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The Semi Orlicz Space of χ^{π} of Analytic

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Abstract

In this paper we introduce a new class of sequence space namely the semi orlicz space of χ^{π} analytic. It is shown that the intersection of all semi orlicz space of χ^{π} of analytic is the semi orlicz space of χ^{π} of analytic.

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1 Introduction

A complex sequence, whose k^{th} term is x_k is denoted by $\{x_k\}$ or simply x. Let w be the set of all sequences $x = (x_k)$ and ϕ be the set of all finite sequences. Let l_∞, c, c_0 be the sequence spaces of bounded, convergent and null sequences $x = (x_k)$ respectively. In respect of l_∞, c, c_0 we have

$\|x\| = \overset{sup}{k} |x_k|$, where $x = (x_k) \in c_0 \subset c \subset l_\infty$. A sequence $x = \{x_k\}$ is said to be analytic if $sup_k \; |x_k|^{1/k} < \infty$. The vector space of all analytic sequences will be denoted by Λ. A sequence x is called entire sequence if $lim_{k\to\infty}|x_k|^{1/k} = 0$. The vector space of all entire sequences will be denoted by Γ.χ was discussed in

Kamthan [19]. Matrix transformation involving χ were characterized by Sridhar [20] and Sirajiudeen [21]. Let χ be the set of all those sequences $x = (x_k)$ such that $(k!|x_k|)^{1/k} \to 0$ as $k \to \infty$. Then χ is a metric space with the metric

$$d(x,y) = sup_k \left\{ (k!|x_k - y_k|)^{1/k} : k = 1,2,3,... \right\}$$

Orlicz [4] used the idea of Orlicz function to construct the space (L^M). Lindenstrauss and Tzafriri [5] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space l_M contains a subspace isomorphic to l_p ($1 \leq p < \infty$). Subsequently, the different classes of sequence spaces defined by Parashar and Choudhary [6], Mursaleen et at. [7], Bektas and Altin[8], Tripathy et al. [9], Rao and subramanian [10] and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in Ref [11].

Recall ([4], [11]) an Orlicz function is a function $M : [0,\infty) \to [0,\infty)$ which is continuous, non – decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$, and $M(x) \to \infty$ *as* $x \to \infty$. If the convexity of Orlicz function M is replaced by $M(x + y) \leq M(x) + M(y)$ then this function is called modulus function, introduced by Nakano [18] and further discussed by Ruckle [12] and Maddox [13] and many others.

An Orlicz function M is said to satisfy Δ_2 -condition for all values of u, if there exists a constant $K > 0$, such that $M(2u) \leq KM(u)$ ($u \geq 0$). The Δ_2-condition is equivalent to $M(lu) \leq KM(u)$ ($u \geq 0$). The Δ_2 – condition is equivalent to $M(lu) \leq KlM(u)$, for all values of u and for $l > 1$. Lindenstrauss and Tzafriri [5] used the idea of Orlicz function to construct Orlicz sequence space

$$l_M = \left\{ x \in w: \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, for\ some\ \rho > 0 \right\}$$

The space l_M with the norm

$$||x|| = inf \left\{ \rho > 0: \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p$, $1 \leq p < \infty$, *the space* l_M coincide with the classical sequence space l_p. Given a sequence $x = \{x_k\}$ its n^{th} section is the sequence $x^{(n)} = \{x_1, x_2, ... x_n, 0.0, ...\}$ $\delta^{(n)} =$ (0, 0,....., 1, 0, ...), 1 in the n^{th} place and zero's elsewhere; and $s^{(n)} =$ (0, 0,....., 1, -1, 0, ...), 1 in the n^{th} place, -1 in the $(n+1)^{th}$ place and zero's elsewhere. An FK – space (Frechet coordinate space) is a Frechet space which is made up of numerical sequences and has the property that the coordinate functional $p_k(x) = x_k$ ($k =$ 1,2,3,....) are continuous. We recall the following definitions [see [15]]

An FK space is a locally convex Frechet space which is made up of sequences and has the property that coordinate projections are continuous. An metric space (X,*d*) is said to have AK (or sectional convergence) if and only if d $(x^{(n)}, x) \to 0$ as $n \to \infty$.[see [15]] The space is said to have AD (or) be an AD space if ϕ is dense in X. We note that AK implies AD by [14].

If X is a sequence space, we define

(i) X' = the continuous dual of X.

(ii) $X^{\alpha} = \{a = (a_k) : \sum_{k=1}^{\infty} |x_k| < \infty$, *for each* $x \in X\}$;

(iii) $X^{\beta} = \{a = (a_k) : \sum_{k=1}^{\infty} a_k x_k$ *is convergent, for each* $x \in X\}$;

(iv) $X^{\gamma} = \{a = (a_k) : \sup_n |\sum_{k=1}^{\infty} a_k x_k| < \infty$, *for each* $x \in X\}$;

(v) *Let X be an FK – space* $\supset \phi$. Then $X^{f} = \{ f(\delta^{(n)}) : f \in X'\}$

$X^{\alpha}, X^{\beta}, X^{\gamma}$ *are called the* $\propto -$ *(or Kö the – T öeplitz) dual of X,* β - *(* or generalized K*ö* the – T*ö*eplitiz) dual of *X*, γ - dual of X. Note that $X^{\alpha} \subset X^{\beta} \subset X^{\gamma}$. If $X \subset Y$ then $Y^{\mu} \subset X^{\mu}$, for $\mu = \alpha, \beta$, or γ.

1.1 Lemma

(Sec (15, Theorem 7.27)). Let X be an *FK* – space $\supset \phi$. Then
(i) $X^{\gamma} \subset X^{f}$. (ii) If X has AK, $X^{\beta} = X^{f}$. (iii) If X has AD. $X^{\beta} = X^{\gamma}$

2 Definitions and Preliminaries

Let w denote the set of all complex sequences $x = (x_k)_{k=1}^{\infty}$ and $M : [0, \infty) \to [0, \infty)$ be an Orlicz function, or a modulus function. Let

$$\chi_M^{\pi} = \left\{ x \in w : \lim_{k \to \infty} \left(M\left(\frac{k!|x_k|^{1/k}}{\pi_k^{1/k} \rho} \right) \right) = 0 \text{ for some } \rho > 0 \right\}$$

$$\Gamma_M^{\pi} = \left\{ x \in w : \lim_{k \to \infty} \left(M\left(\frac{|x_k|^{1/k}}{\pi_k^{1/k} \rho} \right) \right) = 0 \text{ for some } \rho > 0 \right\}$$

$$\Lambda_M^{\pi} = \left\{ x \in w : \sup_k \left(M\left(\frac{|x_k|^{1/k}}{\pi_k^{1/k} \rho} \right) \right) < \infty \text{ for some } \rho > 0 \right\}$$

The space χ_M^{π} is a metric space with the metric

$$d(x,y) = \inf \left\{ \rho > 0 : \sup_k \left(M\left(\frac{k!|x_k - y_k|^{1/k}}{\pi_k^{1/k} \rho} \right) \right) \le 1 \right\}$$

The space Γ_M^π and Λ_M^π is a metric space with the metric

$$d\,(x,y) = inf\left\{\rho > 0: sup_k\left(M\left(\frac{|x_k - y_k|^{1/k}}{\pi_k^{1/k}\,\rho}\right)\right) \leq 1\right\}$$

In this chapter we define semi orlicz space of χ^π of analytic, and hence show that Γ_M^π is smallest semi Orlicz space of x of χ^π of analytic.

3 Main Results

3.1 Proposition

$\chi_M^\pi \subset \Gamma_M^\pi$

Proof : It is easy. Therefore omit the proof.

3.2 Proposition

χ_M^π has AK, where M is a modulus function

Proof :

Let $x = \{x_k\} \in \chi_M^\pi$, then $\left\{M\left(\frac{k!|x_k|^{1/k}}{\pi_k^{1/k}\,\rho}\right)\right\} \in \chi$ and hence

$sup_{k \geq n+1}\left\{M\left(\frac{k!|x_k|^{1/k}}{\pi_k^{1/k}\,\rho}\right)\right\} \to 0\ as\ n \to \infty$. Therefore

$$d\quad \left(x, x^{[n]}\right) = inf\left\{\rho > 0: sup_{k\geq n+1}\left(M\left(\frac{k!|x_k|^{1/k}}{\pi_k^{1/k}\,\rho}\right)\right) \leq 1\right\} \to 0\ as\ n \to \infty \Rightarrow x^{[n]} \to x\ as\ n \to \infty$$

implying that χ_M^π has AK. This completes the proof.

3.3 **Proposition**

$(\chi_M^\pi)^\beta = \Lambda$

Proof : Step 1 : $\chi_M^\pi \subset \Gamma_M^\pi$ by proposition 3.1

$\Rightarrow (\Gamma_M^\pi)^\beta \subset (\chi_M^\pi)^\beta$. But $(\Gamma_M^\pi)^\beta = \Lambda$. (see [22])

$$\Lambda \subset (\chi_M^\pi)^\beta$$

Step 2: Let $y \in (\chi_M^\pi)^\beta$. But $f(x) = \sum_{k=1}^{\infty} x_k y_k$, with $x \in \chi_M^\pi$. *We recall that* s^k *has* $\left(\frac{1}{k!}\right)$ *in the* k^{th} *place and zero's elsewhere, with.*

$$x = S^k, \left\{ M\left(\frac{k!|x_k|^{1/k}}{\pi_k^{1/k}\,\rho} \right) \right\} = \left\{ 0,0,\ldots.\left(M\left(\frac{(0)^{1/k}}{\pi_k^{1/k}\,\rho} \right), 0, \ldots. \right) \right\} \textit{ which}$$

converges

to zero. Therefore $s^k \in \chi_M^\pi$. Hence $d(s^k, 0) = 1$ But $|y_k| \leq ||f||\, d(s^k, 0) < \infty$ for all k. Thus (y_k) is a bounded sequence and hence an analytic sequence, In other word $y \in \Lambda$.

$$(\chi_M^\pi)^\beta = \Lambda$$

From (1) and (2) we obtain $(\chi_M^\pi)^\beta = \Lambda$. This completes the proof.

3.4 Lemma

[15, theorem 8.6.1] $Y \supset X \Leftrightarrow Y^f \subset X^f$ where X is an AD – space and Y an FK – space.

3.5 Proposition

Let Y be any FK – space $\supset \phi$. Then $Y \supset \chi_M^\pi$ if and only if the sequence $s^{(k)}$ is weakly analytic.

Proof : The following implications establish the result.

$Y \supset \chi_M^\pi \Leftrightarrow Y^f \subset (\chi_M^\pi)^f$, *Since* χ_M^π has AD by Lemma 3.4

$\Leftrightarrow Y^f \subset \Lambda$, since $(\chi_M^\pi)^f = \Lambda$

$\Leftrightarrow$ for each $f \in Y$, the topological dual of Y.

$\Leftrightarrow f(s^{(k)})$ is analytic

$\Leftrightarrow s^{(k)}$ is weakly analytic. This completes the proof.

4 Properties of Semi orlicz space of χ^π of analytic

4.1 Definition

An FK-Space X is called "Semi Orlicz space of χ^π of analytic" if its dual $(X)^f \subset \Lambda$. In other

words X is semi Orlicz space of χ^π of analytic if $f(s^{(k)}) \in \Lambda$ for all $f \in (X)$ for each fixed k.

4.2 Example

χ_M^π is semi Orlicz space of χ^π of analytic. Indeed, If χ_M^π is the space of all Orlicz sequence of χ^π, then by Lemma 4.3 $\left(\chi_M^\pi\right)^f = \Lambda$

4.3 Lemma

$$\left(\chi_M^\pi\right)^f = \Lambda$$

Proof $\left(\chi_M^\pi\right)^\beta$ = Λby proposition 3.3 But (χ_M^π) has AK by proposition 3.2 Hence $\left(\chi_M^\pi\right)^\beta = \left(\chi_M^\pi\right)^f$ *Therefore* $\left(\chi_M^\pi\right)^f = \Lambda$. This completes the proof. We recall.

4.4 Lemma

(See 15 Theorem 4.3.7) Let *z* be a sequence. The (z^β, *P*) is an AK space with P = (P_k *: k = 0, 1, 2, ...) where* P_0 *(x) =* $\sup_m |\sum_{k=1}^m z_k x_k|$,$P_n(x) = |x_n|$. *For any k such that* $z_k \neq 0$, P_k *may be omitted. If z* $\in \phi$, P_0 *may be omitted.*

4.5 Proposition

Let *z* be a sequence. z^β is semi orlicz space of χ^π of analytic if and only if *z* is Λ

Proof : Step 1 : Suppose that z^β is semi Orlicz space of χ^π of analytic. z^β has AK by Lemma 4.4 Therefore $z^{\beta\beta} = \left(z^\beta\right)^f$ by theorem 7.2.7 of wilansky

[15} So z^β is semi Orlicz space of χ^π of analytic if and only if $z^{\beta\beta} \subset \Lambda$. But $z \in z^{\beta\beta} \subset \Lambda$.Hence z is Λ.

Step2: Conversely suppose that z is Λ. Then $z^\beta \supset \{\Lambda\}^\beta$ and $z^{\beta\beta} \subset \{\Lambda\}^{\beta\beta} = \Gamma^\beta = \Lambda$, because $\Lambda^\beta = \Gamma$ But $\left(z^\beta\right)^f = z^{\beta\beta}$. Hence $\left(z^\beta\right)^f \subset \Lambda$. Therefore z^β is semi Orlicz space of χ^π of analytic. This completes the proof.

4.6 Proposition

Every semi Orlicz space of χ^π *of analytic contained* Γ_M^π

Proof : Let *X* be any semi orlicz space of χ^π *of analytic. Hence* $(X)^f \subset \Lambda$.

Therefore f ($s^{(k)}$) $\in \Lambda$ for all $f \in$ (X). So, {$s^{(k)}$} is weakly analytic with respect to *X.*

Hence $X \supset \chi_M^\pi$ by Proposition 3.5. But $\chi_M^\pi \subset \Gamma_M^\pi$. Hence $X \subset \Gamma_M^\pi$ .This completes the proof.

4.7 Proposition

The intersection of all semi Orlicz space of χ^π*of analytic*

{X_n *: n* = 1, 2,.....} is semi orlicz space of χ^π*of analytic.*

Proof : Let $X = \cap_{n=1}^\infty X_n$. *Then X* is an FK space which contains ϕ. Also every $f \in (X)$ can be

written as $f = g_1 + g_2 + \ldots + g_m$, where $g_k \in (X_n)$ for some n and for $1 \leq k \leq m$. But f (s^k) = $g_1(s^k) + g_2(s^k) + \ldots + g_m(s^k)$. Since X_n *(n = 1,2....) are semi orlicz space of* χ^π *of analytic, if follows that* $g_i(s^k) \in \Lambda$ for all *i* =1,2,....*m*. Therefore f (s^k) $\in \Lambda$ for all *k* and and for all *f*. Hence *X* is semi Orlicz space of χ^π of analytic. This completes the proof.

4.8 Proposition

The intersection of all semi orlicz space χ^π of analytic Γ_M^π

Proof: Let I be the intersection of all semi Orclicz space of χ^π of analytic. By Proposition 2.14 we see that the intersection

$I \subset \cap \ \{z^{\beta} : z \in \Lambda\} = \{\Lambda\}^{\beta} = \Gamma = \Gamma_M^{\pi}$

By proposition 2.16 it follows that I is semi Orlicz space of χ^π of analytic consequently

$\Gamma_M^{\pi} \subset I$ (by Proposition 4.6)

From (3) and (4) we get I = Γ_M^{π}. This completes the proof.

4.9 Corollary

The smallest semi orlicz space of χ^π of analytic is Γ_M^{π}.

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