Recursiveness in Hilbert Spaces and Application to Mixed $ARMA(p,q)$ Process

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Abstract

In this paper, the gap between the nonhomogeneous linear recurrence relation and the mixed $ARMA(p,q)$ process in Hilbert spaces is provided. We focus ourselves on the case when the $ARMA(p,q)$ process is of Fibonacci type. Notably, involving properties of Fibonacci sequences, we reach to establish the new formulations of some characteristics of the $ARMA(p,q)$ process. For purpose of illustration, some examples are explored.

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1 introduction

Let $(\Omega, \mathcal{A}, P)$ be a probability space and consider the stochastic process $\{X_t; t \in \mathbb{Z}\}$ of a mixed $ARMA(p,q)$ process, satisfying the following difference equation,

$$X_{t+1} = a_0 X_t + \cdots + a_{p-1} X_{t-p+1} + C_{t+1},$$

(1)
such that $C_{t+1} = A_{t+1} + b_0 A_t + \cdots + b_{q-1} A_{t-q+1}$, where $a_0, a_1, \ldots, a_{p-1} \in \mathbb{R}$ are the auto-regressive parameters, $b_0, b_2, \ldots, b_{q-1} \in \mathbb{R}$ are the auto-regressive moving average regressive parameters and $\{A_t; t \in \mathbb{Z}\}$ is a stochastic process describing the white noise (see [4], for example). The mixed $ARMA(p,q)$ process are currently used in the methods of prediction and time series (for more details see [4], [5], for example). For discrete indices $t = n$ and $\omega \in \Omega$, we set $x_n = X_n(\omega)$ and $c_n = C_n(\omega)$. It ensues from Equation (1) that the sequence $\{x_n\}_{n \geq 0}$ satisfies the following nonhomogeneous linear recursive equation,

$$x_{n+1} = a_0 x_n + \cdots + a_{p-1} x_{n-p+1} + c_{n+1}, \text{ for every } n \geq p-1,$$

(2)

where $x_0, \ldots, x_{p-1}$ are the initial data. Solutions of Equation (2) are studied in the literature when $c_n$ is polynomial or factorial polynomial (see [2], [7], [8], [9], for example). Recently, two methods have been proposed, for studying solutions of Equation (2) in [2], [7]. More precisely, in [7] a matrix theoretical approach for solving (2) is developed for a general $\{c_n\}_{n \geq 0}$. Beside in [2], a linearization process is introduced, when $\{c_n\}_{n \geq 0}$ verifies a nonhomogeneous linear recursive relation. When $c_n = 0$, for every $n$, it was proved in [11] that solutions of the homogeneous part of (2) are,

$$x_n^{<n>} = \rho(n,p) S_0 + \rho(n-1,p) S_1 + \cdots + \rho(n-p+1,p) S_{p-1},$$

(3)

for $n \geq p$, where $S_k = a_{p-1} x_k + \cdots + a_k x_{p-1}$ ($0 \leq k \leq p - 1$) and

$$\rho(n,p) = \sum_{k_0+2k_1+\ldots+pk_{p-1}=n-p} \frac{(k_0 + \cdots + k_{p-1})!}{k_0! \cdots k_{p-1}!} a_0^{k_0} \cdots a_{p-1}^{k_{p-1}},$$

(4)

with $\rho(p,p) = 1$ and $\rho(n,p) = 0$ if $n \leq p - 1$.

On the other hand, the closed relationship between stochastic process and the recursive relation (2), has been explored in [10], [11]. Moreover, it was established that some probabilistic properties are related both to recursiveness and Expression (4) in [13], [14]. Particularly, it was shown that Expression (4) is also a special case of Philippou polynomials (see [1], [3]).

In this paper, we are interested in studying a theoretical approach of the mixed $ARMA(p,q)$ (stochastic) process defined by (1) and the nonhomogeneous linear recurrence relation (2) in a real Hilbert space. We extend the matrix method of [7] to the case of a real Hilbert space (Section 2). We apply this extension in order to obtain a general solution of (2). Using the extension of the linearization method of [2] to the case of real Hilbert spaces, we study the case of $ARMA(p,q)$ process of Fibonacci type defined by $C_{t+1} = A_{t+1} = b_0 A_t + b_1 A_{t-1} + \cdots + b_{q-1} A_{t-q+1}$ (Section 3). Properties concerning the computation of some characteristics of the $ARMA(p,q)$ model, via the approach of linear recurrence relations, are explored (Section 4). Final
2 Equation (2) in a Hilbert space

In this section, we extend the fundamental results established in [2], [7] upon the matrix method and the linearization process for solving Equation (2). To this aim, we recall first the following known result established in [7].

**Proposition 2.1** Let \( \{v_{n}\}_{n \geq 0} \) be a real Fibonacci sequence given by \( v_{n+1} = a_0v_n + \cdots + a_{p-1}v_{n-p+1} \) for \( n \geq 0 \), where \( v_0, v_1, \ldots, v_{p-1} \) are specified initial data. Let \( A = (\alpha_{i,j})_{1 \leq i,j \leq p} \) be the matrix defined by \( \alpha_{i,j} = a_{j-i} \), for every \( 1 \leq j \leq p \), and \( \alpha_{i,j} = \delta_{i-j,1} \), for every \( 2 \leq i \leq p, 1 \leq j \leq p \). Then, for \( n \geq 1 \), the entries \( a_{is}^{(n)} \) \( (0 \leq i, s \leq p-1) \) of the matrix powers \( A^n \) are given by \( a_{is}^{(n)} = v_{n-i}^{(s)} \), where \( \{v_{n}^{(s)}\}_{n \geq 0} \) are \( p \) copies of \( \{v_{n}\}_{n \geq 0} \), whose initial conditions are \( v_{j}^{(s)} = \delta_{s,j} \) \( (0 \leq j \leq p-1, 0 \leq s \leq p-1) \). Here, \( \delta_{s,j} \) represents the Kronecker symbol.

In view of (3), we infer that for every \( s \) \( (0 \leq s \leq p-1) \) the sequence \( \{v_{n+1}^{(s)}\}_{n \geq 0} \) can be formulated as follows. For \( 0 \leq n \leq p-1 \) we have \( v_{n}^{(s)} = \delta_{n,s} \)

\[
v_{n}^{(s)} = a_{p-s+1}\rho(n,p) + a_{p-1}\rho(n-1,p) + \cdots + a_{p-1}\rho(n-s+1,p), \quad \text{for } n \geq p,
\]

where the \( \rho(n,p) \) are given by Expression (4).

Consider \( \{T_n\}_{n \geq 0} \) and \( \{C_n\}_{n \geq 0} \) two sequences of \( \mathcal{H} \) satisfying (2). For \( n \geq p-1 \), we set \( Z_n = \left(T_n, \ldots, T_{n-p+1}\right)^\prime \in \mathcal{H}^p \) and \( D_n = \left(C_n, 0, \ldots, 0\right)^\prime \in \mathcal{H}^p \), where \( \left(T_n, \ldots, T_{n-p+1}\right)^\prime \) means the transpose of \( Z \). It’s easy to show that (2) is equivalent to the matrix equation \( Z_{n+1} = AZ_n + D_{n+1} \), for \( n \geq p-1 \), where \( A = (\alpha_{i,j})_{1 \leq i,j \leq p} \) is the matrix exhibited in Proposition 2.1. Thus, we obtain \( Z_n = A^{n-p+1}Z_{p-1} + \sum_{k=p}^{n} A^{n-k}D_k \), for every \( n \geq p \). Hence, similarly to the real case (see [2], [7]), we establish that the solution of Equation (2) is,

\[
T_n = \sum_{s=0}^{p-1} v_{n-p+1-s}^{(s)} T_{p-s-1} + \sum_{k=p}^{n} v_{n-k}^{(0)} C_k, \quad \text{for every } n \geq 0.
\]

The sequence \( \{T_n^{<h>}\}_{n \geq 0} \) defined by \( T_n^{<h>} = \sum_{s=0}^{p-1} v_{n-p+1-s}^{(s)} T_{p-s-1} \) is a solution of the homogeneous part of Equation (2). In addition, a particular solution \( \{T_n^{<p>}\}_{n \geq 0} \) of (2) is \( T_n^{<p>} = \sum_{k=p}^{n} v_{n-k}^{(0)} C_k = -\sum_{s=0}^{p-1} v_{n-p+1-s}^{(s)} T_{p-s-1} + T_n \), where
\( T_n^{<p>} = 0 \) for \( n = 0, 1, \cdots, p - 1 \). The sequence \( \{T_n^{<p>}\}_{n \geq 0} \) is called the fundamental particular solution of (2). Thence, in light of Proposition 2.1 and Equation (6), we get the following general result,

**Theorem 2.2** Let \( \{T_n\}_{n \geq 0} \) be a solution of (2) in a Hilbert space \( \mathcal{H} \). Then, for \( n \geq 0 \), we have

\[
T_n = T_n^{<h>} + T_n^{<p>} = T_n^{<h>} - \sum_{s=0}^{p-1} q^{(s)}_{n-p+1} T_{p-s-1}^{<p>} + T_n^{<p>},
\]

where \( \{T_n^{<p>}\}_{n \geq 0} \) is the fundamental particular solution of (2) and \( \{T_n^{<h>}\}_{n \geq 0} \) is a solution of the homogeneous part of (2) with initial data \( T_0, \cdots, T_{p-1} \).

Now let consider the case when \( \{C_n\}_{n \geq 0} \) is a linear recursive sequence. Set \( \{W_n\}_{n \geq 0} \) a recursive sequence in \( \mathcal{H} \), whose initial data are \( W_0, \cdots, W_{s-1} \) and \( W_{n+1} = b_0 W_n + \cdots + b_s W_{n-s+1} \), for \( n \geq s - 1 \), where \( b_0, \cdots, b_{s-1} \) are fixed in \( \mathbb{R} \). Suppose that \( C_n = W_n \) then, for every \( n \geq p + s - 1 \), Equation (2) yields \( W_{n-j} = T_{n-j} - \sum_{k=0}^{p-1} a_k T_{n-k-j-1} \), for \( 0 \leq j \leq s - 1 \). And the substitution of these expressions in (2), implies that \( T_n \ (n \geq p + s - 1) \) satisfies the following linear recurrence equation of order \( p + s \),

\[
T_{n+1} = \sum_{j=0}^{p-1} a_j T_{n-j} + \sum_{j=0}^{s-1} b_j T_{n-j} - \sum_{j=0}^{s-1} \sum_{k=0}^{p-1} b_j a_k T_{n-j-k-1}.
\]

As a result, we get the following extension of the Linearisation Process of [2] (see Theorem 2.1 of [2]),

**Theorem 2.3** (Linearization Process). Let \( \{T_n\}_{n \geq 0} \) be a sequence of \( \mathcal{H} \) solution of (2) and suppose that \( C_n = W_n \), where \( \{W_n\}_{n \geq 0} \) is a homogeneous linear recursive sequence of order \( s \). Then, the sequence \( \{T_n\}_{n \geq 0} \) satisfies a homogeneous linear recursive equation of order \( m = r + s \) in \( \mathcal{H} \), with initial data \( T_0, \cdots, T_{p+s-1} \) and whose coefficients are those of the polynomial \( p(z) = p_1(z)p_2(z) \), where \( p_1(z) = z^p - \sum_{j=0}^{p-1} a_j z^{p-j-1} \) and \( p_2(z) = z^s - \sum_{j=0}^{s-1} b_j z^{s-j-1} \).

**Remark 2.4** For \( E = \mathbb{R} \), it is well known that the solution of (2) can be written under the form \( T_n = T_n^{<h>} + T_n^{<p>} \), where \( \{T_n^{<h>}\}_{n \geq 0} \) and \( \{T_n^{<p>}\}_{n \geq 0} \) are the solutions of the homogeneous part and the particular solution of (2) respectively. It is easy to verify that this property is still valid for sequences (2) in \( \mathcal{H} \). Following the similar method of [11], we can bring out that the particular solution of (2) is \( T_n^{<p>} = T_n - T_n^{<h>} \), for \( n \geq 0 \).
3 Study of the ARMA\((p, q)\) process equations

3.1 General setting.

In this section we are concerned by the Hilbert space \(\mathcal{H} = L^2(\Omega, \mathcal{A}, P)\), where \((\Omega, \mathcal{A}, P)\) is a probability space. That is, \(\mathcal{H}\) is the collection of the real-valued random variables (\(r.r.v.\) for short) \(X\) with finite variance, viz.,

\[
E(X^2) = \int X^2(\omega)P(d\omega) = \int X^2dP < +\infty.
\]

Here \(\mathcal{H}\) is equipped with the usual inner product \(<X, Y> = E(XY) = \int XYdP\). Expressions (5)-(6) and Theorem 2.2, allow us to formulate the result.

**Theorem 3.1** Let \(\{X_t \in \mathcal{H}; t \in \mathbb{Z}\}\) be an ARMA\((p,q)\) process and \(\{A_t \in \mathcal{H}; t \in \mathbb{Z}\}\) its associated white noise. Then, for every \(t \geq p\), we have

\[
X_t = \rho(t-p+1,p)X_{p-1} + [a_{p-2}\rho(t,p) + a_{p-1}\rho(t-1,p)]X_{p-2} + \cdots + (9)
\]

\[
\left[\sum_{j=0}^{s}a_{p-s+j-1}\rho(t-j,p)\right]X_{p-s+1} + \cdots + \left[\sum_{j=0}^{p-1}a_{j}\rho(t-j,p)\right]X_0 + \sum_{k=p}^{t}\rho(t-k,p)C_k,
\]

where \(X_0, X_1, \cdots, X_{p-1}\) are the \(r.r.v\) of the initial data, the \(\rho(n,p)\) are given by (4) and \(C_t = A_t + b_0A_{t-1} + \cdots + b_{q-1}A_{t-q}\).

Similarly, for \(t \leq 0\), we set \(Y_{p-t-1} = X_t\) and \(b_0 = -\frac{a_0}{a_{p-1}}, \cdots, b_j = -\frac{a_{p-j-1}}{a_{p-1}}, \cdots, b_{p-2} = -\frac{a_{p-2}}{a_{p-1}}, b_{p-1} = 1 / a_{p-1}\). Thus, we have

\[
X_t = \Delta_0X_0 + \Delta_1X_1 + \cdots + \Delta_{s-1}X_{s-2} + \cdots + \Delta_{p-1}X_{p-1} + \Gamma(-t), \quad (10)
\]

where \(\Delta_0 = \rho(t-p+1,p)\), \(\Delta_1 = b_{p-2}\hat{\rho}(-t + p - 1, p) + b_{p-1}\hat{\rho}(-t + p - 2, p)\), \(\Delta_{s-1} = \sum_{j=0}^{s}b_{p-s+j-1}\hat{\rho}(-t + p - 1 - j, p)\), \(\Delta_{p-1} = \sum_{j=0}^{p-1}b_j\hat{\rho}(-t + p - 1 - j, p)\) and \(\Gamma(-t) = \sum_{k=p}^{t}\hat{\rho}(-t + p - 1 - k, p)[\frac{-C_{k+2p-1}}{a_{p-1}}]\), with

\[
\hat{\rho}(n,p) = \sum_{k_0+2k_1+\cdots+p_k=n-p} \frac{(k_0 + \cdots + k_{p-1})!}{k_0! \cdots k_{p-1}!} b_0^{k_0} \cdots b_{p-1}^{k_{p-1}}.
\]

The combinatorial expressions (9)-(10) show that every \(r.r.v\). \(X_t\) (for \(t \geq p\)) can be expressed in terms of the initial \(r.r.v\). \(X_0, X_1, \cdots, X_{p-1}\) and \(A_t\) (\(t \in \mathbb{Z}\)) the \(r.r.v\). of the white noise. As far as we know (9)-(10) are not current in the literature on the ARMA\((p,q)\) models.
Example - Study of the ARMA(2,q) process. Consider \( \{X_t\}_{t \in \mathbb{Z}} \) an ARMA(2,q) process defined by the random vector of initial data \((X_0, X_1)\), and

\[
X_t = a_0 X_{t-1} + a_1 X_{t-2} + C_t,
\]

where \( C_{t+1} = A_{t+1} + b_1 A_t + \cdots + b_q A_{t-q+1} \). Let \( P(z) = z^2 - a_0 z - a_1 = (z - \lambda_1) (z - \lambda_2) \) be the characteristic polynomial of the homogeneous part of this process, where \( \lambda_1 \neq 0, \lambda_2 \neq 0 \) are in \( \mathbb{R} \). For studying the process (11), we discuss the two cases \( \lambda_1 \neq \lambda_2 \), and \( \lambda_1 = \lambda_2 \).

For \( \lambda_1 = \lambda_2 \), a direct computation shows that the Binet formula of \( \rho(t, 2) \) takes the form \( \rho(t, 2) = \lambda^{t-2}(-1 + t) \). Thereby, for every \( t \geq 2 \), we have

\[
X_t = (t - 2)\lambda^{t-3}X_1 + (2\lambda^{t-1}(-1 + t) - (t - 2)\lambda^{t-1})X_0 + \sum_{k=2}^{t} \lambda^{t-k-2}(t - k - 1)C_k.
\]

In the same way, for \( t \leq 0 \) we obtain \( \hat{\rho}(-t, 2) = (\frac{1}{\lambda})^{-t-2}(-1 - t) \) and thus

\[
X_t = \left(\frac{1}{\lambda}\right)^{-t-1}X_0 + [2(-1)^{-t}(\frac{1}{\lambda})^{-t}(-t) + (-1)^{-t}(\frac{1}{\lambda})^{-t}(-1 - t)]X_1 + \Gamma(t),
\]

where \( \Gamma(t) = \sum_{k=2}^{t} \lambda^{-t-k+1}(-t - k)C_{t-k} \).

For \( \lambda_1 \neq \lambda_2 \), a similar calculation leads to the Binet formula of \( \rho(n, 2) \) is

\[
\rho(n, 2) = \frac{\lambda_1^{n-1} - \lambda_2^{n-1}}{\lambda_1 - \lambda_2}.
\]

Hence, for \( t > 0 \), we get,

\[
X_t = \frac{\lambda_1^{t-2} - \lambda_2^{t-2}}{\lambda_1 - \lambda_2}X_1 + \left[(\lambda_1 + \lambda_2)(\frac{\lambda_1^{t-1} - \lambda_2^{t-1}}{\lambda_1 - \lambda_2}) + \lambda_1 \lambda_2 \frac{\lambda_1^{t-2} - \lambda_2^{t-2}}{\lambda_1 - \lambda_2}\right]X_0 + \Gamma(t),
\]

where \( \Gamma(t) = \sum_{k=2}^{t} \frac{\lambda_1^{t-k-1} - \lambda_2^{t-k-1}}{\lambda_1 - \lambda_2}C_k \). For \( t \leq 0 \), we have

\[
\hat{\rho}(-t, 2) = \frac{1}{(\lambda_1 \lambda_2)^{-t-2}} \frac{\lambda_2^{-t-1} - \lambda_1^{-t-1}}{\lambda_2 - \lambda_1}.
\]

Whence,

\[
X_t = \Lambda_1(t)X_0 + \Lambda_2(t)X_1 + \Gamma(t),
\]

where \( \Gamma(t) = \sum_{k=2}^{t} \frac{\lambda_2^{-t-k} - \lambda_1^{-t-k}}{\lambda_2 - \lambda_1} \left(\frac{1}{(\lambda_1 \lambda_2)^{-t-2}}\right)C_{t-k} \), \( \Lambda_1(t) = \frac{1}{(\lambda_1 \lambda_2)^{-t-2}} \left[\frac{\lambda_2^{-t-1} - \lambda_1^{-t-1}}{\lambda_2 - \lambda_1}\right] \), and \( \Lambda_2(t, \lambda_j) = \frac{\lambda_1 + \lambda_2}{(\lambda_1 \lambda_2)} \left[\frac{\lambda_2^{-t-1} - \lambda_1^{-t-1}}{\lambda_2 - \lambda_1}\right] + \left(\frac{-1}{(\lambda_1 \lambda_2)^{-t-2}}\right)\left[\frac{\lambda_2^{-t-1} - \lambda_1^{-t-1}}{\lambda_2 - \lambda_1}\right] \).

3.2 The ARMA\((p,q)\) process of Fibonacci type and linearization process.

We say that an ARMA\((p,q)\) process is of Fibonacci type if in (1) the \( C_{t+1} \) takes the form \( C_{t+1} = A_{t+1} + b_1 A_t + \cdots + b_{q-1} A_{t-q+1} \), where \( \{A_t\}_{t \in \mathbb{Z}} \) is the
r.r.v. of white noise. That is to say, \( \{A_t\}_{t \in \mathbb{Z}} \) satisfies a linear recursive relation of order \( q \) and with initial data \( A_0, \ldots, A_{q-1} \). With the aid of Theorem 2.2 and (8), the process \( \{X_t\}_{t \in \mathbb{Z}} \) satisfies the following linear recursive equations of order \( r = p + q \),

\[
X_{t+1} = (a_0 + b_0)X_t + \sum_{j=0}^{r_1-1} (a_j + b_j - c_j)X_{t-j} + \sum_{j=r_1}^{r_2-1} v_jX_{t-j} - \sum_{j=r_2}^{p+q-1} c_jX_{t-j}, \quad (12)
\]

where \( c_j = \sum_{k+s=j; k \geq 1, s \geq 0} b_{k-1}a_s \) and \( r_1 = \min(p, q) \), \( r_2 = \max(p, q) \) with \( v_j = a_j - c_j \) for \( p > q \), \( v_j = b_j - c_j \) for \( p < q \) and \( v_j = 0 \) for \( p = q \).

**Theorem 3.2** Let \( \{X_t \in \mathcal{H}; t \in \mathbb{Z}\} \) be an ARMA\((p,q)\) process of Fibonacci type and \( \{A_t \in \mathcal{H}; t \in \mathbb{Z}\} \) its associated white noise, with initial data \( X_0, \ldots, X_{p-1} \) and \( A_0, \ldots, A_{q-1} \). Then, the process \( \{X_t\}_{t \in \mathbb{Z}} \) satisfies a homogeneous linear recursive of order \( r = p + q \) in \( \mathcal{H} \), whose coefficients are those of the polynomial \( p(z) = p_1(z)p_2(z) \), where \( p_1(z) = z^p - \sum_{j=0}^{p-1} a_jz^{p-j-1} \) and \( p_2(z) = z^q - \sum_{j=0}^{q-1} b_jz^{q-j-1} \).

For reason of simplicity we suppose that \( p \geq q \), since the same argumentation still valid in the case of \( p \leq q \). It’s easy to see that Expression (12) takes the linear form \( X_{t+1} = \sum_{j=0}^{p+q-1} w_jX_{t-j} \), such that the coefficients \( w_j \), derived from Theorem 2.3, are given by

\[
w_0 = a_0 + b_0, \quad w_j = a_j + b_j - c_j \text{ for } 1 \leq j \leq q-1, \quad w_j = a_j - c_j \text{ for } q \leq j \leq p-1
\]

and \( w_j = c_j \) for \( p \leq j \leq p + q - 1 \),

and its initial data are

\[
R_j = X_j \text{ for } 0 \leq j \leq p - 1 \text{ and } R_j = X_j + A_j \text{ for } p \leq j \leq p + q - 1. \quad (14)
\]

Theorems 2.3, 3.2 and (3) yield,

**Theorem 3.3** Under the data of Theorem 3.2, the system \( \{X_t; t \in \mathbb{Z}\} \) obey to a linear recursive equation of order \( r = p + q \), whose coefficients are given by (13) and its initial data of r.r.v. by (14). Moreover, for every \( t \geq p + q \), we have \( X_t = \rho(t, p + q)S_0 + \rho(t - 1, p + q)S_1 + \cdots + \rho(t - p - q + 1, p + q)S_{p+q-1} \), where \( S_k = w_{p+q-1}X_k + \cdots + w_kX_{p+q-1} \) (0 \( \leq k \leq p + q - 1 \)) and as in (4) the \( \rho(n, p + q) \) are

\[
\rho(n, p + q) = \sum_{k_0 + 2k_1 + \cdots + (p+q)k_{p+q-1} = t-p-q} \frac{(k_0 + \cdots + k_{p+q-1})!}{k_0! \cdots k_{p+q-1}!} w_0^{k_0} \cdots w_{p+q-1}^{k_{p+q-1}},
\]

with \( \rho(n, n) = 1 \) and \( \rho(n, s) = 0 \) if \( n \leq s - 1 \).
Similarly, for \( t \leq 0 \), by setting \( Y_{p+q-t-1} = X_t \) and \( \hat{w}_0 = \frac{-w_0}{w_{p+q-1}}, \ldots, \hat{w}_j = \frac{-w_{p+j-1}}{w_{p+q-1}}, \hat{w}_{p+q-2} = \frac{-w_{p+q-2}}{w_{p+q-1}}, \hat{w}_{p+q-1} = \frac{1}{w_{p+q-1}} \), we obtain

\[
X_t = \rho(p + q - t - 1, p + q)\hat{S}_0 + \hat{\rho}(p + q - t - 2, p + q)\hat{S}_1 + \cdots + \hat{\rho}(-t, p + q)\hat{S}_{p+q-1},
\]

where \( \hat{S}_k = \hat{w}_{p+q}X_{p+q-k-1} + \cdots + \hat{w}_kX_0 \) (\( 0 \leq k \leq p + q - 1 \)) and

\[
\hat{\rho}(n, p + q) = \sum_{k_0 + 2k_1 + \cdots + (p+q)k_{p+q-1} = t-p-q} \frac{(k_0 + \cdots + k_{p+q-1})!}{k_0! \cdots k_{p+q-1}!} \hat{w}_0^{k_0} \cdots \hat{w}_{p+q-1}^{k_{p+q-1}},
\]

with \( \hat{\rho}(n, n) = 1 \) and \( \hat{\rho}(n, s) = 0 \) if \( n \leq s - 1 \).

**Example** - Consider an ARMA\((2, 2)\) process \( \{X_t; t \in \mathbb{Z}\} \) such that \( P_1(z) = (z - \lambda_0)(z - \lambda_1) \) and \( P_2(z) = (z - \lambda_2)(z - \lambda_3) \), where \( \lambda_0, \lambda_1, \lambda_2, \lambda_3 \) are real numbers satisfying \( \lambda_j \neq \lambda_k \) for \( j \neq k \). Hence, we have \( P(z) = (z - \lambda_0)(z - \lambda_1)(z - \lambda_2)(z - \lambda_3) \). Then, for each \( n \geq 0 \), the Binet formula of \( \rho(n, r) \) is

\[
\rho(n, r) = \Gamma_n(\lambda_j) + \Theta_n(\lambda_j),
\]

where \( \Gamma_n(\lambda_j) = \frac{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)\lambda_n^{n-1} - (\lambda_0 - \lambda_2)(\lambda_0 - \lambda_3)(\lambda_2 - \lambda_3)\lambda_n^{n-1}}{(\lambda_0 - \lambda_1)(\lambda_0 - \lambda_2)(\lambda_0 - \lambda_3)(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} \)

and \( \Theta_n(\lambda_j) = \frac{(\lambda_1 - \lambda_2)(\lambda_0 - \lambda_2)(\lambda_0 - \lambda_3)(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)\lambda_n^{n-1}}{(\lambda_0 - \lambda_1)(\lambda_0 - \lambda_2)(\lambda_0 - \lambda_3)(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} \).

Therefore, for \( t \geq 3 \), we obtain,

\[
X_t = \rho(t, 4)S_0 + \rho(t - 1, 4)S_1 + \rho(t - 2, 4)S_2 + \rho(t - 3, 4)S_3,
\]

with \( S_0 = \omega_3X_0 + \omega_2X_1 + \omega_1X_2 + \omega_0X_3, S_1 = \omega_3X_1 + \omega_2X_2 + \omega_1X_3, S_2 = \omega_3X_2 + \omega_2X_3, S_1 = \omega_3X_3, \) where \( \omega_0 = \lambda_0 + \lambda_1 + \lambda_2 + \lambda_3, \omega_1 = \prod_{i \neq j, i, j = 1} \lambda_i \lambda_j, \omega_2 = \prod_{i \neq j, k \neq i, k \neq j} \lambda_i \lambda_j \lambda_k \) and \( \omega_3 = \lambda_0 \lambda_1 \lambda_2 \lambda_3 \).

## 4 More on some characteristics of an ARMA\((p, q)\) process

In this section we are also concerned by the Hilbert space \( \mathcal{H} = L^2(\Omega, \mathcal{A}, P) \), where \( (\Omega, \mathcal{A}, P) \) is a probability space. Consider an ARMA\((p, q)\) process (1), whose white noise \( A_t \) \( (t \in \mathbb{Z}) \) is with zero mean \( E(A_t) = 0 \) and constant variance \( \text{var}(A_t) = \sigma^2 \). Thus the sequence of moments \( \{E(X_t)\}_{t \in \mathbb{Z}} \) satisfies the recursive equation \( E[X_{t+1}] = a_0E[X_t] + \cdots + a_{p-1}E[X_{t-p+1}] \). The Fibonacci sequence’s properties, specially the Binet Formula, permits us to get.

**Proposition 4.1** Under the preceding data, suppose that \( \lambda_1, \lambda_2, \cdots, \lambda_r \) are the distinct roots of the polynomial \( P(z) = z^p - a_0z^{p-1} - \cdots - a_{p-1} \), of multiplicities \( m_1, \cdots, m_p \) (respectively). Then, we have \( E[X_t] = \sum_{i=1}^r P_i(t)\lambda^i \) (Binet formula), for every \( t \geq 0 \), where each \( P_i(z) \) is a polynomial of degree \( m_i - 1 \) \((1 \leq i \leq r)\).
Following (3)-(4), we manage to have the combinatorial form of $E[X_t]$.

**Proposition 4.2** Under the data of Proposition 4.1, we have

$$E[X_t] = \Delta_{p-1}(t)E[X_{p-1}] + \cdots + \Delta_{p-s+1}(t)E[X_{p-1+s}] + \cdots + \Delta_0(t)E[X_0].$$

for every $t \geq p$, where $\Delta_{p-1}(t) = \rho(t-p+1, p)$, $\Delta_{p-s+1}(t) = \sum_{j=0}^{s} a_{p-s+j-1}\rho(t-j, p)$ and $\Delta_0(t) = \sum_{j=0}^{p-1} a_j \rho(t-j, p)$.

Furthermore, we study others characteristics of $X_t$ relying on the precedent results. We begin by the simple case when the $\{X_t; t \in \mathbb{Z}\}$ are mutually independent, the $A_t$ are also mutually independent and $X_{t-j}$ is independent of $A_t$ for $j, t \geq 0$. Thus, we have $\text{var}[X_{t+1}] = a_0^2 \text{var}[X_t] + \cdots + a_p^2 \text{var}[X_{t-p+1}] + (1 + b_0 + \cdots + b_{q-1})\sigma^2$, where $t \in \mathbb{Z}$. Therefore, the sequence $\{\text{var}[X_t]\}_{t \geq 0}$ is a sequence (2) of characteristic polynomial $Q(z) = (z-1)(z^p - a_0 z^{p-1} - \cdots - a_{p-1}^2), whose roots are 1, \lambda_1, \lambda_2, \cdots, \lambda_r$, with multiplicities are $1, m_1, \cdots, m_p$ (respectively). A direct application of the Binet formulas leads to derive the following result.

**Proposition 4.3** Under the data of Proposition 4.1, suppose that the $X_t$ are mutually independent, the $A_t$ mutually independent and $X_{t-i}$ is independent of $A_t$ for $t \geq 0$. Let $\lambda_1, \lambda_2, \cdots, \lambda_r$ be the distinct roots of the polynomial $P(z) = z^p - a_0 z^{p-1} - \cdots - a_{p-1}$, of multiplicities $m_1, \cdots, m_p$ (respectively). Then, we have $\text{var}[X_t] = \sum_{i=1}^{r} P_i(t)\lambda_i^2 + K$, where $K \in \mathbb{R}$ and each $P_i(z)$ ($1 \leq i \leq r$) is a polynomial of degree $m_i - 1$, whose coefficients are arisen from the initial conditions $\text{var}[X_0], \cdots, \text{var}[X_{p-1}], \text{var}[X_p]$, either $K$.

Now, we are going to discuss the general case without setting the independence condition of $\{X_t; t \in \mathbb{Z}\}$. To this aim, we release the combinatorial expression of $\text{var}[X_t]$ from Proposition 4.1, thereby we manage to have the following result.

**Proposition 4.4** Under the data of Proposition 4.1, suppose that the $A_t$ are mutually independent and $X_{t-i}$ is independent of $A_t$ for $t \geq 0$. Then, for every $t \geq p$, we have

$$\text{var}[X_t] = \sum_{i=0}^{p-1} \left( \sum_{j=0}^{p-1} \rho(t-j, p) \right)^2 \text{var}[X_t^2] + 2 \sum_{i<j}(i=1, j=2) \alpha_{ij}\text{cov}(X_{p-i}, X_{p-j}),$$

where $\alpha_{ij} = \left( \sum_{k=0}^{i} a_{p-i+k-1}\rho(t-k, p) \right) \left( \sum_{h=0}^{j} a_{p-j+h-1}\rho(t-h, p) \right)$.

When the r.r.v. $X_t (t \in \mathbb{Z})$ are mutually independent, we have the following corollary.
Corollary 4.5  Under the hypothesis of Proposition 4.4, suppose that \( \{X_t; t \in \mathbb{Z}\} \) are mutually independent. Then, we have

\[
\text{var}[X_t] = \sum_{i=0}^{p-1} \left( \sum_{j=0}^{p-1} \rho(t-j,p) \right)^2 \text{var}[X_t^2].
\]

We point out that we present, in Corollary 4.5, another expression of \( \text{var}[X_t] \).

5 Operator methods and recursiveness

In the real case, difference equations of Fibonacci type are also related to operator methods for studying the ARMA\((p,q)\) process. That is, the equation \( \phi(B)X_t = \theta(B)A_t \), where \( B \) is the auto-regressive operator, shows that \( X_t = \psi(B)A_t \) and \( A_t = \Pi(B)X_t \), where \( \psi(B) = \phi(B)^{-1}\theta(B) = 1 + \sum_{j=1}^{\infty} \psi_j B^j \) and \( \Pi(B) = \theta(B)^{-1}\phi(B) = 1 + \sum_{j=1}^{\infty} \pi_j B^j \) such that \( \sum_{j=1}^{\infty} \psi_j^2 < +\infty \) and \( \sum_{j=1}^{\infty} \pi_j^2 < +\infty \). For computing the \( \psi_j \), in terms of the \( a_j \) \((0 \leq j \leq p-1)\) and \( b_j \) \((0 \leq j \leq q-1)\), we use the equation \( \phi(B)\psi(B) = \theta(B) \). A straightforward computation shows that \( \psi_1 = a_0 - b_0 \), \( \psi_j = \sum_{k=0}^{j-1} a_k \psi_{j-k+1} + a_{j-1} - b_{j-1}(1 < j < p-1) \), \( \psi_p = \sum_{k=0}^{p-1} a_k \psi_{p-k+1} + a_{p-1} - b_{p-1}\epsilon_{p,q} \) (with \( \epsilon_{p,q} = 1 \) for \( p \leq q \) and \( \epsilon_{p,q} = 0 \) otherwise), and

\[
\psi_{n+1} = a_0\psi_n + \cdots + a_{p-1}\psi_{n-p+1}, \quad \text{for} \quad n \geq \text{max}(p, q) - 1. \tag{15}
\]

Similarly, for exhibiting the \( \pi_j \) in terms of the \( a_j \) \((0 \leq j \leq p-1)\) and \( b_i \) \((0 \leq i \leq q-1)\), a direct computation implies that \( \pi_1 = a_0 - b_0 \), \( \pi_j = \sum_{k=0}^{j-1} b_k \pi_{j-k+1} + a_{j-1} - b_{j-1}(1 < j < q-1) \), \( \pi_q = \sum_{k=0}^{q-1} a_k \psi_{p-k+1} + \epsilon_{p,q} a_{p-1} - b_{p-1} \) (with \( \epsilon_{p,q} = 1 \) for \( p \leq q \) and \( \epsilon_{p,q} = 0 \) otherwise), and

\[
\pi_{n+1} = b_0\pi_n + \cdots + b_{q-1}\pi_{n-q+1}, \quad \text{for} \quad n \geq \text{max}(p, q) - 1. \tag{16}
\]

The characteristic polynomial of (15) and (16) are \( P_1(z) = z^p - a_0 z^{p-1} - \cdots - a_{p-1} \) and \( P_2(z) = z^q - b_0 z^{q-1} - \cdots - b_{q-1} \). It’s easy to see that \( P_1(z) = z^p \phi(1/z) \) and \( P_2(z) = z^q \theta(1/z) \). Therefore, the stationary and invertible ARMA\((p,q)\) process can be described as follows.

Proposition 5.1  Let \( \{X_t; t \in \mathbb{Z}\} \) be an ARMA\((p,q)\) process, with associated white noise is \( \{A_t; t \in \mathbb{Z}\} \). Then, the following assertions are equivalent:

(i) \( \{X_t; t \in \mathbb{Z}\} \) is stationary (respectively invertible),

(ii) For every root \( \lambda \) of \( P_1(z) \) (respectively \( P_2(z) \) we have \( |\lambda| < 1 \). Moreover, we have \( \lim_{n \to +\infty} \psi_n = 0 \) and \( \lim_{n \to +\infty} \pi_n = 0 \).

The last affirmation of Proposition 5.1 is an immediate consequence of the Binet formula of (15) and (16) (see [6], [9], for example). Moreover, if we
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suppose that $a_j$ ($0 \leq j \leq p-1$) and $b_j$ ($0 \leq j \leq q-1$) are nonnegative, we can derive the asymptotic behavior of the two sequences $\{\psi_n\}_{n \geq 0}$ and $\{\pi_n\}_{n \geq 0}$, by using results of [6], [11].

References


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