On Poset of Subhypergroup and Hyper Lattices

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Abstract

In this Paper we have described the Poset of Sub(H) for remarkable classes such as rotary closed subhypergroups of Permutation group related to open problem given in [3]. Also we Prove that Sym_n(G) is a hyper group. We prove some necessary and sufficient condition for Poset of subhypergroup to be a hypersemilattice. Also we prove that Sub (G) is coincide with the lattice subgroup of hypergroup of permutation group. Finally we prove Principal Filter is hyper lattice.

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Introduction

The theory of Hyperstructures was introduced in 1934 by Marty [1] at the 8th Congress of Scandinavian mathematicians. This theory has been subsequently developed by Corsini[4], Mittas [2] and by various authors. Basic definitions and propositions are found in [4]. Marius Tarauceanu contributed to the study of Poset of subhypergroup of a hypergroup. He had drawn conclusions on Poset (Sub(H),⊆) and he had also given some open problems on above stated Poset and lattices and description about rotary mapping is given in[6] and [8]. These conclusions and open problems constitute the starting point for our paper. In this section, we study the Poset of subhypergroup of a permutation group using rotary closed subhypergroup and we study the necessary and sufficient condition for Poset to be a hypersemilattice. Finally we study the structure of Principal Filter which constitutes a hyperlattice. Notations and definitions are used from [3],[7],[6].

1. Basic Notations and Terminology

Definition 1.1 [3]: A hyper Operation on H is a map  ⊙:H x H → P*(H).

Definition 1.2[3]: If the hyperoperation” ⊙ “is associative and a ⊙ H=H=H ⊙ a, then (H, ⊙) is a Hypergroup.

Definition 1.3 [3]: A hyper group (H, ⊙) is called a join space if “⊙” is commutative and a/b ∼ c/d ⇒ a ⊙ d ∼ b ⊙ c, where a/b = {x ∈ H/a ∈ b ⊙ x}

Definition 1.4[4 ]: Let L be a non-empty set and ⊕ : L x L → P(L) be a hyperoperation , where P(L) is a power set of L and P*(L) =P(L) -{Ø} and ⊗ : L x L → L be an operation. Then (L, ⊗ ⊕) is a hyperlattice if for all a, b, c ∈ L.
1. a ∈ a ⊕ a, a ⊗ a=a
2. a ⊕ b=b ⊕ a, a ⊗ b=b ⊗ a
3. (a ⊕ b) ⊕ c=a ⊕ (b ⊕ c) ,(a⊗ b) ⊗ c=a ⊗ (b ⊗ c).
4. a ⊙ [a ⊗ (a ⊕ b)] ∩ [a ⊕ (a ⊗ b)]
5. a ∈ a ⊕ b implies a ⊗ b=b.

Definition1.5[7]: Let L be a non-empty set with a hyper operation ⊗ On L satisfying the following conditions, for all a, b, c ∈ L
1. a ∈ a ⊗ a (Idempotent)
2. a \circ b = b \circ a \quad \text{(Commutative)}
3. (a \circ b) \circ c = a \circ (b \circ c). \quad \text{(Associative)}

Then \((L, \circ)\) is called a hypersemilattice.

**Definition 1.6[7]**: Let \((L, \circ)\) be a hypersemilattice. An element a \in L is called absorbent element of L if it satisfies c \circ a a \circ c for all c \in L. An element b \in L is called fixed element of L if it satisfies b \circ c = \{b\} for all c \in L.

### 2. Rotary closed subhypergroups

**Definition 2.1 [6]**: Let G be a group, let Sym(G) be the group of all permutations on G, and let Sym\(_e\)(G) be the stabilizer of the identity e \in G in Sym(G). Given two permutations \(\Phi, \Psi\) from Sym\(_e\)(G) and an element g \in G, we define a new permutation \(\Phi \circ_g \Psi = L_{\Phi(g)}^{-1} \circ L_g \Psi\), where \(L_{\Phi(g)}^{-1} \circ L_g \in\) Sym(G) are left multiplications by the elements \(\Phi(g)\) and \(g\). A subgroup H of Sym\(_e\)(G) closed under taking products of this form is called rotary closed i.e H \subseteq Sym\(_e\)(G) is called rotary close provided \(\Phi \circ_g \Psi \in H\) for all \(\Phi, \Psi \in H\) and \(g \in G\).

**Theorem 2.2**: Let Sym\(_e\)(G) be the stabilizer of the identity e \in G in Sym(G) then Sym\(_e\)(G) with binary operation \(\circ\) is a hyper group.

**Proof**: Let us define \(\Phi \circ_g \Psi = \{\Phi \circ_g \Psi/g \in G\}\). Let \(\Phi\) be arbitrary permutations of Sym\(_e\)(G). Suppose \(\Psi \in\) Sym\(_e\)(G) is arbitrary and let \(\Psi = L_g^{-1} \Phi L_{\Phi(g)} \Psi\). Then \(\Phi \circ_g \Psi = \Psi\). So, Sym\(_e\)(G) = \(\Phi \circ_g\) Sym\(_e\)(G). Let \(\Phi, \Psi, \Psi\) be permutations of Sym\(_e\)(G). \((\Phi \circ_g \Psi) \circ_g \Psi = L_{\Phi(g)} \circ_g \Psi L_g \Psi = L_{\Phi(g)} \Psi L_g = L_{\Phi(g)} \Psi L_g \Psi = L_{\Phi(g)} \Psi L_g \Psi = L_{\Phi(g)} \Psi L_g \Psi = L_{\Phi(g)} \Psi L_g \Psi = L_{\Phi(g)} \Psi L_g \Psi = L_{\Phi(g)} \Psi L_g \Psi\). Hence the result. \(\blacksquare\)

**Theorem 2.3**: For a hypergroup Sym\(_e\)(G), Sub (Sym\(_e\)(G)) = L (Sym\(_e\)(G)) , where Sub(Sym\(_e\)(G)) is a rotary closed subhypergroup of a Permutation group G and \(L(\text{Sym}\(_e\)(G))\) is set of all Subhypergroups of a Hyper group Sym\(_e\)(G).

**Proof**: Let \(k \in\) Sub (Sym\(_e\)(G)) i.e. \(k\) is a subhypergroup of a permutation group. Let \(\Phi \circ \Psi \in K\) and let \(\Psi = L_g^{-1} \Phi L_{\Phi(g)} \Psi\). Then \(\Phi \Psi = \Phi \circ \Psi = L_{\Phi(g)}^{-1} \Phi L \Psi\). So \(\Psi = L_g^{-1} \Phi L_{\Phi(g)}^{-1} \circ L_{\Phi(g)}^{-1} \Phi L \Psi\). It implies \(\Psi = \Phi \circ \Psi = L_{\Phi(g)}^{-1} \Phi L \Psi\). That is \(K \subseteq L (\text{Sym}\(_e\)(G))\). Conversely, Let \(K \subseteq L (\text{Sym}\(_e\)(G))\) and \(\Phi \in\) Sym\(_e\)(G) and for any
element $g \in G$. Consider $\Phi \circ g \psi = L_{\phi(g)}^{-1} \Phi L \psi$ for any $\psi \in \text{Sym}_e (G)$. Let $\psi = L_g^{-1} \Phi^{-1} L_{\phi(g)} \Phi$, which gives $\Phi \circ g \psi = \Phi$, that is $\Phi \in \Phi \circ g \text{Sym}_e (G)$. $\text{Sym}_e (G) \subseteq \Phi \circ g \text{Sym}_e (G)$ Now let us consider $\gamma \in \Phi \circ g \text{Sym}_e (G)$. By definition of rotary closed subhypergroup $\gamma \in \text{Sym}_e (G)$, that is $\Phi \circ g \text{Sym}_e (G) \subseteq \text{Sym}_e (G)$. Therefore by both the results $\text{Sym}_e (G) = \Phi \circ g \text{Sym}_e (G)$. Similarly we can prove $\text{Sym}_e (G) \circ g \Phi = \text{Sym}_e (G)$. This is reproductivity law. By theorem [2.2], it is associative. Therefore $K \in \text{Sub} (\text{Sym}_e (G))$. Hence the result. 

**Theorem 2.4** : A necessary and sufficient condition for a poset of $(\text{Sub}(\text{Sym}_e (G), \circ ))$ is a Hypersemilattice is that $\Phi \circ g \psi = \Phi$. Provided $\text{Sub}(\text{Sym}_e (G))$ is a rotary closed and $\Phi$ is a fixed element of $\text{Sub}(\text{Sym}_e (G))$.

**Proof:** Let $(\text{Sub}(\text{Sym}_e (G), \circ ))$ is a Poset of rotary closed subhypergroup of a hyper group $\text{Sym}_e (G)$. To prove Poset $(\text{Sub}(\text{Sym}_e (G), \circ ))$ is a hypersemilattice, for any $\Phi, \psi \in \text{Sym}_e (G)$, let $\Phi \circ g \psi \in \text{Sub}(\text{Sym}_e (G))$ as it is rotary closed and $\Phi \circ g \psi = L_{\phi(g)}^{-1} \Phi L \psi$ where $g \in G$. And let us define $\psi = L_g^{-1} \Phi^{-1} L_{\phi(g)} \Phi$, then clearly $\Phi \circ g \psi = \Phi$. So it is well-defined. Obviously by the definition $\Phi \in \Phi \circ g \psi$. So $\circ g$ is idempotent. Let $\Phi \circ \psi$ and $\psi \circ \Phi$, therefore $\Phi \circ g \psi = \Phi$ and $\psi \circ g \Phi = \psi$ by well-defined statement. But $(\text{Sub}(\text{Sym}_e (G), \circ ))$ is Poset. By Antisymmetry Property, if $\Phi \circ \psi$ and $\psi \circ \Phi$ belongs to $\text{Sym}_e (G)$, then $\psi = \Phi$. Therefore $\Phi \circ g \psi = \psi \circ g \Phi$. So $\circ g$ is Commutative. Let $(\Phi \circ g \psi \circ g \gamma = \Phi \circ g \gamma\Phi = \Phi$ and similarly $\Phi \circ g (\psi \circ \gamma) = \Phi \circ g \psi = \Phi$. Therefore from both the results $\circ g$ is associative. Hence Poset of $(\text{Sub}(\text{Sym}_e (G), \circ ))$ is Hypersemilattice. Conversely, let $(\text{Sub}(\text{Sym}_e (G), \circ ))$ is Hypersemilattice. To prove it is a Poset. Let $\Phi \in \text{Sym}_e (G)$. As $(\text{Sub}(\text{Sym}_e (G), \circ ))$ is hypersemilattice and as $\Phi$ is a fixed element, by definition of fixed element $\Phi \circ g \Phi = \Phi$. Therefore $\circ g$ is reflexive. Secondly, let $\Phi, \psi \in \text{Sym}_e (G)$. So $\Phi \circ g \psi = \Phi$ and $\psi \circ g \Phi = \psi$, but by commutativity of hypersemilattice $\Phi \circ g \psi = \psi \circ g \Phi$, this implies $\Phi = \psi$. It proves Anti Symmetry. To prove transitivity, let $\Phi, \psi, \gamma \in \text{Sym}_e (G)$ and let $\Phi \circ \psi$ implies $\Phi \circ g \psi = \Phi$ and $\psi \circ g \gamma$ implies $\psi \circ g \gamma = \psi$. Now consider $\Phi \circ \gamma$ that is $\Phi \circ g \gamma = (\Phi \circ g \psi) \circ g \gamma$, by associativity of hypersemilattice $(\Phi \circ g \psi \circ g \gamma = \Phi \circ g (\psi \circ g \gamma)) = \Phi \circ g \psi = \Phi$ so this gives $\Phi \circ g \gamma = \Phi$. Therefore $\Phi \circ \gamma$. This proves Transitivity. Therefore $(\text{Sub}(\text{Sym}_e (G), \circ ))$ is Poset.
3. Principal Filter

Example 3.1[3]: Let (L, \wedge, V) be a complete lattice and for every a \in L, denoted by F(a) the principal filter of L generated by a (i.e. F(a) = \{x \in L / a \leq x\}). Then L is join space under the hyper operation a \circ b = F(a \wedge b), for all a, b \in L.

Property 3.2[3]: For the join space (L, \circ) given by example [3.1], the following equality holds.

\[ \text{Sub (L)} = F(L) = \{F(a) / a \in L\}. \]

Theorem 3.3: (F(L), \otimes, \oplus) is hyper lattice.

Proof: Let x, y \in F(a), then by [3.1] x \leq a and b \leq x. Let x \in F(a) it implies x \in F(a) always. Therefore F(x) \subseteq F(a) \oplus F(x). Let x \in F(a) \oplus F(b). So x \in F(a) or x \in F(b), i.e. a \leq x or b \leq x. or otherwise b \leq x or a \leq x. So x \in F(b) or x \in F(a). Therefore x \in F(b) \oplus F(a). So F(a) \oplus F(b) \subseteq F(b) \oplus F(a). Similarly we can prove F(b) \oplus F(a) \subseteq F(a) \oplus F(b). \oplus is commutative. We know that [F(a) \oplus F(b)] \oplus F(c) = \{x \in L / a \oplus b \leq x\} \oplus F(c) = \{p \in L / x \oplus c \leq p\} = \{a \oplus (b \oplus c) \leq p\} = \{a \oplus y \leq p / b \oplus c \leq y\} = F(a) \oplus [F(b) \oplus F(c)]. \oplus is associative. Similarly we can prove for \otimes. To prove F(a) \in [F(a) \otimes (F(a) \oplus F(b))] \cap [F(a) \oplus (F(a) \otimes F(b))]. Consider let x \in F(a) \otimes (F(a) \oplus F(b)) = \{x \in L/ a \otimes (a \oplus b) \leq x\} = \{x \in L/ a \leq x\} = \{x \in L/ a \oplus (a \otimes b) \leq x\}. Therefore F(a) \in [F(a) \otimes (F(a) \oplus F(b))] and F(a) \in [F(a) \oplus (F(a) \otimes F(b))]. Finally F(a) \in [F(a) \otimes (F(a) \oplus F(b))] \cap [F(a) \oplus (F(a) \otimes F(b))]. Let F(a) \subseteq F(a) \oplus F(b) \oplus F(b). By definition, F(a) \oplus F(b) = \{x \in L/ a \oplus b \leq x\}. As x \in F(b) means b \leq x and b \oplus (a \otimes b) \leq x. So b \leq x or (a \otimes b) \leq x. So a \leq x and b \leq x. This implies x \in F(a) \otimes F(b). Therefore F(a) \subseteq F(a) \otimes F(b). Similarly we can prove F(a) \otimes F(b) \subseteq F(a). Therefore (F(L), \otimes, \oplus) is hyper lattice.

References


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