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The Properties of Probability of Normal Chain

Li Chen

School of Mathematics and Statistics, Zhengzhou Normal University
Zhengzhou City, Hennan Province, 450044, China
clxuu6697@sina.com
Mailing address: No.6, Yingcai stree, North University City
Zhengzhou City, Henan Prov, China, 450044

Zhong-guang Fan

School of Mathematics and Statistics Zhengzhou Normal University.
Zhengzhou City, Hennan Province, 450044, China

Abstract

In order to better study the Markov chain of track structure typical bilateral birth-death process, nature has regular chain of in-depth study, this paper mainly discusses some probabilistic properties of normal chain.

Keywords: Normal chain; Stopping time; Jumping track

1 Introduction

For the Markov chain of stochastic process model, bilateral birth-death process is one of typical Markov chain. However, how to find out all solutions of bilateral birth-death process attracts the interesting of many experts, and it needs a lot of work. Academician Wang Zikun et. al proposed the method of combing the function structure and probability methods, and this method is considered to be more reasonable and attracted a lot of attention[1]-[3]. Therefore, it is necessary to study the probabilistic properties of normal chain corresponding to Q matrix. This paper will discuss probabilistic properties of normal chain.

2 Preliminary notes

Let $E=\{\cdots,-2,-1,0,1,2,\cdots\}$, $a_i, b_i, i=0,\pm1,\pm2,\cdots$ is a group of positive, $Q=(q_{i,j})$ is bilateral birth-death matrix of E, where

$$q_{i,j}=\begin{cases} b_i & j=i+1 \\ a_i & j=i-1 \\ -(a_i+b_i) & j=i \\ 0 & else \end{cases}$$

Let $p_{i,j}^{\min}(t)$ be a minimal transfer function, which is identified by the bilateral birth-death matrix $Q=(q_{i,j})$. And

$$\overline{p}_{i,j}(t)=\begin{cases} \delta_{i,j} & if \quad i=\Delta, j\in E_\Delta \\ p_{i,j}^{\min}(t) & if \quad i,j\in E \\ 1-\sum_k p_{i,j}^{\min}(t) & if \quad i\in E, j=\Delta \end{cases}$$

where $\Delta\notin E, E_\Delta=\{\Delta\}\cup E$, $\overline{p}_{i,j}(t)$ is honest minimal transition function, $\overline{Q}=(\overline{q}_{i,j})$ is density of $\overline{p}_{i,j}(t)$. For $\forall i,j\in E$, $\overline{p}_{i,j}(t)=p_{i,j}^{\min}(t)$, thus $\overline{q}_{i,j}=q_{i,j}$. Obviously, Δ is absorbing state, and $\forall i\in E, \overline{q}_{i\Delta}=0$.

$X=(\Omega, \mathrm{F}, \mathrm{F}_t, X_t, \theta_t, P^x)$, which is normal chain corresponding to $\overline{p}_{i,j}(t)$. Let T_n is the n jump times of $\{X_t\}$, $n=1,2,\cdots$, For convenience, assume T_0=0. For $\forall i\in E_\Delta$, $\delta_i=\inf\{t \mid X_t=i\}$.

$\forall i, j \in E$, $t>0$, $F_{i,j}^{(n)}(t) = P^i\left\{X_t = j, T_n > t\right\}$, thus:

$$F_{i,j}^{(1)}(t) = e^{-q_i t}\delta_{i,j},$$

$$F_{i,j}^{(n+1)} = P^i\left\{X_t = j, T_1 \geq t\right\} + P^i\left\{X_t = j, T_n \geq t, T_1 \leq t, X_{T_1} \in E\right\}$$

$$= e^{-q_i t}\delta_{i,j} + \int_0^t e^{-q_i s}\sum_{k\in E} q_{i,k}F_{k,j}^{(n)}(t-s)ds,$$

By the definition of minimal transfer function, $\lim_{n\to\infty} F_{i,j}^{(n)}(t) = p_{i,j}^{\min}(t)$.

$\left\{X_t = j, T_n > t\right\} \uparrow \left\{X_t = j, \delta > t\right\}$, so that

$$P^i\left\{X_t = j, \delta > t\right\} = \lim_{n\to\infty} F_{i,j}^{(n)}(t) = p_{i,j}^{\min}(t) = P^i\left\{X_t = j\right\},$$

and $P^i\left\{X_t \in E, \delta > t\right\} = P^i\left\{X_t \in E\right\}$, that is almost certainly getting $\delta > t$ from $X \in E$. Then, that is almost certainly getting $X_t = \Delta$ from $\delta \leq t$. Therefore $X \equiv \Delta$ is almost in $[\delta, \infty)$.

3. Probabilistic properties of normal chain

Property 1 $P^i\left\{\delta_m < \delta\right\} = \dfrac{z - z_i}{z - z_m}, i \geq m, P^i\left\{\delta < \delta_m\right\} = \dfrac{z_i - z_m}{z - z_m}$.

Prove: For $\forall m, n \in E, n \geq m+1$, let

$$x_i = P^i\left\{\delta_n < \delta_m\right\}, i = m, m+1, \cdots, n.$$

Obviously, $x_m = 0, x_n = 1$. And for $\forall i, m < i < n$,

$$x_i = P^i\left\{\delta_n < \delta_m\right\}$$

$$= P^i\left\{X_{T_1} = i-1, \delta_n < \delta_m\right\} + P^i\left\{X_{T_1} = i+1, \delta_n < \delta_m\right\}$$

$$= \frac{a_i}{a_i + b_i} P^{i-1}\{\delta_n < \delta_m\} + \frac{b_i}{a_i + b_i} P^{i+1}\{\delta_n < \delta_m\}$$

$$= \frac{a_i}{a_i + b_i} x_{i-1} + \frac{b_i}{a_i + b_i} x_{i+1}$$

Thus

$$x_{i+1} - x_i = \frac{a_i}{b_i}(x_i - x_{i-1}) = \cdots = \frac{a_{m+1}a_{m+2}\cdots a_i}{b_{m+1}b_{m+2}\cdots b_i}(x_{m+1} - x_m)$$

$$= \frac{a_{m+1}a_{m+2}\cdots a_i}{b_{m+1}b_{m+2}\cdots b_i} x_{m+1}$$

so that

$$x_i = \left[\sum_{k=m+1}^{i-1} \frac{a_{m+1}a_{m+2}\cdots a_k}{b_{m+1}b_{m+2}\cdots b_k} + 1\right] x_{m+1}$$

$$= \frac{b_0 b_1 \cdots b_m}{a_1 a_2 \cdots a_m}[z_i - z_m] x_{m+1}$$

$$= cons\tan t \cdot [z_i - z_m]$$

Because of $x_n = 1$, so $x_i = \frac{z_i - z_m}{z_n - z_m}$. Let $n \to \infty$, get

$$P^i\{\delta < \delta_m\} = \frac{z_i - z_m}{z - z_m}, P^i\{\delta_m < \delta\} = \frac{z - z_i}{z - z_m}, i \geq m.$$

Property2

$$P^i\{\delta_m < \delta_\Delta\} = \frac{z_i}{z_m}, P^i\{\delta_\Delta < \delta_m\} = \frac{z_m - z_i}{z_m}, i = 0,1,\cdots,m.$$

Prove: Assume $a_0>0, m>1$. For $\forall i, i \le m$, let $x_i = P^i\left\{\delta_m < \delta_\Delta\right\}$, so that $x_m = 1$.According to Property 1, get

$$x_0 = \frac{b_0}{a_0 + b_0} x_1$$

$$x_i = \frac{a_i}{a_i + b_i} x_{i-1} + \frac{b_i}{a_i + b_i} x_{i+1}, 1 \le i < m$$

Thus,

$$x_{i+1} - x_{i-1} = \frac{a_i}{b_i}\left(x_i - x_{i-1}\right) = \cdots = \frac{a_i a_{i-1} \cdots a_0}{b_i b_{i-1} \cdots b_0} x_0$$

$$= \frac{a_0 a_1 \cdots a_i}{b_0 b_1 \cdots b_i} x_0$$

so that, $x_i = z_i a_0 x_0$. If $x_m = 1$, then $x_0 = \dfrac{1}{a_0 z_m}$, that is $x_i = \dfrac{z_i}{z_m}$.

Therefore, $P^i\left\{\delta_m \langle \delta_\Delta\right\} = \dfrac{z_i}{z_m}, P^i\left\{\delta_\Delta \langle \delta_m\right\} = \dfrac{z_m - z_i}{z_m}, i = 0,1,\cdots,m$.

Note 1: property 1 and 2 describe the probability of $z_i, i = 0,1,2,\cdots$.

Property3 Assume $R < \infty$, let $y_i = 1 - E^i\left\{e^{-\lambda\sigma_\Delta}\right\}$, so that $\lim_{i\to\infty} y_i = 0$.

And $y_i, i \in E$ satisfies equations:

$$y_i = \begin{cases} \lambda\sum_{k=i}^{\infty}\mu_k(1-y_k)(z-z_k)+\lambda(z-z_i)\sum_{k=1}^{i-1}\mu_k(1-y_k) \\ \lambda\frac{z_i}{z}\left[\sum_{k=i}^{\infty}\mu_k(1-y_k)(z-z_k)\right]+\lambda(z-z_i)\left[\sum_{k=1}^{i-1}\mu_k(1-y_k)\frac{z_k}{z}\right] \end{cases}$$

The process of proving can be seen in reference [4].

Property 4 Assume $R < \infty$, so that

$$E^i\{\sigma_\Delta\} = \begin{cases} \sum_{k=i}^{\infty}\mu_k(z-z_k)+(z-z_i)\sum_{k=1}^{i-1}\mu_k \\ \frac{z_i}{z}\left[\sum_{k=i}^{\infty}\mu_k(z-z_k)\right]+(z-z_i)\left[\sum_{k=1}^{i-1}\mu_k\frac{z_k}{z}\right] \end{cases}$$

Prove: Based on Property 3, get

$$P^i\{\sigma_\Delta < \infty\} = \lim_{\lambda\downarrow 0} E^i\{e^{-\lambda\sigma_\Delta}\} = 1$$

While $\lambda \downarrow 0$, $\frac{1-e^{-\lambda\sigma_\Delta}}{\lambda} \uparrow \sigma_\Delta$,thus,

$$E^i\{\sigma_\Delta\} = \lim_{\lambda\downarrow 0}\frac{1-E^i\{e^{-\lambda\sigma_\Delta}\}}{\lambda}$$

Combining formula y_i of property 3 and above formula, so that

$$E^i\{\sigma_\Delta\} = \begin{cases} \sum_{k=i}^{\infty}\mu_k(z-z_k)+(z-z_i)\sum_{k=1}^{i-1}\mu_k \\ \frac{z_i}{z}\left[\sum_{k=i}^{\infty}\mu_k(z-z_k)\right]+(z-z_i)\left[\sum_{k=1}^{i-1}\mu_k\frac{z_k}{z}\right] \end{cases}$$

We can get the following property from property 3 and 4.

Property 5 If $R = \infty$, $P^i\{\sigma = \infty\} = 1$.

Property 6 For $\forall k \in E, P^k\{\sigma = \infty\} = 1$ or $P^k\{\sigma < \infty\} = 1$, and these two kinds of probabilities are reflected by $R = \infty, \& R < \infty$.

Note 2: Property 5 and 6 describe the probability of canonical measure.

4 Conclusion

Bilateral birth-death process is one of important Markov chain. It is widely used in the actual models of chemistry, physics, medicine, etc. And bilateral birth-death process has important theoretical significance. Thereby, to provide theoretical basis for finding out all solutions of the irregular matrix Q based on the research on probability property of normal chain corresponding to bilateral birth-death matrix Q of set E.

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