Decomposition of Complete Graphs into Union of Stars

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Abstract

Let S_{k+1} denote a star with k edges. Tarsi and Yamamoto et al. have characterized the S_{k+1} -decomposability of K_n the complete graph. In this paper we study the edge decomposition of both K_n and the complete bipartite graph $K_{m,n}$ into copies of the union of two edge disjoint stars S_{p+1} and S_{q+1} where $p \neq q$ and $p, q \geq 2$ and obtain the necessary and sufficient conditions for the $S_{p+1} \cup S_{q+1}$ -decomposability of K_n and $K_{m,n}$.

Keywords: Decomposition, Complete graphs, Star

1. Introduction

The graphs considered are finite and undirected. The terminology adopted agrees with that in any standard book on graph theory. [2]

K_n denotes the complete graph on n vertices. It has nC_2 edges. $K_{m,n}$, the complete bipartite graph has $m+n$ vertices and mn edges. A star S_{k+1} with $k+1$ vertices and k edges is nothing but the complete bipartite graph $K_{1,k}$. The star S_{k+1} with its centre at 1 and edges joining 1 and $i, i=2,3,\dots,k$, is denoted by $S\{1:2,3,\dots,k\}$. A p -subset of V is a subset of size p . The join $G \vee H$ of two vertex disjoint graphs G and H is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv : u \in V(G) \text{ and } v \in V(H)\}$.

Let G be any graph and $F=\{G_1, G_2, \dots, G_k\}$ be a family of subgraphs of G . An F -decomposition of G is an edge-disjoint decomposition of G into copies of G_i for $i=1,2,\dots,k$, provided, no G_i has isolated vertices. Obviously, k_i is a positive integer. If each G_i is isomorphic to a graph H then G is H -decomposable or G has an H -decomposition, denoted by $H \mid G$. Clearly, when $H \mid G$, then $e(H) \mid e(G)$. If G is F -decomposable, it necessarily satisfies the condition $\sum_{i=1}^k k_i e(H_i) = e(G)$. For a family F of stars, Lin and Shyu [1] have given a characterization of the F -decomposability of K_n .

Tarsi and Yamamoto et al.[4,6] have established the following result.” Let k and n be positive integers. There exists a S_{k+1} decomposition of K_n if and only if $2k \leq n$ and $n(n-1) \equiv 0 \pmod{2k}$.”

We concentrate on H -decompositions of $K_{m,n}$, the complete bipartite graph and K_n , the complete graph where, $H = S_{p+1} \cup S_{q+1}$, the union of two vertex disjoint stars.

We assume that $p \neq q$, for otherwise it reduces to star-decomposition. Even then, if either p or $q=1$, the problem reduces to decomposition of complete graphs into paths and stars.[5] Hence $p \geq 2$ and $q \geq 3$. Regi T.[3] gives the conditions for the $S_2 \cup S_3$ decomposability of K_n and $K_{m,n}$.

In this paper the $S_{p+1} \cup S_{q+1}$, $p \geq 2, q \geq 2$, decomposability of complete graphs is characterized. The main results are given in (2.3) and (3.4), theorems A and B.

2. Decomposition of $K_{m,n}$, the complete bipartite graph into $S_{p+1} \cup$

S_{q+1} , $p \neq q$

The complete bipartite graph $K_{m,n}$ has $m+n$ vertices and mn edges while $S_{p+1} \cup S_{q+1}$ has $p+q+2$ vertices and $p+q$ edges. The stars S_{p+1} and S_{q+1} are vertex disjoint and $K_{m,n}$ is connected. If $K_{m,n}$ is $S_{p+1} \cup S_{q+1}$ -decomposable, then $K_{m,n}$ will be the edge disjoint union of at least two copies of $S_{p+1} \cup S_{q+1}$. For any value of p and q ,

there exists the minimal $S_{p+1} \cup S_{q+1}$ decomposable $K_{m,n}$ with the value two for either m or n . Hence, we fix $m, n \geq 2$. The same condition is chosen in [3] also. There are $S_{p+1} \cup S_{q+1}$ -non-decomposable complete bipartite graphs for m or $n < 2$.

2.1 Example: Let $m=1, n=5, p=2, q=3$. $K_{1,5}$ is not $S_3 \cup S_4$ -decomposable, whereas

$K_{2,5}$ is decomposable.

The main result of this section is given below.

2.2 Theorem A:

Let m, n, p, q be positive integers such that $m, n \geq 2$ and $p \neq q$. The necessary and sufficient condition that the complete bipartite graph $K_{m,n}$ is $S_{p+1} \cup S_{q+1}$ -decomposable is (i) $m+n \geq p+q+2$ and (ii) $mn \equiv 0 \pmod{p+q}$

Proof: Necessity The conditions (i) and (ii) result immediately from the

$S_{p+1} \cup S_{q+1}$ -decomposability of $K_{m,n}$

Sufficiency: When $mn \equiv 0 \pmod{p+q}$, either $m \equiv 0 \pmod{p+q}$ or $n \equiv 0 \pmod{p+q}$. It is sufficient to consider any one of them.

$m \equiv 0 \pmod{p+q}$ implies, $m = k(p+q)$, k a positive integer and $k \geq 2$.

Let V_1 and V_2 be the subsets into which $V(K_{m,n})$ is bipartitioned. Let $|V_1| = m$, $|V_2| = n$ and $V_1 = \{u_1, u_2, \dots, u_p, u_{p+1}, \dots, u_{p+q}, \dots, u_{2(p+q)}, \dots, u_{k(p+q)}\}$, $V_2 = \{v_1, v_2, \dots, v_n\}$

V_1 is again partitioned into k subsets of p vertices and k subsets of q vertices.

Case I: n is even. $n = 2r$, for some integer $r \geq 1$.

Partition V_2 into $r = \frac{n}{2}$ subsets of size 2 each.

A p -subset of V_1 , a q -subset of V_1 and a 2-subset of V_2 span two disjoint copies of $S_{p+1} \cup S_{q+1}$.

When all of the $p, q, 2$ -subsets are considered, they span a $K_{m,n}$ which is decomposed into $2 \times k \times r = nk$ copies of $S_{p+1} \cup S_{q+1}$.

Case II: n is odd. $n = 2r+3$. $r \geq 0$.

Partition V_2 into $\frac{n-3}{2}$ 2-subsets and a 3-subset. As in case I, the 2-subsets span $2rk$ copies of $S_{p+1} \cup S_{q+1}$. From the 3-subset, two vertices are chosen in three ways, each pair spanning k copies of $S_{p+1} \cup S_{q+1}$. $K_{m,n}$ is decomposed into $2rk+3k = nk$ copies of $S_{p+1} \cup S_{q+1}$.

This completes the proof. $\square$

2.3 Note:

The conditions $m, n \geq 2$ and $m+n \geq p+q+2$ guarantee the existence of a $S_{p+1} \cup S_{q+1}$ decomposable $K_{m,n}$.

3. Decomposition of K_n , the complete graph into $S_{p+1} \cup S_{q+1}$, $p \neq q$

The proof methodology adopted here depends on the principle of mathematical induction. Given any p and q we have to establish the relation between p , q and n to check whether K_n is $S_{p+1} \cup S_{q+1}$ -decomposable. The proof for the initial values of p and q (2 and 3) is given separately, even though it is a part of the main theorem. Theorem A is used in proving Theorem B, the main result here.

3.1 Lemma: If $H \mid G_1$ and $H \mid G_2$ then $H \mid G_1 \cup G_2$.

3.2 Theorem [3]: K_n is $K_{1,2} \cup K_2$ -decomposable if and only if $n > 4$ and $n \equiv 0, 1 \pmod{3}$.

3.3 Theorem:

The complete graph K_n is $S_3 \cup S_4$ -decomposable if and only if $n \geq 10$ and $n \equiv 0, 1 \pmod{5}$.

Proof: Suppose K_n is $S_3 \cup S_4$ -decomposable

Then $5 \mid n$, implies either $5 \mid n$ or $5 \mid n-1$. It follows that $n \equiv 0, 1 \pmod{5}$.

which is equivalent to $n \equiv 0, 1 \pmod{10}$, since 2 and 5 are relatively prime. Also,

$n > 7$, for $S_3 \cup S_4$ has seven vertices. $n \equiv 0, 1 \pmod{5}$ implies that $n \geq 10$.

It is noted that both K_5 and K_6 are not $S_3 \cup S_4$ -decomposable.

Sufficiency: Let K_n be a complete graph on $n \geq 10$ vertices and $n \equiv 0, 1 \pmod{5}$. Then, $n=5k$ or $n=5k+1$ for some positive integer $k \geq 2$.

Case 1: $n=5k$. for the least value of n , we have K_{10} , which is $S_3 \cup S_4$ -decomposable, illustrated below.

Let $V(K_{10}) = \{1, 2, \dots, 10\}$. The nine copies of $S_3 \cup S_4$ are,

$S\{8: 9, 10\} \cup S\{4: 5, 6, 7\}$, $S\{9: 1, 10\} \cup S\{8: 2, 4, 5\}$, $S\{9: 4, 5\} \cup S\{10: 2, 3, 7\}$,
 $S\{2: 7, 9\} \cup S\{10: 1, 4, 5\}$, $S\{7: 8, 9\} \cup S\{3: 4, 5, 6\}$, $S\{6: 9, 10\} \cup S\{1: 5, 7, 8\}$,
 $S\{6: 1, 2\} \cup S\{3: 7, 8, 9\}$, $S\{5: 6, 7\} \cup S\{1: 2, 3, 4\}$, $S\{6: 7, 8\} \cup S\{2: 3, 4, 5\}$

When $k=3$, K_{15} is $S_3 \cup S_4$ -decomposable, giving 21 copies of $S_3 \cup S_4$,

$S\{1: 13, 14, 15\} \cup S\{10: 2, 11\}$, $S\{13: 4, 5\} \cup S\{2: 14, 15\}$, $S\{1: 10, 11, 12\} \cup S\{2: 5, 6\}$,
 $S\{1: 6, 7, 8\} \cup S\{2: 9, 10\}$, $S\{2: 7, 8, 11\} \cup S\{3: 4, 5\}$, $S\{3: 6, 7, 8\} \cup S\{2: 12, 13\}$,
 $S\{3: 10, 11, 12\} \cup S\{1: 2, 9\}$, $S\{4: 13, 14, 15\} \cup S\{3: 9, 13\}$, $S\{4: 5, 6, 7\} \cup S\{3: 5, 14\}$,
 $S\{4: 8, 9, 10\} \cup S\{5: 6, 7\}$, $S\{12: 11, 13, 14\} \cup S\{5: 8, 9\}$, $S\{5: 10, 11, 12\} \cup S\{13: 14, 15\}$
 $S\{5: 13, 14, 15\} \cup S\{6: 7, 8\}$, $S\{6: 9, 10, 11\} \cup S\{7: 14, 15\}$, $S\{7: 8, 9, 10\} \cup S\{6: 12, 13\}$,
 $S\{7: 11, 12, 13\} \cup S\{6: 14, 15\}$, $S\{8: 13, 14, 15\} \cup S\{9: 10, 11\}$, $S\{11: 13, 14, 15\} \cup S\{9: 8, 12\}$,
 $S\{8: 10, 11, 12\} \cup S\{9: 13, 14\}$, $S\{10: 13, 14, 15\} \cup S\{4: 11, 12\}$ and
 $S\{15: 9, 12, 14\} \cup S\{2: 3, 4\}$.

For $k>3$, we have, $E(K_{5k}) = E(K_{5(k-2)}) \cup E(K_{10}) \cup E(K_{5(k-2), 10}) \dots (*)$

When $k = 4$, equation (*) yields the edge set of K_{20} as the disjoint union of the edge sets of two K_{10} s and a $K_{10, 10}$, each of which is $S_3 \cup S_4$ -decomposable by 2.1 (Theorem A) and the $S_3 \cup S_4$ -decomposition of K_{10}

By lemma 3.1, K_{5k} is $S_3 \cup S_4$ -decomposable for $k>3$.

Case 2: $n=5k+1$, $k \geq 2$.

We adopt the same inductive procedure as in case 1. When $k=2$ and $k=3$ we get K_{11} and K_{16} , both $S_3 \cup S_4$ -decomposable. The relation corresponding to (*) is

$E(K_{5k+1}) = E(K_{5(k-2)}) \cup E(K_{11}) \cup E(K_{5(k-2), 11}) \dots (**)$

By lemma 3.1, K_{5k+1} is $S_3 \cup S_4$ -decomposable.

The proof is completed.

3.4 Theorem B

The necessary and sufficient condition that the complete graph K_n is $S_{p+1} \cup S_{q+1}$ -decomposable, $p \neq q$ is

$$n > p+q+2 \quad \text{and} \quad (ii) \quad n \equiv 0, 1 \pmod{2(p+q)}.$$

Proof : (necessity) If K_n is $S_{p+1} \cup S_{q+1}$ -decomposable, then $S_{p+1} \cup S_{q+1}$ is a proper subgraph of K_n . Hence, $n > p+q+2$.

$$e(S_{p+1} \cup S_{q+1}) \mid e(K_n) \text{ implies } (p+q) \mid n \text{ or } n \equiv 1 \pmod{2(p+q)}, \text{ equivalently } n \equiv 0, 1 \pmod{2(p+q)}.$$

(sufficiency) The proof is by induction on n . The existence of such a K_n is proved for $p=2$ and $q=3$ (Theorem 3.3)

Let $p > 2$ and $q > 3$ and satisfying (i) and (ii) and the graph K_m is $S_{p+1} \cup S_{q+1}$ -decomposable

$$\text{Let } m \equiv 0, \pmod{2(p+q)}.$$

The next higher value of n is $m+2(p+q) \equiv 0 \pmod{2(p+q)}$ by the induction hypothesis.

$$E(K_{m+2(p+q)}) = E(K_m) \cup E(K_{2(p+q)}) \cup E(K_{m, 2(p+q)})$$

The graphs on the r.h.s. are $S_{p+1} \cup S_{q+1}$ -decomposable, by the induction hypothesis.

By Lemma 3.1, $K_{m+2(p+q)}$ is $S_{p+1} \cup S_{q+1}$ -decomposable, When $m \equiv 0 \pmod{2(p+q)}$, $m+1 \equiv 1 \pmod{2(p+q)}$.

$$\text{From, } K_{m+1} = K_m \vee K_1 \text{ we have, } E(K_{m+1}) = E(K_m) \cup E(K_{1,m}).$$

By the induction hypothesis, Theorem A and lemma 3.1, K_{m+1} is $S_{p+1} \cup S_{q+1}$ -decomposable.

By induction principle, for any n satisfying the conditions ((i) $n > p+q+2$ and (ii) $n \equiv 0, 1 \pmod{2(p+q)}$) K_n is $S_{p+1} \cup S_{q+1}$ -decomposable.

Conclusion: We have given, in this paper, a characterization for the $S_{p+1} \cup S_{q+1}$,

$p \geq 2, q \geq 2$, decomposability of complete graphs. Similar results for other graphs will be given in another paper.

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