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A Note on Cohomology of a Riemannian Manifold

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Abstract

In this paper we use the vanishing of first cohomology group of a Riemannian manifold (M, g) to find a sufficient condition for a closed vector field ξ on M to be a concircular vector field.

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1. Introduction

Let (M, g) be an n - dimensional Riemannian manifold. A diffeomorphism $\varphi : M \longrightarrow M$ is said to be a concircular transformation if it maps circle to a circle, and a smooth vector field ξ on (M, g) is said to be concircular vector field if its flow consists of concircular transformation $(cf.\ [1], [4])$. concircular vector fields have been used in finding characterizations of different Riemannian manifolds $(cf.[3])$ and are also important in the general theory of relativity. Recall that circular vector fields are closed vector fields; and on a Riemannian manifold (M, g) whose first cohomology group $H^1(M, \mathbb{R}) = 0$, the closed vector fields are in abundance. Since, concircular vector fields are important in geometry as well as physical sciences, its an interesting question to find sufficient conditions for a closed vector field ξ on a Riemannian manifold to be a concircular vector fields. Recall that on Riemannian manifold (M, g), a smooth vector field X is said to be an eigen vector of the Laplacian operator Δ (also called the rough Laplacian) if there exists a constant λ such that $\Delta X = -\lambda X$, $\lambda \geq 0$.

Note that the position vector field $X = \sum \frac{x^i \partial}{\partial x^i}$ on the Euclidian space $(\mathbb{R}^n, \langle\ ,\rangle)$ satisfies $\Delta X = 0$. Also if Z is a constant vector field on $(\mathbb{R}^{n+1}, \langle\ ,\rangle)$, then its tangential projection u on the unit sphere S^n satisfies

$$\Delta u = -u, \text{ where } Z = u + \rho N,\ \rho = \langle Z, N \rangle$$

N being the unit normal vector field to S^n in $(\mathbb{R}^{n+1}, \langle\ ,\rangle)$.

In this paper, we consider the question " under what conditions a closed vector field ξ on a Riemannian manifold (M, g) satisfying $H^1(M, \mathbb{R}) = 0$ is a concircular vector field?". Note that in the examples discussed above the vector fields X and u are closed vector fields on $(\mathbb{R}^n, \langle\ ,\rangle)$ and the unit sphere S^n respectively and that $H^1(\mathbb{R}^n, \mathbb{R}) = 0$, $H^1(S^n, \mathbb{R}) = 0$ holds.

We answer the above question by proving the following

Theorem: *Let (M, g) be an $n-$ dimensional compact Riemannian manifold with $H^1(M, \mathbb{R}) = 0$. If ξ is smooth closed vector field on M satisfying $\Delta\xi = -\lambda\xi$ and $(\operatorname{div}\xi)^2 \geq n\lambda \|\xi\|^2$, then ξ is a concircular vector field .*

2. Preliminaries

Let (M, g) be an $n-$ dimensional Riemannian manifold. A smooth vector field ξ on M is said to be a concircular vector field if

$$\nabla_X \xi = fX\ ,\ X \in \mathfrak{X}(M), \tag{2.1}$$

where f is a smooth function, ∇ the Riemannian connection on M and $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on M.Suppose that the first cohomology group $H^1(M, \mathbb{R}) = 0$ for the Riemannian manifold (M, g). Since each closed 1 - form η on M defines a cohomology class in the deRham cohomology group $H^1_{dR}(M)$ which is isomorphic to $H^1(M, \mathbb{R}) = 0$, the closed smooth 1 - form η must be exact. If $\xi \in \mathfrak{X}(M)$ is dual to the smooth closed 1 - form η, that is $\eta(X) = g(X, \xi)$, $X \in \mathfrak{X}(M)$, then the conditions that η is closed and exact imply that $\eta = d\rho$ for some function ρ on (M, g) , that is $\xi = \nabla\rho$, where $\nabla\rho$ is the gradient of ρ on (M, g) . If we define an operator $A : \mathfrak{X}(M) \to \mathfrak{X}(M)$, BY

$$g(A(X), Y) = \frac{1}{2}(\pounds_\xi g)(X, Y)\ ,\ X, Y \in \mathfrak{X}(M)$$

then A is symmetric that is $g(A(X), Y) = g(Y, A(X))$ holds and using Koszul's formula we have

$$\nabla_X \xi = AX, \quad X \in \mathfrak{X}(M)\ , \tag{2.2}$$

where we have used $d\eta, = 0$, that is the smooth 1 - form η is closed. The curvature tensor field R of the Riemannian manifold (M, g) is given by

$$R(X,Y) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

$X, Y, Z \in \mathfrak{X}(M)$ and the Ricci curvature Ric is given by

$$Ric(X,Y) = \sum_{i=1}^{n} g(R(e_i, X) Y, e_i), \quad X, Y \in \mathfrak{X}(M).$$

where $\{e_1, e_2, \ldots, e_n\}$ is a local orthonormal frame on M. The Ricci operator Q is a symmetric operator defined by $Q : \mathfrak{X}(M) \to \mathfrak{X}(M)$

$$Ric(X,Y) = g(QX, Y), \quad X, Y \in \mathfrak{X}(M).$$

For a closed vector field $\xi \in \mathfrak{X}(M)$, using (2.2,), we get

$$R(X,Y)\xi = (\nabla A)(X,Y) - (\nabla A)(Y, X,) \tag{2.3}$$

where the covariant derivative $(\nabla A)(X,Y)$ is defined by

$$(\nabla A)(X,Y) = \nabla_X AY - A(\nabla_X Y), \;\; X, Y \in \mathfrak{X}(M) ()$$

Using the equation (2.3) we get

$$Ric(X,\xi) = \sum_{i=1}^{n} g((\nabla A)(e_i, X) - (\nabla A)(X, e_i), e_i), \; X \in \mathfrak{X}(M). \tag{2.4}$$

We define a smooth function α on M by

$$\alpha = \sum_{i=1}^{n} g(Ae_i, e_i),$$

then using symmetry of the operator A in the equation (2.4), we get

$$Ric(X,\xi) = \sum_{i=1}^{n} g(X, (\nabla A)(e_i, e_i)) - X(\alpha), \;\; X \in \mathfrak{X}(M). \tag{2.5}$$

The rough Laplacian operator Δ on a Riemannian manifold (M, g) is a self adjoint operator $\Delta : \mathfrak{X}(M) \to \mathfrak{X}(M)$ with respect to the inner product $(\, ,) : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathbb{R}$

$$(X,Y) = \int_M g(X,Y)$$

for compactly supported vector fields X, Y; defined by

$$\Delta X = \left(\sum_i \nabla e_i \nabla e_i X - \nabla_{\nabla_{e_i}} e_i X \right), \quad X \in \mathfrak{X}(M) \tag{2.6}$$

and that the operator Δ being elliptic has non - negative eigenvalues, a non - negative number λ satisfying

$$\Delta X = -\lambda X$$

is called eigenvalue of Δ corresponding to eigen vector X.

3. Proof of the Theorem

Let (M, g) be an n - dimensional compact Riemannian manifold with first singular homology group $H^1(M, \mathbb{R}) = 0$. Let $\xi \in \mathfrak{X}(M)$ be a closed vector field on M that satisfies

$$\Delta\xi = -\lambda\xi \tag{3.1}$$

and

$$(\operatorname{div}\xi)^2 \geq n\lambda \|\xi\|^2 \tag{3.2}$$

as required by the theorem, where λ is a constant.

If $\{e_1, e_2, \dots, e_n\}$ is a local orthonormal frame on M and $\alpha = \sum_{i=1}^{n} g(Ae_i, e_i)$, then equation (2.2) gives

$$\operatorname{div}\xi = \alpha \tag{3.3}$$

We have

$$\operatorname{div}(\alpha\xi) = \xi(\alpha) + \alpha^2$$

which by Stoke's theorem gives

$$\int_M \xi(\alpha) = -\int_M \alpha^2 \tag{3.4}$$

Now using the equation (2.5), we get

$$Q(\xi) = \sum_{i=1}^{n} \nabla(A)(e_i, e_i) - \nabla\alpha \tag{3.5}$$

Also, using the definition (2.6), and the equation (2.2) we get

$$\Delta(\xi) = \sum_{i=1}^{n} \nabla(A)(e_i, e_i) \tag{3.6}$$

Thus using the equations (3.1), (3.5) and (3.8), we conclude

$$Q(\xi) = -\lambda\xi - \nabla\alpha$$

and taking the inner product with ξ in the above equation, we get

$$Ric(\xi.\xi) = -\lambda \|\xi\|^2 - \xi(\alpha) \tag{3.7}$$

Note that the smooth 1- form η dual to ξ is closed and as the singular cohomology $H^1(M, \mathbb{R})$ is isomorphic to the deRham cohomology group $H^1_{dR}(M)$, we have $H^1_{dR}(M) = 0$ and consequently λ is exact, that is there exists a smooth function $\rho : M \to \mathbb{R}$ such that $\eta = d\rho$. Hence, we get $\xi = \nabla\rho$. We know by (2.2)

$$AX = \nabla_X \xi = \nabla_X \nabla\rho, \quad X \in \mathfrak{X}(M) \ ,$$

that is A is the Hessian operator of the smooth function ρ. Then by Bochner formula gives

$$\int_M (Ric(\nabla\rho, \nabla\rho) + \|A\|^2 - (\Delta\rho)^2) = 0 \tag{3.8}$$

Note that $\Delta\rho = \operatorname{div}(\nabla\rho) = \operatorname{div}\xi = \alpha$, where we used equation (3.3). Thus the equations $(3.7,)$, (3.8) and $\nabla\rho = \xi$ give

$$\int_M (\lambda \|\xi\|^2 + \xi(\alpha) + \alpha^2 - \|A\|^2) = 0 \ ,$$

which together with the equation (3.4) gives

$$\int_M (\|A\|^2 - \lambda \|\xi\|^2) = 0 \ .$$

The above equation could be re -arranged as

$$\int_M (\|A\|^2 - \frac{1}{n}\alpha^2) + \frac{1}{n}\left(\alpha^2 - n\lambda \|\xi\|^2\right) = 0 \tag{3.9}$$

Not that the Shwarz's inequality for the symmetric operator A states that $\|A\|^2 \geq \frac{1}{n}(trA)^2 = \frac{1}{n}\alpha^2$ and the equality holds if and only if $A = \frac{\alpha}{n}I$. Moreover, the inequality (3.2) gives

$$\alpha^2 = (\operatorname{div}\xi)^2 \geq n\lambda \|\xi\|^2$$

consequently, both terms in (3.9) are non - negative and we have

$$\|A\|^2 = \frac{1}{n}\alpha^2 \quad \text{and} \quad \alpha^2 = n\lambda \|\xi\|^2 \tag{3.10}$$

The equation is the equality in the Shwarz's inequality and hence we have $A = \frac{\alpha}{n} I$.Hence equation (2.2) becomes

$$\nabla_X \xi = fX \quad , \qquad X \in \mathfrak{X}(M),$$

where $f = \frac{\alpha}{n}$ is a smooth function on M. This proves that ξ is a concircular vector field.

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