On the Basis Property of Eigenfunction
of the Frankl Problem with Nonlocal Parity Conditions
of the Third Kind

A. Sameripour and A. Ghaedrahmati

Math. Department, Lorestan University
Khoram Abad, Lorestan, Iran

Copyright © 2014 A. Sameripour and A. Ghaedrahmati. This is an open access article distributed under the
Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any
medium, provided the original work is properly cited.

Abstract

In the present paper, we obtain the eigenvalues and eigenfunctions of the Frankl problem
with a nonlocal parity condition of the third kind. We prove the minimalist, the completeness and
Riesz basis of the eigenfunctions corresponding to the eigenvalues of the problem in the space
$L_2(D_+)$.

1. INTRODUCTION

The primary Frankl problem was inquired in [1]. The problem with a nonlocal boundary
condition of the second kind was studied in [2]. In the present paper, we assume boundary
conditions of the third kind that when $y$ is limited to zero and also in $x = o \ x = o$ the function values
are linearly dependent in the elliptic and Hyperbolic regions. In the proof of principal theorem we
investigate the minimalist, the completeness and Riesz basis of a specified system of cosines.

Definition 1. System $\{x_n\}_{n \in N} \subset X$ is called complete in $X$ if $L[\{x_n\}_{n \in N}] = X$.

Definition 2. System $\{x_n\}_{n \in N} \subset X$ is called minimal in $X$ if $x_k \notin L[\{x_n\}_{n \in N}], \forall k \in N$. 
Remark. If the system \( \{x_n\}_{n \in \mathbb{N}} \) is minimal in \( L_p(I) \), then it is also minimal in \( L_p(J) \) for \( J \supset I \); and if it is complete in \( L_p(I) \), then it is also complete in \( L_p(J) \) for \( J \subset I \).

2. THE FRANKL PROBLEM WITH NONLOCAL CONDITION OF THE THIRD KIND

The Frankl problem is to seek a solution for equation
\[
u_x + \text{sgn}(y)u_y + \mu^2 \text{sgn}(x + y)u = 0
\]
in \( D_1 \cup D_2 \cup D_3 \) with the boundary conditions
\[
u(1, \theta) = 0, \quad \theta \in [0, \frac{\pi}{2}]
\]
\[rac{\partial u}{\partial x} (0, y) = 0, \quad y \in (-1, 1)
\]
\[rac{\partial u}{\partial y} (x, +0) = \frac{\partial u}{\partial y} (x, -0)
\]
\[\kappa u(0, y) = u(0, -y), \quad y \in [0, 1]
\]
\[\kappa u(x, +0) = u(x, -0)
\]
The function \( u(x, y) \in C^0(D_1 \cup D_2 \cup D_3) \cap C^2(D_1) \cap C^2(D_2) \). which areas \( D_1, D_2 \) and \( D_3 \) are defined as follows:
\[
D_1 = \left\{ (r, \theta) : \quad 0 < r < 1, \quad 0 < \theta < \frac{\pi}{2} \right\}
\]
\[
D_2 = \left\{ (x, y) : \quad -y < x < y + 1, \quad \frac{1}{2} < y < 0 \right\}
\]
\[
D_3 = \left\{ (x, y) : \quad x - 1 < y < -x, \quad 0 < x < \frac{1}{2} \right\}
\]

Theorem 1. The eigenvalues and eigenfunctions of problem (1)-(6) show by two series. In the first series, the eigenvalues \( \lambda_{nk} = \mu_{nk}^2 \) are found from the equation
\[
J_{\frac{\lambda_k}{\mu}}(\mu_{nk}) = 0
\]
such that \( n = 0, 1, 2, \ldots, k = 1, 2, \ldots \) and the \( J_{\frac{\lambda}{\mu}} (z) \) are the Bessel functions [3, p. 12], and the eigenfunctions are provided by the regulations
\[
u_{nk}(r, \theta) = A_{nk} J_{\frac{\lambda_k}{\mu}}(\mu_{nk} r) \cos n\left(\frac{\pi}{2} - \theta\right) \quad \text{in} \quad D_1.
Frankl problem with nonlocal parity conditions

\[
u_{nk} (\rho,\psi) = \kappa A_n J_{4n} (\mu \rho) \cosh 4n \psi \quad \text{in} \quad D_{-1}
\]
\[
u_{nk} (R,\varphi) = \kappa A_n J_{4n} (\mu R) \cosh 4n \varphi \quad \text{in} \quad D_{-2}
\]

that we use of polar coordinate system

\[r^2 = x^2 + y^2 \quad x = r \cos \theta \quad y = r \sin \theta\]

for \(0 \leq \theta \leq \frac{\pi}{2}\) and \(0 \leq r \leq 1\) in \(D_{+}\),

of cartesian coordinate system

\[\rho^2 = x^2 - y^2 \quad x = \rho \cosh \psi \quad y = \rho \sinh \psi\]

for \(-\infty < \psi < 0\) and \(0 < \rho < 1\) in \(D_{-1}\), and

\[R^2 = y^2 - x^2 \quad x = R \sin \varphi \quad y = -R \cosh \varphi\]

for \(0 < \varphi < \infty\) and \(0 < R < 1\) in \(D_{-2}\). In the second series, the eigenvalues \(\tilde{\lambda}_{nk} = \tilde{\mu}_{nk}^2\) are resulted from the equation

\[J_{\mu(n+\Delta)} (\tilde{\mu}_{nk}) = 0 \quad n = 0,1,... \quad k = 1,2,...\]

and the eigenfunctions are determined by the relations

\[\tilde{u}_{nk} (r,\theta) = \tilde{A}_{nk} J_{\mu} (\tilde{\mu}_{nk} r) \cos \tilde{\alpha}_n (\frac{\pi}{2} - \theta) \sin D_{+}
\]
\[\tilde{u}_{nk} (\rho,\psi) = \tilde{A}_{nk} J_{\mu} (\tilde{\mu}_{nk} \rho) (\kappa^2 - 1) \cosh \tilde{\alpha}_n \psi - 2\kappa \cosh \tilde{\alpha}_n \psi \quad \text{in} D_{-1}
\]
\[\tilde{u}_{nk} (R,\varphi) = \kappa \tilde{A}_{nk} J_{\mu} (\tilde{\mu}_{nk} R) \cosh \tilde{\alpha}_n \varphi \quad \text{in} D_{-2}
\]

where;

\[\tilde{\alpha}_n = 4(n + \Delta) \quad , \quad \Delta = \frac{1}{\pi} \arcsin \frac{\kappa}{\sqrt{1 + \kappa^2}} \quad , \quad \Delta \in (-\frac{1}{2}, \frac{1}{2})\]

Theorem 2. The system of functions

\[\left\{ \cos 4n (\frac{\pi}{2} - \theta) \right\}_{n=0}^{\infty} \quad , \quad \left\{ \cos 4(n + \Delta)(\frac{\pi}{2} - \theta) \right\}_{n=1}^{\infty}\]

is complete and a Riesz basis in the space \(L_2 (0, \frac{\pi}{2})\) for \(\Delta \in (-\frac{1}{4}, \frac{1}{2})\).

for \(\Delta < -\frac{1}{4}\) the system is not complete but is minimal, for \(\Delta > \frac{3}{4}\) is complete but is not minimal, and if \(\Delta = -\frac{1}{4}\) is complete and minimal.

Proof. The proof of this theorem we use the convergence function

\[f(\theta) = \sum_{n=0}^{\infty} A_n \cos 4n (\frac{\pi}{2} - \theta) + \sum_{n=1}^{\infty} B_n \cos 4(n + \Delta)(\frac{\pi}{2} - \theta)\]
in \( L^2(0, \frac{\pi}{2}) \), Riesz basis the system \{\sin(n + \Delta)(\pi - 4\theta)\}_{n=0}^{\infty}\) for \(\Delta \in (-\frac{1}{4}, \frac{3}{4})\) and [3].

**Theorem 3.** The system eigenfunction

\[
\begin{align*}
    u_{nk}(r, \theta) &= A_{nk} J_{2n}(\mu_{nk} r) \cos n(\frac{\pi}{2} - \theta) \\
    \tilde{u}_{nk}(r, \theta) &= \tilde{A}_{nk} J_{2n}(\tilde{\mu}_{nk} r) \cos \tilde{n}(\frac{\pi}{2} - \theta)
\end{align*}
\]

is complete and basis in the space \( L^2(D_+) \), therefore

\[
\begin{align*}
    \int_{D_+} f(r, \theta) u_{nk}(r, \theta) r d\theta dr &= 0, \\
    \int_{D_+} f(r, \theta) \tilde{u}_{nk}(r, \theta) r d\theta dr &= 0
\end{align*}
\]

and \( f \in L^2(D_+) \) then \( f = 0 \) in \( D_+ \).

**Proof.** Using fubini theorem and Lebesgue’s integral for any \( n, k = 1, 2, \ldots \) we have

\[
0 = \int_{D_+} f(r, \theta) u_{nk}(r, \theta) r d\theta dr = \int_0^\pi \left( r J_{2n}(\mu_{nk} r) \int_0^\pi f(r, \theta) \cos n(\frac{\pi}{2} - \theta) d\theta \right) dr
\]

again since \( f \in L^2(D_+) \) so;

\[
\int_0^\pi \left( \int_0^\pi r f(r, \theta) \cos n(\frac{\pi}{2} - \theta) d\theta \right) dr < \infty
\]

Inasmuch system \( \{\sqrt{r} J_{2n}(\mu_{nk} r)\}_{k=1}^{\infty} \) in \( L^2(0,1) \) is orthogonal and complete, it is enough to prove;

\[
\sqrt{r} \int_0^\pi f(r, \theta) \cos n(\frac{\pi}{2} - \theta) d\theta \in L^2(0,1)
\]

Using the Holder inequality

\[
\left( \int_0^\pi f(r, \theta) \cos n(\frac{\pi}{2} - \theta) d\theta \right)^2
\]
Frankl problem with nonlocal parity conditions

\[ \int_0^\infty \left| \sqrt{r} \left( \int_0^\pi f^2(r, \theta) d\theta \right) \right|^2 \frac{1}{2} \left( \int_0^\pi \cos^2 4n \left( \frac{\pi}{2} - \theta \right) d\theta \right) \right]^{\frac{1}{2}} \]

\[ = r \left( \int_0^\pi f^2(r, \theta) d\theta \right) \left( \int_0^\pi \cos^2 4n d\theta \right) \left( \int_0^\pi \frac{1 + \cos 8n \theta}{2} d\theta \right) \]

\[ < r \left( \int_0^\pi f^2(r, \theta) d\theta \right) \left( \int_0^\pi \frac{1 + \cos 8n \theta}{2} d\theta \right) \]

\[ < \frac{1}{2} r \left( \int_0^\pi f^2(r, \theta) d\theta \right) \left( \int_0^\pi \frac{1 + \cos 8n \theta}{2} d\theta \right) \]

\[ = \frac{\pi}{4} r \left( \int_0^\pi f^2(r, \theta) d\theta \right) \left( \int_0^\pi \frac{1 + \cos 8n \theta}{2} d\theta \right) \]

with the integration interval \((0,1)\).

\[ \int_0^\infty \left| \sqrt{r} \left( \int_0^\pi f(r, \theta) \cos 4n \left( \frac{\pi}{2} - \theta \right) d\theta \right) \right|^2 \frac{1}{2} \left( \int_0^\pi d\theta \right) \right]^{\frac{1}{2}} \]

Thus

\[ \int_0^\infty \left| \sqrt{r} \left( \int_0^\pi f(r, \theta) \cos 4n \left( \frac{\pi}{2} - \theta \right) d\theta \right) \right|^2 \frac{1}{2} \left( \int_0^\pi d\theta \right) \right]^{\frac{1}{2}} < \infty \]

This inequality is equivalent to

\[ \left\{ \int_0^\infty \left| \sqrt{r} \left( \int_0^\pi f(r, \theta) \cos 4n \left( \frac{\pi}{2} - \theta \right) d\theta \right) \right|^2 \right\}^{\frac{1}{2}} < \infty \]

Also system \(\{\sqrt{r} J_{4n}(\mu_{nk} r)\}\) is orthogonal and complete for \(k = 1, 2, \ldots\) in \(L^2(0,1)\) of relation

\[ \int_0^\infty \left( \sqrt{r} J_{4n}(\mu_{nk} r) \sqrt{r} \left( \int_0^\pi f(r, \theta) \cos 4n \left( \frac{\pi}{2} - \theta \right) d\theta \right) \right) dr = 0 \]
A. Sameripour and A. Ghaedrahmati

imply that
\[
\sqrt{r} \int_0^\pi f(r, \theta) \cos 4n\left(\frac{\pi}{2} - \theta\right) d\theta = 0
\]

According to theorem 2, we conclude that \( f(r, \theta) = 0 \) in \( L^2(0,1) \). Similarly, if we consider the above calculations for sequence \{\cos[4(n + \Delta)]\left(\frac{\pi}{2} - \theta\right)\} for \( n = 1,2,... \) we have;

\[
\sqrt{r} \int_0^\pi f(r, \theta) \cos[4(n + \Delta)]\left(\frac{\pi}{2} - \theta\right) d\theta = 0
\]

Because completeness \( [4(n+\Delta)](2-)_n=0^\Lambda, f(r, \theta) = 0 \) in \( L^2(0,1) \). The proof of the theorem is complete.

**Theorem 3.** The system of eigenfunctions \( u_{nk} \) and \( \tilde{u}_{nk} \) of the problem (1)-(6) is a Riesz basis in the space \( L^2(D_e) \) where,

\[
A_{nk} = \left( \int_0^1 J_{4n}(\mu_{nk} r) r dr \right)^{-1}
\]

\[
\tilde{A}_{nk} = \left( \int_0^1 J_{4n}(\mu_{nk} r) \tilde{r} dr \right)^{-1}
\]

**Proof.** Theorem 3 results from Theorem 2 and the completeness and orthogonality of the system \( \{A_{nk} \sqrt{r} J_{4n}(\mu_{nk} r)\}_{k=1}^{\infty} \) for \( n \neq 0 \) and \( \{\tilde{A}_{nk} \sqrt{r} J_{4n}(\mu_{nk} r)\}_{k=1}^{\infty} \) for \( n \neq 1 \) in \( L^2(0,1) \).

**REFERENCES**


Received: December 11, 2013