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Some Estimates for the $3D$ Non-autonomous Linearization Brinkman-Forchheimer Equation with Singularly Oscillating

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Abstract

In this paper, we derive some estimates for the $3D$ non-autonomous linearization Brinkman-Forchheimer equation with singularly oscillating forces together with the averaged equation.

Keywords: linearization Brinkman-Forchheimer equation, singularly oscillating forces

1 Introduction

Let $\rho \in [0,1)$ be a fixed parameter, $\Omega \subset R^3$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$. We consider the 3D non-autonomous Brinkman-Forchheimer equation with singularly oscillating forces that governs the motion of fluid in a saturated porous medium:

$$u_t - \nu \triangle u + \alpha u + \beta |u| u + \gamma |u|^2 u + \nabla p = f_0(t,x) + \varepsilon^{-\rho} f_1(t/\varepsilon, x), \quad (1)$$

$$\nabla \cdot u = 0, \quad x \in \Omega, \quad (2)$$

$$u(t,x)|_{\partial\Omega} = 0, \quad (3)$$

$$u(\tau,x) = u_\tau(x), \ \tau \in R, \quad (4)$$

Here $u = u(t,x) = (u_1(t,x), u_2(t,x), u_3(t,x))$ is the velocity vector field, p is the pressure, $\nu > 0$ is the Brinkman kinematic viscosity coefficient, $\alpha > 0$ is the Darcy coefficient, $\beta > 0$ and $\gamma > 0$ are the Forchheimer coefficients.

Along with (1)-(4), we consider the averaged Brinkman-Forchheimer equation

$$u_t - \nu\triangle u + \alpha u + \beta|u|u + \gamma|u|^2 u + \nabla p = f_0(t,x), \quad x \in \Omega, \tag{5}$$
$$\nabla \cdot u = 0, \quad x \in \Omega, \tag{6}$$
$$u(t,x)|_{\partial\Omega} = 0, \tag{7}$$
$$u(\tau,x) = u_\tau(x), \ \tau \in R. \tag{8}$$

formally corresponding to the case $\varepsilon = 0$.

The function

$$f^\varepsilon(x,t) = \begin{cases} f_0(x,t) + \varepsilon^{-\rho} f_1(x, t/\varepsilon), & 0 < \varepsilon < 1, \\ f_0(x,t), & \varepsilon = 0 \end{cases} \tag{9}$$

represents the external forces of problem (1)-(4) for $\varepsilon > 0$ and problem (5)-(8) for $\varepsilon = 0$ respectively.

The functions $f_0(x,s)$ and $f_1(x,s)$ are taken from the space $L^2_b(R,H)$ of translational bounded functions in $L^2_{loc}(R,H)$, namely,

$$\|f_0\|^2_{L^2_b} := \sup_{t\in R} \int_t^{t+1} \|f_0(s)\|^2 ds = M_0^2, \tag{10}$$
$$\|f_1\|^2_{L^2_b} := \sup_{t\in R} \int_t^{t+1} \|f_1(s)\|^2 ds = M_1^2, \tag{11}$$

for some constants $M_0, M_1 \geq 0$. We denote

$$Q^\varepsilon = \begin{cases} M_0 + 2M_1\varepsilon^{-\rho}, & 0 < \varepsilon < 1, \\ M_0, & \varepsilon = 0. \end{cases}$$

As a straightforward consequence of (9), we have

$$\|f^\varepsilon\|_{L^2_b} \leq Q^\varepsilon. \tag{12}$$

Note that Q^ε is of the order $\varepsilon^{-\rho}$ as $\varepsilon \to 0^+$.

In this paper, we shall derive some estimates for the $3D$ non-autonomous linearization Brinkman-Forchheimer equation with singularly oscillating forces together with the averaged equation to arrive the convergence of corresponding equations.

2 Main Results and Discussion

Throughout this paper, C will stand for a generic positive constant, depending on Ω and some constants, but independent of the choice of the initial time $\tau \in R$ and t.

The Hausdorff semidistance in X from one set B_1 to another set B_2 is defined as $dist_X(B_1, B_2) = \sup_{b_1 \in B_1} \inf_{b_2 \in B_2} \|b_1 - b_2\|_X$.

$L^p(\Omega)(1 \leq p \leq +\infty)$ is the generic Lebesgue space, $H^s(\Omega)$ is the usual Sobolev space. We set $E := \{u | u \in (C_0^\infty(\Omega))^3, divu = 0\}$, H is the closure of the set E in $(L^2(\Omega))^3$ topology, V is the closure of the set E in $(H_0^1(\Omega))^3$ topology.

The problem (1)-(2) can be written as an abstract form

$$u_t + \nu Au + \alpha u + B(u) = \sigma(t, x), \tag{13}$$

$$divu = 0, \tag{14}$$

where the pressure p has disappeared by force of the application of the Leray-Helmholtz projection P, $B(u) = PF(u)$, $F(u) = \beta |u| u + \gamma |u|^2 u$. Clearly, system (13)-(14) is equivalent to (1)-(2).

The existence and uniqueness of global solution for the initial boundary value problem to (13)-(14) can be derived by standard method as in [8], [2] or [4]:

Theorem 2.1 *Assume $\sigma \in L^2_{loc}(R, H)$, $u_\tau \in H$, then problem (13)-(14) possesses a unique global solution $u(t, x)$ which satisfies*

$$u \in C([\tau, +\infty); H) \cap L^2(\tau, T; V) \cap L^4(\tau, T; (L^4(\Omega))^3). \tag{15}$$

Firstly, we shall consider the auxiliary linear equation with non-autonomous singularly oscillating external force for 3D non-autonomous Brinkman-Forchheimer equation and give some useful estimates in H and V.

Considering the linear equation for the 3D non-autonomous Brinkman-Forchheimer equation with singularly oscillating forces as

$$Y_t + \nu AY + \alpha Y = K(t), \quad Y|_{t=\tau} = 0, \tag{16}$$

we obtain the following theorems.

Theorem 2.2 *Assume $K \in L^2_{loc}(R, H)$, then problem (16) has a unique solution*

$$Y \in L^2((\tau, T); V) \cap C((\tau, T); H), \tag{17}$$

$$\partial_t Y \in L^2((\tau, T); V'). \tag{18}$$

Moreover, the following inequalities

$$\|Y(t)\|_V^2 \leq C\int_\tau^t e^{-C\nu(t-s)}\|K(s)\|_H^2 ds, \tag{19}$$

$$\int_t^{t+1}\|Y(s)\|_H^2 ds \leq C\Big(\|Y(t)\|_H^2+\int_t^{t+1}\|K(s)\|_H^2 ds\Big) \tag{20}$$

hold for every $t\geq\tau$ and some constant $C=C(\lambda)>0$, independent of the initial time $\tau\in R$.

Proof. Using Galerkin approximation method, we can deduce the existence of global solution, here we omit the details.

Then multiplying (16) by Y and AY respectively, using the Poincaré inequality we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|Y\|^2+\nu\lambda^{1/2}\|Y\|^2+\alpha\|Y\|^2 &= \frac{1}{2}\frac{d}{dt}\|Y\|^2+\nu\|\nabla Y\|^2+\alpha\|Y\|^2\\ &= (K(t),Y)\leq\frac{1}{\alpha}\|K(t)\|^2+\alpha\|Y\|^2\end{aligned} \tag{21}$$

and

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla Y\|^2+\nu\lambda^{1/2}\|\nabla Y\|^2+\alpha\|\nabla Y\|^2 &= \frac{1}{2}\frac{d}{dt}\|\nabla Y\|^2+\nu\|AY\|^2+\alpha\|\nabla Y\|^2\\ &= (K(t),AY)\leq\frac{1}{\alpha}\|K(t)\|^2+\alpha\|AY\|^2.\end{aligned} \tag{22}$$

By the Gronwall inequality, we can easily prove the lemma.

Setting $K(t,\tau)=\int_\tau^t k(s)ds,\ t\geq\tau,\ \tau\in R$, we have the following theorem.

Theorem 2.3 *Let $k\in L^2_{loc}(R,H)$. Assume that*

$$\sup_{t\geq\tau,\tau\in R}\Big\{\|K(t,\tau)\|_H^2+\int_t^{t+1}\|K(s,\tau)\|_H^2 ds\Big\}\leq l^2, \tag{23}$$

for some constant $l\geq 0$. Then the solution $Y(t)$ to the following Cauchy problem

$$Y_t+\nu AY+\alpha Y=k(t/\varepsilon),\quad Y|_{t=\tau}=0, \tag{24}$$

with $\varepsilon\in(0,1)$ satisfies the inequality

$$\|Y(t)\|_V^2+\int_t^{t+1}\|Y(s)\|_H^2 ds\leq Cl^2\varepsilon^2,\quad \forall\ t\geq\tau, \tag{25}$$

where constant $C>0$ is independent of K.

Proof. Noting that

$$K_\varepsilon(t) = \int_\tau^t k(s/\varepsilon)ds = \varepsilon \int_{\tau/\varepsilon}^{t/\varepsilon} k(s)ds = \varepsilon K(t/\varepsilon, \tau/\varepsilon), \tag{26}$$

we can derive the following estimates of $K_\varepsilon(t)$ from (23)

$$\sup_{t\geq\tau} \|K_\varepsilon(t)\|_H \leq l\varepsilon, \tag{27}$$

$$\begin{aligned}\int_t^{t+1} \|K_\varepsilon(s)\|_H^2 ds &= \varepsilon^2 \int_t^{t+1} \|K(s/\varepsilon, \tau/\varepsilon)\|_H^2 ds \\ &\leq C\varepsilon^2 \sup_{t\geq\tau} \left\{ \int_t^{t+1} \|K(s,\tau)\|_H^2 ds \right\} \\ &\leq Cl^2\varepsilon^2.\end{aligned} \tag{28}$$

From Theorem 2.2, we have

$$\begin{aligned}&\int_\tau^t e^{-C\nu(t-s)} \|K_\varepsilon(s)\|_H^2 ds \\ &\leq \int_{t-1}^t e^{C\nu(s-t)} \|K_\varepsilon(s)\|^2 ds + \int_{t-2}^{t-1} e^{C\nu(s-t)} \|K_\varepsilon(s)\|^2 ds + \cdots \\ &\leq \int_{t-1}^t \|K_\varepsilon(s)\|^2 ds + e^{-C\nu} \int_{t-2}^{t-1} \|K_\varepsilon(s)\|^2 ds + e^{-2C\nu} \int_{t-3}^{t-2} \|K_\varepsilon(s)\|^2 ds + \cdots \\ &\leq \left(1 + e^{-C\nu} + e^{-2C\nu} + \cdots\right) \|K_\varepsilon(s)\|_{L_b^2(R;H)}^2 \\ &\leq \frac{1}{\left(1 - e^{-C\nu}\right)} \|K_\varepsilon(s)\|_{L_b^2(R;H)}^2 \\ &\leq \frac{1}{\left(1 - e^{-C\nu}\right)} \sup_{t\geq\tau} \int_t^{t+1} \|K_\varepsilon(s)\|_H^2 ds \\ &\leq Cl^2\varepsilon^2.\end{aligned} \tag{29}$$

Hence, from the Poincaré inequality, (19)-(20) and (29), we derive

$$\|Y(t)\|_V^2 \leq Cl^2\varepsilon^2, \tag{30}$$

$$\begin{aligned}\int_t^{t+1} \|Y(s)\|_H^2 ds &\leq C \left(\|Y(t)\|_H^2 + \int_t^{t+1} \|K(s)\|_H^2 ds \right) \\ &\leq Cl^2\varepsilon^2.\end{aligned} \tag{31}$$

Integrating (24) with respect to time from τ to t, we see that $Y(t)$ is a solution to the problem

$$\partial_t Y(t) + \nu AY(t) + \alpha Y(t) = K_\varepsilon(t), \quad Y(t)|_{t=\tau} = 0, \tag{32}$$

such that we can deduce that

$$\begin{aligned} &\|Y(t)\|_H^2 + \|\nabla Y(t)\|_H^2 + \int_t^{t+1} \|Y(s)\|_H^2 ds \\ &= \|Y(t)\|_V^2 + \int_t^{t+1} \|Y(s)\|_H^2 ds \\ &\leq Cl^2\varepsilon^2 \end{aligned} \tag{33}$$

from (30) and (31). The proof for the Lemma is finished.

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