On a Multisection Style Binomial Summation Identity for Fibonacci Numbers

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Abstract

A lattice path enumeration approach is exploited in order to derive a binomial summation formula for the number of paths in the lattice \((0, 1, \ldots, d)\) with start 0 and \(n\) steps. This approach reveals multisection style binomial summation identities and, particularly, a novel relationship between Fibonacci numbers and the Pascal triangle.

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1 Introduction

It is well known that the generation rule \(F_{k+2} = F_{k+1} + F_k, \ F_1 = 1, \ F_2 = 1\) of the Fibonacci numbers \(F_k, \ k = 1, 2, \ldots\) can be represented in matrix form by

\[
\begin{pmatrix}
F_{k+1} \\
F_k
\end{pmatrix}
= \begin{pmatrix}
1 & 1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
F_k \\
F_{k-1}
\end{pmatrix}.
\]

(1)

Equation (1) leads to the relation

\[
\begin{pmatrix}
F_{k+1} \\
F_k
\end{pmatrix}
= \begin{pmatrix}
1 & 1 \\
1 & 0
\end{pmatrix}^k
\begin{pmatrix}
F_k \\
F_{k-1}
\end{pmatrix}.
\]

(2)

for \(k \in \mathbb{N}\), from which a variety of Fibonacci properties can be derived, see, e.g., [4, 5, 8].
This paper is motivated by a different matrix approach, which is motivated by the following observation. Let us denote by $\mathbf{e}_i$ the $i$-th unit vector and by $\mathbf{1}$ the vector $(1,1,1,1)^T$. The sequence $\mathbf{a}_k = (a_1^{(k)}, a_2^{(k)}, a_3^{(k)}, a_4^{(k)})^T \in \mathbb{N}_0^4$ given by $\mathbf{a}_0 = (1,0,0,0)^T$ and

$$
a_1^{(k+1)} = a_2^{(k)}, a_2^{(k+1)} = a_1^{(k)} + a_3^{(k)}, a_3^{(k+1)} = a_2^{(k)} + a_4^{(k)}, a_4^{(k+1)} = a_3^{(k)}
$$

implies $a_3^{(2k)} = a_2^{(2k-1)} + a_4^{(2k-1)} = a_2^{(2k-1)} + a_3^{(2k-2)}$, and $a_2^{(2k+1)} = a_1^{(2k)} + a_3^{(2k)} = a_3^{(2k)} + a_2^{(2k-1)}$ for $k \in \mathbb{N}$. Hence, $F_{2k} = a_3^{(2k)}$, $F_{2k-1} = a_2^{(2k-1)}$. (3) is equivalent to the matrix equation $\mathbf{a}_{k+1} = \mathbf{Qa}_k$, where

$$
\mathbf{Q} = (q_{i,j}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \end{pmatrix}.
$$

The elements of the power $\mathbf{Q}^k = (q_{i,j}^{(k)})$ obey the symmetry relation ($i, j \in \{1, 2, 3, 4\}$)

$$
q_{i,j}^{(k)} = q_{5-i,5-j}^{(k)}.
$$

For a proof see the Appendix A. Note, that $\mathbf{a}_k = \mathbf{Q}^k \mathbf{a}_0$. Particularly, we have $\mathbf{a}_0 = \mathbf{e}_1$, $\mathbf{a}_1 = \mathbf{e}_2$. This, together with (5) implies

$$
\mathbf{Q}^{(2k-1)} = \begin{pmatrix} 0 & F_{2k-1} & 0 & F_{2k-2} \\
F_{2k-1} & 0 & F_{2k} & 0 \\
0 & F_{2k} & 0 & F_{2k-1} \\
F_{2k-2} & 0 & F_{2k-1} & 0 \end{pmatrix},
$$

$$
\mathbf{Q}^{(2k)} = \begin{pmatrix} F_{2k-1} & 0 & F_{2k} & 0 \\
0 & F_{2k+1} & 0 & F_{2k} \\
F_{2k} & 0 & F_{2k+1} & 0 \\
0 & F_{2k} & 0 & F_{2k-1} \end{pmatrix}.
$$

Independently from whether $k$ is even or odd, (6) implies that $\mathbf{1}^T \mathbf{Q}^k \mathbf{e}_1 = F_{k+1}$.

As pointed out by [6, 10], the Toeplitz matrix (4) and its higher dimensional versions $\mathbf{Q}_d$ are adjacency matrices of walks on bounded lattices $L_d = (0,1, \ldots, d-1)$. [7] exploits the adjacency matrix in order to derive enumeration formulae for various classes of lattice paths, and relates these formulae to special numbers like Catalan, Motzkin, Fibonacci, Delannoy, Schröder and Pell numbers. This paper continues the approach of [7] and focuses on paths with start at 0, length $k$ and arbitrary destinations in $L_d$. Due to [7] the total number of such paths equals $F_{d,k} = \mathbf{1}^T \mathbf{Q}_d^k \mathbf{e}_1$. While [7] stops with a trigonometric representation, we go a step further by providing a discrete mathematical presentation in terms of binomial coefficients. The resulting identity resembles multisection summation and differs from the binomial sums for Fibonacci numbers presented in the literature, see, e.g., [1, 2, 3].
In Section 2 we provide a combinatorial interpretation of \( F_{d,k} \) in terms of lattice path enumeration. Linear recurrence relations, in Section 3.1, and a trigonometric representation based on an eigenvalue decomposition, in Section 3, prepare the representation of the numbers \( F_{d,k} \) as binomial sums in Section 3.2.

2 Lattice Path Enumeration

Consider a walk on the lattice \( L_d = (0, 1, \ldots, d - 1) \), starting at 0 and proceeding with steps \(-1\) or 1. A sequence of admissible steps \( x = (x_1, \ldots, x_k) \in \{-1, 1\}^k \) generates the path \( \gamma_x = (i, \sum_{j=1}^{i} x_j)_{i=1,\ldots,k} \) with \( 0 \leq \sum_{j=1}^{i} x_j \leq d \) for \( i \in \{1, \ldots, k\} \). \((k, j)\) can only be element of \( \gamma_x \) if either \((k - 1, j - 1) \in \gamma_x\) or \((k - 1, j + 1) \in \gamma_x\). This means, that the total number of possible visits \( v_{k,j}(d) \),

\[
v_{k,j}(d) = \#\{(x_1, \ldots, x_k) \in \{-1, 1\}^k | \sum_{i=1}^{k} x_i = j, \forall r : 0 \leq \sum_{i=1}^{r} x_i \leq d\}, \quad (6)
\]

at lattice point \( j \) after \( k \) steps obeys the recurrence relation \((0 < j < d)\)

\[
\begin{align*}
v_{k,j}(d) &= v_{k,j-1}(d) + v_{k-1,j+1}(d), \\
v_{k,0}(d) &= v_{k-1,1}(d), \\
v_{k,d}(d) &= v_{k-1,d-1}(d).
\end{align*} \quad (7)
\]

Let \( d \in \mathbb{N} \) and denote by \( Q_d \) the \( d \times d \) tridiagonal matrix of type (4).

This means, \( Q_d \) is tridiagonal with 0 in the main diagonal and 1 in the upper and sub diagonal. Then, the recurrence relation (7) is represented by \( v_k(d) = Q_d v_{k-1}(d) \), where \( v_k(d) = (v_{k,0}(d), \ldots, v_{k,d}(d))^T \). Therefore, we obtain the interpretation that \( F_{d,k} \) equals the number of possible different paths on the lattice \( L_d \) with start 0 and length \( k \), i.e.,

\[
F_{d,k} = \#\{(x_1, \ldots, x_k) \in \{-1, 1\}^k | \forall r \in \{1, \ldots, k\} : 0 \leq \sum_{i=1}^{r} x_i \leq d\}.
\]

3 Factorization of \( Q_d \)

Due to [11] we get the factorization \( Q_d = V_d \Lambda_d V_d^T \), where \( V_d = (v_1, \ldots, v_d) \) is an orthogonal and \( \Lambda_d \) is a diagonal matrix with diagonal elements \( \lambda_1, \ldots, \lambda_d \) with \( d \) rows and \( d \) columns given by

\[
\lambda_j = 2 \cos \left( \frac{j \pi}{d + 1} \right), \quad (8)
\]

\[
v_j = \frac{\sqrt{2}}{\sqrt{d + 1}} \left( \sin \left( \frac{1 \cdot j \pi}{d + 1} \right), \ldots, \sin \left( \frac{d \cdot j \pi}{d + 1} \right) \right)^T.
\]
(8) entails

\[ F_{d,k} = \frac{2^{k+2}}{d+1} \sum_{j=1}^{\lfloor d/2 \rfloor} \cos^k \left( \frac{2j-1}{d+1} \right) \cos^2 \left( \frac{2j-1}{2} \frac{\pi}{d+1} \right). \] (9)

Due to \( \cos(\pi/5) = (1 + \sqrt{5})/4, \cos(3\pi/5) = (1 - \sqrt{5})/4, \cos(\pi/10)^2 = (5 + \sqrt{5})/8 \) and \( \cos(3\pi/10)^2 = (5 - \sqrt{5})/8 \) Equation (9) yields Binet’s formula \( F_{d,k} = F_k = ((1 + \sqrt{5})^k - (1 - \sqrt{5})^k)/(2\sqrt{5}) \) for \( d = 4 \). Therefore, (9) can be looked at as the analogon of Binet’s formula for the sequences \( (F_n) \).

As a direct consequence of (9) we obtain \( \lim_{k} F_{d,k}/\lambda_1 = 4/(d+1) \cos^2(\pi/(2(d+1))) \), hence, \( \lim_{k} F_{d,k+1}/F_{d,k} = 2 \cos(\pi/(d+1)) \).

### 3.1 Representation of \( (F_{d,k})_k \) by Recurrence Relations

According to the Cayley-Hamilton theorem, the matrix \( Q_d \) satisfies the characteristic polynomial equation, \( p(x) = U_d(x/2) = 0 \), where \( U_d \) is the Chebychev polynomial of the second kind

\[ U_d(x) = \sum_{k=0}^{\lfloor d/2 \rfloor} (-1)^k B(d - k, k)(2x)^{d-2k}. \] (10)

The polynomial (10) is of order \( d \) and induces linear recurrence rules \( F_{d,k+d} = \sum_{j=0}^{d-1} \alpha_j F_{d,k+j} \) with \( d \) constant coefficients \( \alpha_j \). The sum (9) gives rise to look for less complex recurrence formulae of only degree \( [(d+1)/2] \). For this, consider the following eigenvalues \( \tilde{\lambda}_j \) and eigenvectors \( \tilde{\nu}_j, j \in \{1, \ldots, [(d+1)/2]\} \):

\[ \tilde{\lambda}_j = 2 \cos \left( (2j-1) \frac{\pi}{d+1} \right), \] (11)

\[ \tilde{\nu}_j = \frac{2}{\sqrt{d+1}} \left( \cos \left( 1 \cdot \frac{2j-1}{d+1} \cdot \frac{\pi}{2} \right), \ldots, \cos \left( n_d \cdot \frac{2j-1}{d+1} \cdot \frac{\pi}{2} \right) \right)^T, \]

where \( n_d = 2[(d+1)/2] - 1 \). Note, that the vectors \( \tilde{\nu}_j, j \in \{1, \ldots, [(d+1)/2]\} \), establish an orthonormal basis. Consequently, we regain the sum (9) by means of \( e_1^T \tilde{\nu}_j^T \tilde{\Lambda}_{\tilde{d}} \tilde{\nu}_j e_1 \), where \( \tilde{d} = [(d+1)/2] \), \( \tilde{\Lambda}_{\tilde{d}} \) denotes the diagonal matrix with \( \tilde{\lambda}_j \) as diagonal elements, and \( \tilde{\nu}_{\tilde{d}} = (\tilde{\nu}_1, \ldots, \tilde{\nu}_{\tilde{d}}) \). The eigenvalues (11) lead to Chebychev polynomials \( T_n \) of the first kind \([12]\)

\[ T_n(x) = 2^{n-1} \prod_{j=1}^{n} \left( x - \cos \left( \frac{(2j-1)\pi}{2n} \right) \right) = \sum_{j=0}^{\lfloor n/2 \rfloor} B(n, 2j)(x^2 - 1)^j x^{n-2j}. \] (12)

For odd \( d \), therefore, we obtain

\[ \chi_{\tilde{d}}(x) = 2T_{\tilde{d}}(x/2) \] (13)
as characteristic polynomial of $\tilde{\Lambda}_d$. For even $d$ we use the identity $U_n(\cos(\theta)) = \sin((n+1)\theta)/\sin(\theta)$, see [12]. Because of $\sin((d/2+1)k\pi/(d+1))=\sin(d/2k\pi/(d+1))=0$, we obtain the characteristic polynomial for even $d$, $\tilde{d}=d/2$,

$$\chi_{\tilde{d}}(x) = U_{\tilde{d}}(x/2) - U_{\tilde{d}-1}(x/2) = \sum_{r=0}^{d/2} B([(d-r)/2], [r/2])x^{[d/2]-r}. \quad (14)$$

For example, (13) and (14) yield the following characteristic polynomials for $d = 1, 2, 3, 4, 5$: $x = 0, x = 1, x^2 - 2 = 0, x^2 - x - 1 = 0, x^2 - 3 = 0$, inducing the recurrence relations ($k \geq 1$):

$$F_{1,k} = 0, F_{2,k} = 1, F_{3,k} = 2^{[k/2]}, F_{4,k} = F_{k+1}, F_{5,2k} = 23^{k-1}, F_{5,2k+1} = 3^k. \quad (15)$$

### 3.2 Representation by Binomial Coefficients

Using (19) we represent (9) in the following way ($k \geq 0$)

$$F_{d+1,k+1} = \frac{1}{2} 2^{k+2} \left( \sum_{k \geq 0} \cos^{k} \left( (2j-1) \frac{\pi}{d+2} \right) \right) \sum_{j=1}^{[d+2]} \left( \sum_{q=0}^{\lfloor d/2 \rfloor} \sin^{q} \left( \frac{\pi (d+2)}{2} \right) \right). \quad (16)$$

Due to (23), (24), (27) and (29) we obtain for Equation (16) the following multisection style summation formulae (18):

$$F_{d+1,k+1} = \begin{cases} \left( \frac{k}{2} \right) + 2 \sum_{q=1}^{[d/2]} (-1)^q \left( \frac{k}{2} - q^{d+2} \right) & \ldots \ 2|k, 2|d \\ \left( \frac{k}{2} \right) + 2 \sum_{q=1}^{[d/2]} \left( \frac{k}{2} - q^{d+2} \right) \\ - \sum_{q=1}^{[d/2]} \left( \frac{k+1}{2} - q^{d+2} \right) \left( \frac{k+1}{2} - q^{d+2} \right) & \ldots \ 2 \not| k, 2 \not| d \\ \frac{1}{2} \left( \frac{k+1}{2} \right) + \sum_{q=1}^{[d/2]} (-1)^q \left( \frac{k+1}{2} - q^{d+2} \right) \left( \frac{k+1}{2} - q^{d+2} \right) & \ldots \ 2 |k, 2|d \\ \frac{1}{2} \left( \frac{k+1}{2} \right) - 2 \sum_{q=1}^{[d/2]} \left( \frac{k+1}{2} - q^{d+2} \right) \left( \frac{k+1}{2} - q^{d+2} \right) \\ + \sum_{q=1}^{[d/2]} \left( \frac{k+1}{2} - q^{d+2} \right) \left( \frac{k+1}{2} - q^{d+2} \right) \left( \frac{k+1}{2} - q^{d+2} \right) & \ldots \ 2 |k, 2|d \\ \end{cases} \quad (18)$$
Figure 1: Illustration of Pascal triangle identity for $d = 2$ with row sums on the left due to Equations (18).

Note, that, in contrast to identities regarding sums of binomial coefficients that result from series multisection, the sums under consideration have alternating signs [9]. With $1^T Q^d_k e_1 = 0$ and (18) we regain the well-known binomial identity $\sum_{k=0}^{n} (-1)^k B(n, k) = 0$, where $B(n, k)$ refers to the binomial coefficient $n$ over $k$. For $d \geq 2$ we obtain novel Pascal identities. See (15) and the Figures 1, 2 and 3.

4 Conclusion

We have shown how the factorization of the adjacency matrix of a walk on a bounded lattice leads to non-trivial identities in the Pascal triangle. This illustrates the potential of a path enumeration approach for revealing discrete mathematical identities. In future, we will also consider walks on unbounded lattices and ask for a combinatorial characterization of the number of walks with a given range.

A Proof of Property (5)

The proof is based on induction. Obviously, $Q$, see (4), satisfies (5). Suppose that $Q^{(k)} = (q_{i,j}^{(k)})$ satisfies (5) for some $k > 0$, that is $q_{i,j}^{(k)} = q_{5-i,5-j}^{(k)}$. From this, by considering $Q^{k+1} = QQ^k$, it follows that $q_{1,1}^{(k+1)} = q_{2,1}^{(k)} = q_{3,4}^{(k)} = q_{4,4}^{(k+1)}$. 


Multisection style binomial summation identity

Figure 2: Illustration of Pascal triangle identity for $d = 3$ with row sums on the left due to Equations (18).

Figure 3: Illustration of Pascal triangle identity, $d = 4$, with Fibonacci sequence as row sums due to Equation (18).
By means of (19) Equation (17) can be represented for even \( m \)

\[
q_{2,1}^{(k+1)} = q_{1,1}^{(k)} + q_{3,1}^{(k)} = q_{4,1}^{(k)} + q_{2,1}^{(k+1)} = q_{3,4}^{(k+1)}, \quad q_{3,1}^{(k+1)} = q_{2,1}^{(k)} + q_{4,1}^{(k)} = q_{3,4}^{(k)} + q_{1,4}^{(k)} = q_{2,4}^{(k+1)}, \quad q_{4,1}^{(k+1)} = q_{3,1}^{(k)} = q_{2,4}^{(k)} = q_{1,4}^{(k+1)}, \quad q_{1,2}^{(k+1)} = q_{2,2}^{(k)} = q_{3,3}^{(k+1)}, \quad q_{2,2}^{(k+1)} = q_{1,2}^{(k)} + q_{3,2}^{(k)} = q_{3,3}^{(k+1)} = q_{3,2}^{(k)} = q_{2,2}^{(k)} = q_{3,3}^{(k+1)} = q_{3,2}^{(k)} = q_{1,2}^{(k+1)}, \quad \text{and}
\]

\[
q_{4,2}^{(k+1)} = q_{3,2}^{(k)} = q_{2,3}^{(k)} = q_{1,3}^{(k+1)}. \]

### B Representation of (9) in Terms of Binomial Coefficients

For the following transformations we need the well-known trigonometric formulae

\[
\cos^m(\theta) = \left\{ \begin{array}{ll}
\frac{1}{2^m} \left( \begin{array}{c} m \\ m/2 \end{array} \right) + \frac{2}{2^m} \sum_{k=0}^{m-1} \left( \begin{array}{c} m \\ k \end{array} \right) \cos((m-2k)\theta) & \text{for even } m \\
\frac{2}{2^m} \sum_{k=0}^{m-1} \left( \begin{array}{c} m \\ k \end{array} \right) \cos((m-2k)\theta) & \text{for odd } m
\end{array} \right. \tag{19}
\]

\[
\sum_{k=0}^{n} \cos(k\theta) = \frac{\sin \left( \frac{n+1}{2} \theta \right) \cos \left( \frac{n}{2} \theta \right)}{\sin \left( \frac{\theta}{2} \right)}, \quad \sum_{k=0}^{n} \sin(k\theta) = \frac{\sin \left( \frac{n+1}{2} \theta \right) \sin \left( \frac{n}{2} \theta \right)}{\sin \left( \frac{\theta}{2} \right)}. \tag{20}
\]

### B.1 \( \Sigma_{m,d} \), Equation (17), in Terms of Binomial Coefficients for \( m \) Even

By means of (19) Equation (17) can be represented for even \( m \) by

\[
\Sigma_{m,d} = \left[ \frac{d+2}{2} \right] \frac{1}{2^m} \left( \begin{array}{c} m \\ m/2 \end{array} \right) + \frac{2}{2^m} \sum_{j=0}^{m-1} \left( \begin{array}{c} m \\ j \end{array} \right) \xi_{j,m,d}^{(1)}, \tag{21}
\]

where \( \xi_{j,m,d}^{(1)} = \sum_{k=1}^{\left[ \frac{d+2}{2} \right]} \cos \left( (m-2j)(2k-1)\frac{\pi}{d+2} \right) \). With \( \alpha = 2k(m-2j)\frac{\pi}{d+2} \), \( \beta = -(m-2j)\frac{\pi}{d+2} \) and applying \( \cos(\alpha + \beta) = \cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta) \) we get

\[
\xi_{j,m,d}^{(1)} = \cos \left( (m-2j)\frac{\pi}{d+2} \right) \sigma_{j,m,d}^{(1,1)} + \sin \left( (m-2j)\frac{\pi}{d+2} \right) \sigma_{j,m,d}^{(1,2)}, \tag{22}
\]

where \( \sigma_{j,m,d}^{(1,1)} = \sum_{k=1}^{\left[ \frac{d+2}{2} \right]} \cos(k \cdot 2(m-2j)\frac{\pi}{d+2}) \) and \( \sigma_{j,m,d}^{(1,2)} = \sum_{k=1}^{\left[ \frac{d+2}{2} \right]} \sin(k \cdot 2(m-2j)\frac{\pi}{d+2}) \). Let us consider \( d \) odd and let us assume that \((d+2)\not|(m-2j)\), then \( \left[ \frac{d+2}{2} \right] = \frac{d+1}{2} \), which, by setting \( \theta = (m-2j)\frac{\pi}{d+2} \), implies

\[
\sigma_{j,m,d}^{(1,1)} + 1 = \frac{\sin \left( \left( \frac{d+2}{2} + \frac{1}{2} \right)\theta \right) \cos \left( \left( \frac{d+2}{2} - \frac{1}{2} \right)\theta \right)}{\sin(\theta)}. \]
Since \( \sin\left(\frac{d+3}{2}\theta\right) = \cos\left(\frac{d+2}{2}\theta\right) \sin\left(\frac{1}{2}\theta\right) \) and \( \cos\left(\frac{d+1}{2}\theta\right) = \cos\left(\frac{d+2}{2}\theta\right) \cos\left(\frac{1}{2}\theta\right) \) we get \( \sigma_{j,m,d}^{(1,1)} + 1 = \sin\left(\frac{1}{2}\theta\right) \cos\left(\frac{1}{2}\theta\right) / \sin(\theta) \), which implies \( \sigma_{j,m,d}^{(1,1)} = -\frac{1}{2} \). Analogously, we get

\[
\sigma_{j,m,d}^{(1,2)} = \frac{\sin\left(\left(\frac{d+2}{2} + \frac{1}{2}\right)\theta\right) \sin\left(\left(\frac{d+2}{2} - \frac{1}{2}\right)\theta\right)}{\sin(\theta)} ,
\]

which by \( \sin\left(\frac{d+3}{2}\theta\right) = \cos\left(\frac{d+2}{2}\theta\right) \sin\left(\frac{1}{2}\theta\right) \), \( \sin\left(\frac{d+1}{2}\theta\right) = -\cos\left(\frac{d+2}{2}\theta\right) \sin\left(\frac{1}{2}\theta\right) \) leads to \( \sigma_{j,m,d}^{(1,2)} = -\sin^2\left(\frac{1}{2}\theta\right) / \sin(\theta) \). As a result for even \( m \) and odd \( d \), we, therefore, obtain

\[
\sigma_{j,m,d}^{(1,1)} = \begin{cases} 
-\frac{1}{2} & \cdots \ (d+2) \not| \ (m-2j) \\
\frac{d+2}{2} & \cdots \ (d+2) \ | \ (m-2j),
\end{cases}
\]

\[
\sigma_{j,m,d}^{(1,2)} = \begin{cases} 
-\frac{\sin^2\left(\left(\frac{1}{2}(m-2j)\right)\frac{\pi}{d+2}\right)}{\sin\left((m-2j)\frac{\pi}{d+2}\right)} & \cdots \ (d+2) \not| \ (m-2j) \\
0 & \cdots \ (d+2) \ | \ (m-2j).
\end{cases}
\]

This leads to

\[
\xi_{j,m,d}^{(1)} = \begin{cases} 
-\frac{1}{2} \cos\left(\frac{\pi(m-2j)}{d+2}\right) - \sin^2\left(\frac{\pi(m-2j)}{2(d+2)}\right) & \cdots \ (d+2) \not| \ (m-2j) \\
\left|\frac{d+2}{2}\right| \cos\left(\frac{(m-2j)\pi}{d+2}\right) & \cdots \ (d+2) \ | \ (m-2j),
\end{cases}
\]

\[
= \begin{cases} 
-\frac{1}{2} & \cdots \ (d+2) \not| \ (m-2j) \\
\left|\frac{d+2}{2}\right| (-1)^{\frac{m-2j}{d+2}} & \cdots \ (d+2) \ | \ (m-2j)
\end{cases}
\]

\[
\therefore \Sigma_{m,d} = \left[\frac{d+2}{2}\right] \left\{ \frac{1}{2m} \left( m \atop m/2 \right) \right\} + \frac{2}{2^m} \sum_{j=0}^{\left\lfloor \frac{m-2j}{d+2} \right\rfloor} \left( m \atop j \right) \left( \frac{d+2}{2} \right) \left( -1 \right)^{m-2j} \\
+ \frac{2}{2^m} \sum_{j=0}^{\left\lfloor \frac{m}{d+2} \right\rfloor - 1} \left( m \atop j \right) \left( -\frac{1}{2} \right).
\]

For even \( m \) and odd \( d \) we, therefore, obtain \( (d+2) \not| \ (m-2j) \iff \exists p \in \mathbb{N} : m-2j = p(d+2) \iff \exists q \in \mathbb{N} : m-2j = 2q(d+2) \) which is equivalent to \( \frac{m-2j}{d+2} = 2q \iff q \in \left\{ 1, \ldots, \left\lfloor \frac{m}{2(d+2)} \right\rfloor \right\} \). Observe that \( \sum_{j=0}^{m-2} \left( m \atop j \right) = \frac{1}{2} \left( 2^m - \left( m \atop m/2 \right) \right) \) and, therefore,

\[
\sum_{j=0}^{\left\lfloor \frac{m}{d+2} \right\rfloor - 1} \left( m \atop j \right) = \frac{1}{2} \left( 2^m - \left( m \atop m/2 \right) \right) - \sum_{q=1}^{\left\lfloor \frac{m}{d+2} \right\rfloor} \left( m/2 - q(d+2) \right) .
\]
From this it follows that
\[
\Sigma_{m,d} = \left[\frac{d + 2}{2}\right] \frac{1}{2^m} \left(\frac{m}{2}\right) + \left[\frac{d + 2}{2}\right] \sum_{q=1}^{\left\lfloor \frac{m}{4}\right\rfloor} \left(\frac{m}{2} - q(d + 2)\right)
\]

\[\left[-\frac{1}{2}\sum_{j=0}^{\left\lfloor \frac{d}{2}\right\rfloor} \left(\frac{m}{2}\right)\right],\]

hence, for \(m\) even, \(d\) odd
\[
\Sigma_{m,d} = -\frac{1}{2} + \frac{d + 2}{2} \left\{ \left(\frac{m}{2}\right) + 2 \sum_{q=1}^{\left\lfloor \frac{m}{4}\right\rfloor} \left(\frac{m}{2} - q(d + 2)\right) \right\}. \quad (23)
\]

For \(d\) even we immediately obtain from (22)
\[
\xi_{j,m,d}^{(1)} \cdot \left\{ 0 \left[\frac{d+2}{2}\right] \cos\left((m-2j)\frac{\pi}{d+2}\right) \ldots \frac{(d+2)}{\left(\frac{m-2j}{d+2}\right)} \right\}
\]

Therefore, we get for \(m\) and \(d\) even
\[
\Sigma_{m,d} = \frac{d + 2}{2} \left\{ \frac{1}{2^m} \left(\frac{m}{2}\right) + \frac{1}{2^m} \sum_{j=0}^{\left\lfloor \frac{d}{2}\right\rfloor} \left(\frac{m}{2}\right) \right\}
\]

\[\left[\frac{d}{2}\right] \sum_{j=0}^{\left\lfloor \frac{d}{2}\right\rfloor} \left(\frac{m}{2}\right) \right\} \sum_{k=1}^{\left\lfloor \frac{m}{d+2}\right\rfloor} (-1)^k \left(\frac{m}{2} - k\frac{d+2}{2}\right) \right\}. \quad (24)
\]

B.2 \(\Sigma_{m,d}\), Equation (17), in Terms of Binomial Coefficients for \(m\) Odd

By means of (19) \(\Sigma_{m+1,d}^{(1)}\), with even \(m\), can be represented by
\[
\Sigma_{m+1,d} = \frac{2}{2m+1} \sum_{j=0}^{\left\lfloor \frac{d+2}{2}\right\rfloor} \left(\frac{m+1}{j}\right) \xi_{j,m,d}^{(2)}, \quad (25)
\]

where \(\xi_{j,m,d}^{(2)} = \sum_{k=1}^{\left\lfloor \frac{d+2}{2}\right\rfloor} \cos((m+1-2j)(2k-1)\frac{\pi}{d+2})\). With \(\alpha = 2k(m+1-2j)\frac{\pi}{d+2} \cdot \beta = -(m+1-2j)\frac{\pi}{d+2}\) and applying \(\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)\) we get
\[
\xi_{j,m,d}^{(2)} = \cos\left((m+1-2j)\frac{\pi}{d+2}\right) \sigma_{j,m,d}^{(1,1)} + \sin\left((m+1-2j)\frac{\pi}{d+2}\right) \sigma_{j,m,d}^{(1,2)}
\]
where \( \sigma_{j,m,d}^{(2,1)} = \sum_{k=1}^{\left\lfloor \frac{d+2}{2} \right\rfloor} \cos(k \cdot 2(m + 1 - 2j) \frac{\pi}{d+2}) \) and \( \sigma_{j,m,d}^{(2,2)} = \sum_{k=1}^{\left\lfloor \frac{d+2}{2} \right\rfloor} \sin(k \cdot 2(m + 1 - 2j) \frac{\pi}{d+2}) \). For odd \( d \) we obtain

\[
\sigma_{j,m,d}^{(2,1)} = \begin{cases} 
\frac{-1}{2} & \ldots \ (d + 2) \mid (m - 2j + 1) \\
\left\lfloor \frac{d+2}{2} \right\rfloor & \ldots \ (d + 2) \mid (m - 2j + 1)
\end{cases},
\]

\[
\sigma_{j,m,d}^{(2,2)} = \begin{cases} 
\frac{\cos(\left(\frac{1}{2}(m-2j+1)\frac{\pi}{d+2}\right))}{\sin((m-2j+1)\frac{\pi}{d+2})} & \ldots \ (d + 2) \mid (m - 2j + 1) \\
0 & \ldots \ (d + 2) \mid (m - 2j + 1)
\end{cases}.
\]

In analogy to Subsection B.1 this leads to

\[
\xi_{j,m,d}^{(2)} = \begin{cases} 
\frac{1}{2} & \ldots \ (d + 2) \mid (m + 1 - 2j) \\
\left\lfloor \frac{d+2}{2} \right\rfloor (-1)^{\left\lfloor \frac{m+1-2j}{d+2} \right\rfloor} & \ldots \ (d + 2) \mid (m + 1 - 2j)
\end{cases},
\]

hence,

\[
\Sigma_{m+1,d} = \frac{2}{2m+1} \sum_{j=0, \ldots, \frac{m+2}{d+2}} \binom{m+1}{j} \left\lfloor \frac{d+2}{2} \right\rfloor (-1)^{\left\lfloor \frac{m-2j+1}{d+2} \right\rfloor} + \frac{2}{2m+1} \sum_{j=0, \ldots, \frac{m+2}{d+2}} \binom{m+1}{j} \left( \frac{1}{2} \right) .
\] (26)

For even \( d \) the points \((m + 1 - 2j\pi/(d + 2))\) are symmetric w.r.t. \( \pi/2 \), which implies

\[
\Sigma_{m+1,d} = 0.
\] (27)

Therefore, let us consider \( d \) odd, which implies \( \frac{m-2j+1}{d+2} = q, q, j \in \mathbb{N}, q \in \{1, \ldots, \left\lfloor \frac{m+2}{d+2} \right\rfloor\} \). Observe that

\[
\sum_{j=0, \ldots, \frac{m+2}{d+2}} \binom{m+1}{j} = 2^m - \sum_{q=1, odd} \binom{m+1}{\frac{m+1}{2} - q\frac{d+2}{2}} .
\] (28)

This implies that \( \Sigma_{m+1,d} \) equals

\[
-\frac{1}{2^m} \left\lfloor \frac{d+2}{2} \right\rfloor \sum_{q=1, odd} \binom{m+1}{\frac{m+1}{2} - q\frac{d+2}{2}} + \frac{1}{2^m} \sum_{j=0, \ldots, \frac{m+2}{d+2}} \binom{m+1}{j} ,
\]

hence, by taking (28) into account,

\[
\Sigma_{m+1,d} = \frac{1}{2} - \frac{d+2}{2} \frac{1}{2^m} \sum_{q=1, odd} \binom{m+1}{\frac{m+1}{2} - q\frac{d+2}{2}} .
\] (29)

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References


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