_{ij}(\lambda) = \delta_{ij} \quad (8)$$

$$\lim_{\lambda \to \infty} |\lambda R_{ij}(\lambda) - \delta_{ij}| = q_{ij} \quad (9)$$

Let $E\{1, 2, \cdots\}, P_{ij}(t)$ is the honesty transfer function on $E$, $R_{ij}(\lambda)$ is the resolvent of $P_{ij}(t)$, The norm of $M$ is defined as: for arbitrary $f \in M$,

$$\|f\| = \sup_{i \in E} |f(i)|$$. Let the function $i \to \sum_{k \in E} R_{ik}(\lambda) f(k)$ is $R_\lambda f$. Obviously, $R_\lambda f$ is linerator on $M$ and $\|R_\lambda\| = \frac{1}{\lambda}$. For any $G \subset M$, let

$$u(G) = \sum_{i=1}^{n} u_i R_{\lambda_i} f_i |n \in \mathbb{N}, \lambda_i > 0, u_i \geq 0, f_i \in G, \forall i \leq n\}$$. $

$$\Lambda(G) = \{f_1 \Lambda \cdots \Lambda f_n \mid n \in \mathbb{N}, f_1, \cdots, f_n \in G\}.$$

Take $H = H = \{E(k) \mid k \in E\} \cup \{1\}$, then $H$ is countable subset of $G$, let

$$R^{(1)} = u(H), \quad R^{(n+1)} = \Lambda(R^{(n)} + u(R^{(n)}))$$, $R = \bigcup_{n=1}^{\infty} R^{(n)}$. let $\{g_m\}_{m=1}^{\infty}$ is the dense subset of $R$, $d(x, y) = \sum_{m=1}^{\infty} \frac{1}{2^m} |[g_m(x) - g_m(y)]1|, \forall x, y \in S$. (10)

Then $d(\cdot, \cdot)$ is the measure on $E$. $\overline{E}$ is the completion of the $E$ under $d(\cdot, \cdot)$. Apparently, $E$ is compact metric space, called Ray-Knight compactification of $E$. $R_{ij}(\lambda)$ can expand into $U^\alpha(x, dy)$ which is the ray resolvent on $\overline{E}$, $P_t(x, dy)$
is the ray semi-group corresponding to $U^\alpha(x,dy)$. Let

$$D = \{x \in \mathbb{E} | P_0(x,\cdot) = \delta_x(\cdot)\},$$

then $D$ is called non-branch point sets, the point of $D$ is called non-branch point.

### 3. Example of Ray-Knight compactification

Let $E = \{1, 2, \ldots\}$, $Q = \begin{pmatrix}
-q_1 & q_1 & 0 & 0 & \cdots & \cdots \\
0 & -q_2 & q_2 & 0 & \cdots & \cdots \\
0 & 0 & -q_3 & q_3 & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots
\end{pmatrix}$

$q_1, q_2, \ldots$ are a list of integers. The minimum transfer function of $Q$ is named $P^n(t)$, the resolvent of $P^n(t)$ defined by

$$R_{ij}^{\min} = \begin{cases}
0 & \text{if } j < i \\
(\lambda + q_i)^{-1} \prod_{i \leq k < j} (1 + \lambda q_k^{-1})^{-1} & \text{if } j > i \\
(1 + \lambda q_i^{-1})^{-1} & \text{if } j = i
\end{cases} \quad (11)$$

and $P^n(t)$ is honest if and only if $\sum_{i=1}^{\infty} q_i^{-1} = \infty$ (references [9]).

**Example 1.** Let the minimum transfer function $P^n(t)$ is honest, then

$$\sum_{i=1}^{\infty} q_i^{-1} = \infty, R_{ij}^{\min}(\lambda)$$ to satisfy the following assumptions:

1. $E$ is topological space which is the locally compact and has countable topological base (Referred to as L.C.C.B);

2. $\{R_\lambda\}_{\lambda > 0}$ is Markov and $R_\lambda C_b(E) \subseteq C_b(E)$, for arbitrary $\lambda > 0$;

3. For arbitrary $f \in C_b(E)$ and $x \in E$, $\lim_{\lambda \to \infty} \lambda R_\lambda f(x) = f(x)$, $\lim_{i \to \infty} R_i^{\min}(\lambda) = 0$.

**Proof.** For arbitrary $f \in R^{(i)}$, $\lim_{i \to \infty} f(i) = 0$. Using mathematical induction, it is
easy to prove that for arbitrary \( f \in R^{(n)} \), \( \lim_{i \to \infty} f(i) = 0 \). By denoting \( \overline{E} = E \cup \{\infty\} \),
then \( \overline{E} \) is the one point compactification of \( E \). For arbitrary \( \alpha > 0 \), then
\( U^{(\alpha)}(\infty, j) = \lim_{i \to \infty} R_j(\alpha) = 0 \). so \( U^{(\alpha)}(\infty, \{\infty\}) = 1 \). \( \infty \) is a non-branch point.

\( E = E_\mathcal{R} \), \( D = \overline{E} \).

**Example 2.** Let \( \mu_i, i=1,2,\ldots \) is a probability measure on \( E \). \( \mu_i < 1 \), for arbitrary \( i \in E \), the minimum transfer function \( \min \{ \lambda \} \) is interrupt, \( P(t)(t \geq 0) \) is transfer function of Doob process which is \((Q,\pi)\) type.

**Proof.** From references【9】 , we know that \( \sum_{i=1}^{\infty} q_i^{-1} = \infty \), the orbit of \( \{X_t\} \) is right continue. The conditions distribution of \( P\{X_0=i\} \) is remembered to \( P^i(\cdot) \), the corresponding conditional expectation is remembered to \( E^i(\cdot) \).

Let \( \sigma = \inf \{s\|X_s=\infty\} \), then \( \sigma \) is the first leap point and the following equation holds. \( E^i(e^{-\lambda \sigma}) = \prod_{j>\sigma}(1+\lambda q_i^{-1})^{-1} \), \( \forall i \in E, \lambda > 0 \) (12). The resolvent of \( P(t) \) is as follows: \( R_y(\lambda) = R_{\lambda y}^\min(\lambda) + \mathbb{E}\{e^{-\lambda \sigma}\}, \sum_{k \in E} \mu_k R_{\lambda k}^\min(\lambda) \sum_{k \in E} \mu_k [1-E^k \{e^{-\lambda \sigma}\}] \}, \forall i, j \in E \) (13).

For any \( i, j \in E, \lambda > 0 \), obviously,

\[
\lim_{i \to \infty} E^i(e^{-\lambda \sigma}) = 1, \lim_{i \to \infty} R_{\lambda y}^\min(\lambda) = 0, \lim_{i \to \infty} R_{\lambda y}(\lambda) = \sum_{k \in E} \mu_k R_{\lambda k}^\min(\lambda) \sum_{k \in E} \mu_k [1-E^k \{e^{-\lambda \sigma}\}] .
\]

Modeled on the example one, it is easy to prove that for arbitrary \( f \in R^{(n)} \), \( \lim_{i \to \infty} f(i) \) is exist. By the definition of \( \overline{E} \), then \( \overline{E} = E \cup \{\infty\} \). For arbitrary \( \alpha > 0 \), by the definition of \( U^{(\alpha)} \), then
\[ U^a(\infty, j) = \lim_{i \to \infty} R_j(\alpha) = \frac{\sum_{k \in E} \mu_k R_{ij}^{\min}(\alpha)}{\sum_{k \in E} \mu_k [1 - E^k \{e^{-\lambda \sigma}\}]} \quad \forall j \in E. \]

Therefore, \( U^a(\infty, E) = \frac{1}{\alpha} P_0(\infty, j) = \mu_j \), for arbitrary \( j \in E \). \( \infty \) is a non-branch point. \( E_R = E \cup \{\infty\} \), \( D = E \).

**Remark:** In the same way to prove the following conclusion, if there is \( i_0 \in E \) make that \( \mu_{i_0} = 1 \), then the Ray-Knight compactification of \( E \) is named \( \overline{E} \) which meets the following equation, \( \overline{E} = E \), and \( i_0 \) is the limit point of the sequence point of \( 1, 2, 3, \ldots \). At the same time, \( \overline{E} = E = E_R = D \).

**Example 3.** Let \( \mu_i, i = 1, 2, \ldots \) is a probability measure on \( E \). \( \mu_i < 1 \), for arbitrary \( i \in E \), the minimum transfer function minium \( P^{*n}(t) \) is interrupt, The Markov chain \( \{X_i\} \) corresponding to \( P(t)(t \geq 0) \) is not Doob process.

**Proof.** From references \([10]\), we know that \( \sum_{i=1}^{\infty} q^{-1} < \infty \),

\[
\text{lim} E^t(e^{-\lambda \sigma}) = \text{lim} \prod_{i \to \infty} \frac{q_k}{\lambda + q_k} = 1. \quad \text{Since} \quad Q \quad \text{is single outflow zero inflow, according to the general conclusions of single outflow of Markov chain, so there is}
\]

\[ \mu_i, i = 1, 2, \ldots, \text{such that} \quad \sum_{k=1}^{\infty} \mu_k = \infty, \sum_{k} \mu_k [1 - E^k \{e^{-\lambda \sigma}\}] < \infty, \forall \lambda > 0 \quad (14) \]

Moreover, \( R_j(\lambda) \) is the resolvent of the transfer function of \( P(t)(t \geq 0) \) and

\[
R_j(\lambda) = R_{ij}^{\min}(\lambda) + E \{e^{-\lambda \sigma}\} \cdot \frac{\sum_{k \in E} \mu_k R_{ij}^{\min}(\lambda)}{\sum_{k \in E} \mu_k [1 - E^k \{e^{-\lambda \sigma}\}]} \quad i, j \in E, \lambda > 0 \quad (15)
\]

Modeled on the example two, we can prove that \( \overline{E} = E \cup \{\infty\} \). For arbitrary
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\[ \alpha > 0, \quad U^\alpha (\infty, j) = \lim_{i \to \infty} R_i(\alpha) = \frac{\sum_{k \in E} \mu_k R^\alpha_{kj}(\alpha)}{\sum_{k \in E} \mu_k [1 - E\{e^{-\alpha}\}]} \cdot j \in E. \]  

(16)

Use the formula of (14), then \( \lim \alpha U^\alpha (\infty, j) = 0 \), that is \( \lim_{i \to 0} P'(\infty, \cdot) = \delta_\infty (\cdot) \). So \( \overline{E} = E_R = D = E \cup \{\infty\} \).

For the transfer function which contains instantaneous state, it’s Ray-Knight compactification structure is more complicated, please look at the case.

**Example 4.** If \( q_i, i = 1, 2, \cdots \) is a list of positive number, consider the following matrix \( Q \).

\[
Q = \begin{pmatrix}
-\infty & 1 & 1 & 1 & \ldots \\
q_2 & -q_2 & 0 & 0 & \ldots \\
q_3 & 0 & -q_3 & 0 & \ldots \\
q_4 & 0 & 0 & -q_4 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

with \( \sum_{i=2}^\infty q_i^{-1} < \infty \).

The matrix is called **Kolmogorov** matrix. It corresponds to the resolvent

\[ R_{11}(\lambda) = \frac{1}{\lambda} \left( 1 + \sum_{k=2}^{\infty} \frac{1}{\lambda + q_k} \right) \]

\[ R_{ij}(\lambda) = R_{11}(\lambda) \cdot \frac{1}{\lambda + q_j}, \quad j \geq 2 \]

\[ R_{ii}(\lambda) = \frac{q_i}{\lambda + q_i} \cdot R_{11}(\lambda), \quad i \geq 2 \]

\[ R_{ij}(\lambda) = \frac{q_i}{\lambda + q_i} \cdot R_{11}(\lambda) \cdot \frac{1}{\lambda + q_j} + \frac{\delta_{ij}}{\lambda + q_j}, \quad i, j \geq 2 \]

Obviously, \( \lim_{\lambda \to \infty} R_{ij}(\lambda) = R_{ij}(\lambda) \), it is easy to prove \( \overline{E} = E \), and under the topology of Ray-Knight, \( 1 \) is the limit point of the sequence point of \( 2, 3, \cdots \).
References


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