A Note on Sums of Greatest (Least) Prime Factors

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Abstract

Let \( a_m(n) \) be the \( m \)-th power of the least prime factor in the prime factorization of \( n \). We prove the asymptotic formula

\[
\sum_{i=2}^{n} a_m(i) \sim \frac{1}{m+1} \frac{n^{m+1}}{\log n}.
\]

Let \( b_m(n) \) be the \( m \)-th power of the greatest prime factor in the prime factorization of \( n \). We prove the asymptotic formula

\[
\sum_{i=2}^{n} b_m(i) \sim \frac{\zeta(m+1)}{m+1} \frac{n^{m+1}}{\log n},
\]

where \( \zeta(s) \) is the Riemann’s Zeta Function. Consequently

\[
\lim_{n \to \infty} \frac{\sum_{i=2}^{n} b_m(i)}{\sum_{i=2}^{n} a_m(i)} = \zeta(m+1).
\]

In particular if \( m = 1 \) we obtain

\[
\lim_{n \to \infty} \frac{\sum_{i=2}^{n} b_1(i)}{\sum_{i=2}^{n} a_1(i)} = \zeta(2) = \frac{\pi^2}{6}.
\]

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1 Introduction and Lemmas

Let $m$ be a positive integer and let $b_m(n)$ be the $m$-th power of the greatest prime factor in the prime factorization of $n$. For example if $n = 12$ then $b_1(12) = 3$ and $b_4(12) = 3^4$, if $n = 18$ then $b_1(18) = 3$ and $b_2(18) = 3^2$, if $n = 5$ then $b_1(5) = 5$ and $b_4(5) = 5^4$. In this note we prove the asymptotic formula

$$\sum_{i=2}^{n} b_m(i) \sim \frac{\zeta(m+1) n^{m+1}}{m+1 \log n}. \quad (1)$$

If $m = 1$ is asymptotic formula is well-known (see either [1] or [4]). In the proof of (1) we use a similar method of proof already used in the proof of other theorems (see [3]).

The following lemma is a consequence of the prime number theorem (see for example [2]).

**Lemma 1.1** Let $m$ be a nonnegative integer and let $s_m(x)$ be the sum of the $m$-th powers of the primes not exceeding $x$. We have the following asymptotic formula

$$s_m(x) = \sum_{p \leq x} p^m = \frac{x^{m+1}}{(m+1) \log x} + h(x) \frac{x^{m+1}}{\log x}, \quad (2)$$

where $p$ denotes a positive prime and $h(x) \to 0$. Note that $h(x)$ depends of $m$.

Note that if $m = 0$ equation (2) becomes the Prime Number Theorem. That is, $s_0(x) = \pi(x)$, where $\pi(x)$ is the prime counting function.

Let $m$ be a positive integer and let $a_m(n)$ be the $m$-th power of the least prime factor in the prime factorization of $n$. For example if $n = 12$ then $a_1(12) = 2$ and $a_4(12) = 2^4$, if $n = 18$ then $a_1(18) = 2$ and $a_2(18) = 2^2$, if $n = 5$ then $a_1(5) = 5$ and $a_4(5) = 5^4$. In this note we prove the asymptotic formula

$$\sum_{i=2}^{n} a_m(i) \sim \frac{n^{m+1}}{(m+1) \log n}. \quad (3)$$

We also shall need the following lemma.

**Lemma 1.2** Let $m$ be a positive integer. We have the following formula

$$\sum_{j=1}^{\infty} j \left( \frac{1}{j^{m+1}} - \frac{1}{(j+1)^{m+1}} \right) = \sum_{j=1}^{\infty} \left( \frac{1}{j^m} - \frac{j}{(j+1)^{m+1}} \right) = \zeta(m+1),$$

where $\zeta(s)$ is the Riemann's Zeta Function.
Proof. We have
\[
\sum_{j=1}^{n} \left( \frac{1}{j^m} - \frac{j}{(j+1)^{m+1}} \right) - \sum_{j=1}^{n} \frac{1}{j^{m+1}} = \sum_{j=1}^{n} \left( \frac{j-1}{j^{m+1}} - \frac{j}{(j+1)^{m+1}} \right).
\]
Therefore
\[
\sum_{j=1}^{n} \left( \frac{1}{j^m} - \frac{j}{(j+1)^{m+1}} \right) = \left( \sum_{j=1}^{n} \frac{1}{j^{m+1}} \right) - \frac{n}{(n+1)^{m+1}}.
\]
Now
\[
\zeta(m+1) = \sum_{j=1}^{\infty} \frac{1}{j^{m+1}} = \lim_{n \to \infty} \sum_{j=1}^{n} \frac{1}{j^{m+1}}.
\]
The lemma is proved.

Note that a consequence of equation (2) is the following inequality
\[
s_m(x) = \sum_{p \leq x} p^m < h \frac{x^{m+1}}{(m+1) \log x}, \quad (4)
\]
where \( h > 1 \). This inequality holds for \( x \geq x_0 \), where \( x_0 \) depend of \( m \).

## 2 Main Results

Now, we shall prove the mentioned results. Namely, formulas (1) and (3).

**Theorem 2.1** We have the following asymptotic formula
\[
\sum_{i=2}^{n} a_m(i) \sim \frac{1}{m+1} \frac{n^{m+1}}{\log n}, \quad (5)
\]
where \( m \) is an arbitrary but fixed positive integer.

Proof. Let \( A(n,p) \) be the number of positive integer not exceeding \( n \) such that their least prime factor is the prime \( p \). Therefore
\[
\sum_{2 \leq p \leq n} A(n,p) = n - 1.
\]
We have
\[
\sum_{i=2}^{n} a_m(i) = \sum_{2 \leq p \leq n} p^m A(n,p) = \sum_{2 \leq p \leq \sqrt[n]{x}} p^m A(n,p) + \sum_{x^{1/3} < p \leq n} p^m A(n,p), \quad (6)
\]
where \( k \geq 2 \) is a positive integer.

Consider the first sum in (6). Namely

\[
\sum_{2 \leq p \leq \frac{n}{k}} p^m A(n, p).
\]

We have the following trivial inequality

\[
A(n, p) \leq \left\lfloor \frac{n}{p} \right\rfloor \leq \frac{n}{p}.
\]

Therefore (see (4))

\[
\sum_{2 \leq p \leq \frac{n}{k}} p^m A(n, p) \leq \sum_{2 \leq p \leq \frac{n}{k}} p^m \frac{n}{p} = n \sum_{2 \leq p \leq \frac{n}{k}} p^{m-1} \leq nh \frac{(\frac{n}{\lambda})^m}{m \log \frac{n}{\lambda}}
\]

\[
= \frac{h(m+1)}{mk^m} \frac{1}{1 - \frac{\log k}{\log n}} (m+1) \log n
\]

\[
\leq \left( \frac{h(m+1)}{mk^m} + \lambda \right) \frac{n^{m+1}}{(m+1) \log n} \quad (\lambda > 0)
\]

That is

\[
\sum_{2 \leq p \leq \frac{n}{k}} p^m A(n, p) = g(n) \frac{n^{m+1}}{(m+1) \log n}, \quad (7)
\]

where

\[
0 < g(n) < \frac{h(m+1)}{mk^m} + \lambda \quad (\lambda > 0). \quad (8)
\]

Consider the second inequality in (6). Namely

\[
\sum_{\frac{n}{k} < p \leq n} p^m A(n, p).
\]

If \( n \) is large then \( k < p \). On the other hand \( kp > n \). Consequently the unique multiple of \( p \) less than or equal to \( n \) such that \( p \) is its least prime factor is \( p \).

That is, we have \( A(n, p) = 1 \). Therefore (see lemma 1.1)

\[
\sum_{\frac{n}{k} < p \leq n} p^m A(n, p) = \sum_{2 \leq p \leq n} p^m - \sum_{2 \leq p \leq \frac{n}{k}} p^m = \frac{n^{m+1}}{(m+1) \log n} + h(n) \frac{n^{m+1}}{\log n}
\]

\[
- \left( \frac{n}{\lambda} \right)^{m+1} - h \left( \frac{n}{k} \right) \left( \frac{n}{\lambda} \right)^{m+1} \frac{(\frac{n}{\lambda})^{m+1}}{\log (\frac{n}{\lambda})} = \left( 1 - \frac{1}{k^{m+1}} \right) \frac{n^{m+1}}{(m+1) \log n}
\]

\[
+ \left( h(n) - h \left( \frac{n}{k} \right) \frac{1}{k^{m+1}} \frac{1}{\log n} \right) \frac{n^{m+1}}{\log n} = \left( 1 - \frac{1}{k^{m+1}} + q_k(n) \right) \frac{n^{m+1}}{(m+1) \log n}
\]

\[
+ p_k(n) \frac{n^{m+1}}{\log n} = \frac{n^{m+1}}{(m+1) \log n} - \frac{1}{k^{m+1}} \frac{n^{m+1}}{(m+1) \log n} + r_k(n) \frac{n^{m+1}}{(m+1) \log n}.
\]
Sums of greatest (least) prime factors

where \( h(n) \to 0 \), \( q_k(n) \to 0 \), \( p_k(n) \to 0 \) and \( r_k(n) \to 0 \). That is

\[
\sum_{\frac{2}{k} < p \leq n} p^m A(n, p) = \frac{n^{m+1}}{(m+1) \log n} - \frac{1}{k^{m+1}} \frac{n^{m+1}}{(m+1) \log n} + r_k(n) \frac{n^{m+1}}{(m+1) \log n}.
\]  

(9)

where \( r_k(n) \to 0 \).

We have

\[
\sum_{i=2}^{n} a_m(i) = \frac{1}{m+1} \frac{n^{m+1}}{\log n} + f(n) \frac{1}{m+1} \frac{n^{m+1}}{\log n}.
\]

(10)

Substituting equations (7) and (9) into (6) we obtain

\[
\sum_{i=2}^{n} a_m(i) = \frac{n^{m+1}}{(m+1) \log n} + \left( -\frac{1}{k^{m+1}} + r_k(n) + g(n) \right) \frac{n^{m+1}}{(m+1) \log n}.
\]

Consequently

\[
f(n) = -\frac{1}{k^{m+1}} + r_k(n) + g(n).
\]

(11)

Let \( \epsilon > 0 \). If we choose \( k \) sufficiently large then

\[
\left| -\frac{1}{k^{m+1}} \right| < \frac{\epsilon}{3}, \quad |r_k(n)| < \frac{\epsilon}{3}, \quad 0 < g(n) < \frac{\epsilon}{3}.
\]

Therefore we have (see (11))

\[
|f(n)| < \epsilon,
\]

if \( n \) is sufficiently large.

Now, \( \epsilon \) is arbitrarily little. Therefore

\[
\lim_{n \to \infty} f(n) = 0.
\]

(12)

Equations (10) and (12) give (5). The theorem is proved.

**Theorem 2.2** We have the following asymptotic formula

\[
\sum_{i=2}^{n} b_m(i) \sim \frac{\zeta(m+1)}{m+1} \frac{n^{m+1}}{\log n},
\]

(13)

where \( m \) is an arbitrary but fixed positive integer.

Proof. Let \( B(n, p) \) be the number of positive integers not exceeding \( n \) such that their greatest prime factor is the prime \( p \). Therefore

\[
\sum_{2 \leq p \leq n} B(n, p) = n - 1.
\]
We have
\[
\sum_{i=2}^{n} b_m(i) = \sum_{2 \leq p \leq n} p^m B(n, p) = \sum_{2 \leq p \leq \frac{n}{k+1}} p^m B(n, p) + \sum_{\frac{n}{k+1} < p \leq \frac{n}{k}} p^m B(n, p) + \sum_{\frac{n}{k} < p \leq n} p^m B(n, p).
\]  
(14)

Consider the first sum in (14). Namely
\[
\sum_{\frac{n}{k+1} < p \leq \frac{n}{k}} p^m B(n, p).
\]

We have the following trivial inequality
\[
B(n, p) \leq \left\lfloor \frac{n}{p} \right\rfloor \leq \frac{n}{p}.
\]

As in theorem 2.1 we obtain
\[
\sum_{2 \leq p \leq \frac{n}{k+1}} p^m B(n, p) = \frac{g(n)}{n^{m+1}} (m+1) \log n,
\]  
(15)

where
\[
0 < g(n) < \frac{h(m+1)}{m(k+1)^m} + \lambda \quad (\lambda > 0).
\]  
(16)

Now, consider the sum (see (14))
\[
\sum_{\frac{n}{j+1} < p \leq \frac{n}{j}} p^m B(n, p) \quad (j = 1, 2, \ldots, k).
\]  
(17)

If \( n \) is large then \( j \leq k < p \). On the other hand \( jp \leq n \) and \( (j + 1)p > n \). Consequently the multiples of \( p \) less than or equal to \( n \) such that \( p \) is their greatest prime factor are \( p, 2p, \ldots, jp \). That is, we have \( A(n, p) = j \). Consequently (see (17))
\[
\sum_{\frac{n}{j+1} < p \leq \frac{n}{j}} p^m B(n, p) = \frac{n^m}{j} (m+1) \log \frac{n}{j} \quad (j = 1, 2, \ldots, k).
\]  
(18)

Lemma 1.1 gives
\[
\sum_{\frac{n}{j+1} < p \leq \frac{n}{j}} p^m = \frac{\left( \frac{n}{j} \right)^{m+1}}{(m+1) \log \left( \frac{n}{j} \right)} + \frac{\left( \frac{n}{j} \right)^{m+1}}{(m+1) \log \left( \frac{n}{j} \right)} - \frac{\left( \frac{n}{j+1} \right)^{m+1}}{(m+1) \log \left( \frac{n}{j+1} \right)}
\]
where \( h(n) \to 0 \), \( q_j(n) \to 0 \), \( p_j(n) \to 0 \) and \( r_j(n) \to 0 \). That is

\[
\sum_{\frac{p}{m+1} < p \leq \frac{m}{j}} p^m = \left( \frac{1}{j^{m+1}} - \frac{1}{(j+1)^{m+1}} \right) \frac{n^{m+1}}{(m+1) \log n} + r_j(n) \frac{n^{m+1}}{(m+1) \log n}, \tag{19}
\]

where \( r_j(n) \to 0 \).

Substituting (19) into (18) we obtain

\[
\sum_{\frac{p}{m+1} < p \leq \frac{m}{j}} p^m B(n, p) = \left( \frac{1}{j^m} - \frac{j}{(j+1)^m} \right) \frac{n^{m+1}}{(m+1) \log n} + r_j'(n) \frac{n^{m+1}}{(m+1) \log n}, \quad (j = 1, 2, \ldots, k), \tag{20}
\]

where \( r_j'(n) = j \ r_j(n) \to 0 \).

We have

\[
\sum_{i=2}^{n} b_m(i) = \frac{\zeta(m+1) n^{m+1}}{m+1 \log n} f(n) \frac{n^{m+1}}{m+1 \log n}. \tag{21}
\]

Substituting (15) and (20) into (14) we find that (see lemma 1.2)

\[
\sum_{i=2}^{n} b_m(i) = \left( g(n) + \sum_{j=1}^{k} \left( \frac{1}{j^m} - \frac{j}{(j+1)^m} \right) + \sum_{j=1}^{k} r_j'(n) \right) \frac{n^{m+1}}{(m+1) \log n}
\]

\[
= \frac{\zeta(m+1) n^{m+1}}{m+1 \log n} f(n) \frac{n^{m+1}}{m+1 \log n}.
\]

\[
= \frac{\zeta(m+1) n^{m+1}}{m+1 \log n} f(n) \frac{n^{m+1}}{m+1 \log n}.
\]
Consequently
\[ f(n) = g(n) - \sum_{j=k+1}^{\infty} \left( \frac{1}{j^m} - \frac{j}{(j + 1)^{m+1}} \right) + \sum_{j=1}^{k} r'_j(n). \] (22)

Let \( \epsilon > 0 \). If we choose \( k \) sufficiently large then
\[ 0 < \sum_{j=k+1}^{\infty} \left( \frac{1}{j^m} - \frac{j}{(j + 1)^{m+1}} \right) < \frac{\epsilon}{3}, \quad 0 < g(n) < \frac{\epsilon}{3}. \]

On the other hand, since
\[ r'_j(n) \to 0 \quad (j = 1, 2, \ldots, k), \]
if \( n \) is sufficiently large then we have
\[ \left| r'_j(n) \right| < \frac{\epsilon}{3k} \quad (j = 1, 2, \ldots, k). \]

Therefore we have (see (22))
\[ \left| f(n) \right| < \epsilon. \]

Now, \( \epsilon \) is arbitrarily little. Hence
\[ \lim_{n \to \infty} f(n) = 0. \] (23)

Equations (21) and (23) give (13). The theorem is proved.

**Corollary 2.3** The following limits hold
\[ \lim_{n \to \infty} \frac{\sum_{i=2}^{n} b_m(i)}{\sum_{i=2}^{n} a_m(i)} = \zeta(m + 1). \]

In particular if \( m = 1 \) we obtain
\[ \lim_{n \to \infty} \frac{\sum_{i=2}^{n} b_1(i)}{\sum_{i=2}^{n} a_1(i)} = \zeta(2) = \frac{\pi^2}{6}. \]

Proof. It is an immediate consequence of Theorem 2.1 and Theorem 2.2. The corollary is proved.

Let \( c \) be a composite number. If we consider only composite numbers in Corollary 2.3 then we have the following corollary.

**Corollary 2.4** We have the following limit
\[ \lim_{n \to \infty} \frac{\sum_{c \leq n} b_m(c)}{\sum_{c \leq n} a_m(c)} = \infty. \]
Proof. Let $p$ be a prime number. We have $a_m(p) = b_m(p) = p^m$. Therefore (Theorem 2.1, Theorem 2.2 and Lemma 1.1)

$$\frac{\sum_{c \leq n} b_n(c)}{\sum_{c \leq n} a_m(c)} = \frac{\sum_{i=2}^{n} b_m(i) - \sum_{p \leq n} b_m(p)}{\sum_{i=2}^{n} a_m(i) - \sum_{p \leq n} a_m(p)} = \frac{\sum_{i=2}^{n} b_m(i) - \sum_{p \leq n} p^m}{\sum_{i=2}^{n} a_m(i) - \sum_{p \leq n} p^m}$$

$$= \frac{\zeta(m+1)-1}{m+1} \frac{n^{m+1}}{\log n} + o \left( \frac{n^{m+1}}{\log n} \right) = \frac{\zeta(m+1)-1}{m+1} + o(1).$$

The corollary is proved.

Let $p^k$ be a prime power. We have $a_m(p^k) = b_m(p^k) = p^m$. On the other hand, if $d$ is not a prime power then $a_m(d) < b_m(d)$. We have the following corollary

**Corollary 2.5** The following limit holds

$$\lim_{n \to \infty} \frac{\sum_{d \leq n} b_m(d)}{\sum_{d \leq n} a_m(d)} = \infty.$$ 

Proof. We have

$$\sum_{p \leq n} p^m \leq \sum_{p^k \leq n} p^m \leq \sum_{i=2}^{n} a_m(i).$$

Consequently (Lemma 1.1 and Theorem 2.1)

$$\sum_{p^k \leq n} p^m \sim \frac{1}{m+1} n^{m+1}.$$ 

(24)

Therefore (Theorem 2.1, Theorem 2.2 and equation (24))

$$\frac{\sum_{d \leq n} b_m(d)}{\sum_{d \leq n} a_m(d)} = \frac{\sum_{i=2}^{n} b_m(i) - \sum_{p^k \leq n} b_m(p^k)}{\sum_{i=2}^{n} a_m(i) - \sum_{p^k \leq n} a_m(p^k)} = \frac{\sum_{i=2}^{n} b_m(i) - \sum_{p^k \leq n} p^m}{\sum_{i=2}^{n} a_m(i) - \sum_{p^k \leq n} p^m}$$

$$= \frac{\zeta(m+1)-1}{m+1} \frac{n^{m+1}}{\log n} + o \left( \frac{n^{m+1}}{\log n} \right) = \frac{\zeta(m+1)-1}{m+1} + o(1).$$

The corollary is proved.

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**References**


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