Solution of System of Fredholm Integro-Differential Equations by RKHS Method

Mohammed H. AL-Smadi\textsuperscript{1} and Zuhier K. Altawallbeh

Department of Mathematics and Computer Science
Tafila Technical University, Tafila 66110, Jordan

Copyright © 2013 Mohammed AL-Smadi and Zuhier Altawallbeh. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

In this paper, an application of reproducing kernel Hilbert space (RKHS) method is applied to solve system of Fredholm integro-differential equations. The exact solutions are represented in the form of series in the reproducing kernel space. Moreover, the approximate solutions $u_n(x), v_n(x)$ are proved to converge to the exact solutions $u(x), v(x)$, respectively. The results reveal that the RKHS is simple and effective.

Mathematics Subject Classification: 35F50, 45J05, 47B32

Keywords: Nonlinear system, Fredholm Integro-differential equation, Reproducing kernel space

1 Introduction

In applied mathematics, many problems lead to the following nonlinear system of Fredholm integro-differential equations

\[
\begin{aligned}
  u'(x) &= f_1(x, u(x), v(x)) + \int_a^b K_1(x, t, u(t), v(t)) dt, \\
  v'(x) &= f_2(x, u(x), v(x)) + \int_a^b K_2(x, t, u(t), v(t)) dt, \\
  u(a) &= \alpha_1, \\
  v(a) &= \alpha_2,
\end{aligned}
\]

1mhm.smadi@yahoo.com
where \( a, b \) and \( \alpha_i, i = 0, 1 \), are real finite constants, \( u(x), v(x) \in W^2_2[a, b] \) are unknown functions to be determined, \( f_1, f_2 \) are continuous functions and \( K_1, K_2 \) are also continuous functions that satisfy adequate lipschitzian conditions.

In recent years, system of integro-differential equations have been of interest in mathematics, physics, engineering, and so on (see [7-8], [12-13] and [19-21]). IDEs are usually difficult to be solved analytically so it is required to obtain an efficient approximate solution. Therefore, they have been of great interest by several authors. Reproducing kernel theory has important application in numerical analysis, differential equations, integral equations, probability and statistics, and so on (see [1-2], [9-12] and [20-21]).

In this paper, we apply the RKHS method to develop a novel numerical method in the space \( W^2_2[a, b] \) for obtaining the exact and approximate solutions. The method has the following advantages: small computational requirements, high precision and it is possible to pick any point in the interval of integration and as well the approximate solutions and its derivatives will be applicable. Indeed, Numerical examples are presented to demonstrate the computation efficiency of the presented method.

We will give the representation of exact and approximate solution of Eq. (1) in the reproducing kernel Hilbert space under the assumption that the solution of Eq. (1) is unique smooth solution. If the solution of Eq. (1) is not unique, then the solution that we obtain is a least-norm solution.

Let \( Lu = u', L : W^2_2[a, b] \rightarrow W^1_2[a, b] \). After homogenization of the initial conditions, then Eq. (1) can be converted into the following form

\[
\begin{align*}
Lu(x) &= F_1(x, u(x), v(x), Tu(x)), \quad a < x < b, \\
Lv(x) &= F_2(x, u(x), v(x), Sv(x)), \quad a < x < b, \\
u(a) &= v(a) = 0,
\end{align*}
\]

where \( Tu(x) = \int_a^b K_1(x, t, u(t), v(t))dt \), \( Sv(x) = \int_a^b K_2(x, t, u(t), v(t))dt \), \( F_1, F_2 \in W^1_2[a, b] \) and \( u(x), v(x) \in W^2_2[a, b] \). \( W^1_2[a, b] \) and \( W^2_2[a, b] \) are defined in the following section.

2 Construction of RKH spaces

**Definition 2.1 (Reproducing kernel)** Let \( E \) be a nonempty abstract set. A function \( K : E \times E \rightarrow \mathbb{C} \) is a reproducing kernel of the Hilbert space \( H \) iff

1. \( \forall t \in E, K(\cdot, t) \in H \).
2. \( \forall t \in E, \forall \varphi \in H, (\varphi, K(\cdot, t)) = \varphi(t) \).

The last condition is called “the reproducing property”: the value of the function \( \varphi \) at the point \( t \) is reproducing by the inner product of \( \varphi \) with \( K(\cdot, t) \).
2.1 The reproducing kernel Hilbert space \( W_2^2 [a, b] \)

The inner product space \( W_2^2 [a, b] \) is defined as \( W_2^2 [a, b] = \{ u(x) : u, u' \) are absolutely continuous real valued functions on \( [a, b] \), \( u, u', u'' \) are \( L^2 [a, b] \), and \( u(a) = 0 \} \). The inner product in \( W_2^2 [a, b] \) is given by

\[
\langle u, v \rangle_{W_2^2} = u(a)v(a) + u'(a)v'(a) + \int_a^b u''(y)v''(y) \, dy
\]

and the norm \( \| u \|_{W_2^2} \) is denoted by \( \| u \|_{W_2^2} = \sqrt{\langle u, u \rangle_{W_2^2}} \), where \( u, v \in W_2^2 [a, b] \).

**Theorem 2.1** The space \( W_2^2 [a, b] \) is a complete reproducing kernel space. That is, for each fixed \( x \in [a, b] \), there exists \( R_x(y) \in W_2^2 [a, b] \) such that \( \langle u(y), R_x(y) \rangle_{W_2^2} = u(x) \) for any \( u(y) \in W_2^2 [a, b] \) and \( y \in [a, b] \). The reproducing kernel function \( R_x(y) \) can be denoted by

\[
R_x(y) = \begin{cases} 
\sum_{i=1}^{4} p_i(x) y^{i-1}, & y \leq x, \\
\sum_{i=1}^{4} q_i(x) y^{i-1}, & y > x. 
\end{cases}
\]

The method of obtaining coefficients of the reproducing kernel \( R_x(y) \) and the proof of Theorem 2.1 are given in Theorem 2.4 in [20]. By using Mathematica 7.0 software package, the representation of \( R_x(y) \) is given by

\[
R_x(y) = \begin{cases} 
\frac{1}{6} (y-a) (2a^2 - y^2 + 3x (2 + y) - a (6 + 3x + y)), & y \leq x, \\
\frac{1}{6} (x-a) (2a^2 - x^2 + 3y (2 + x) - a (6 + x + 3y)), & y > x.
\end{cases}
\]

2.2 The reproducing kernel Hilbert space \( W_2^1 [a, b] \)

The inner product space \( W_2^1 [a, b] \) is defined as \( W_2^1 [a, b] = \{ u(x) : u \) is absolutely continuous real valued function on \( [a, b] \), \( u' \in L^2 [a, b] \} \). The inner product in \( W_2^1 [a, b] \) is given by \( \langle u, v \rangle_{W_2^1} = \int_a^b u(t) v(t) + u'(t) v'(t) \, dt \) and the norm \( \| u \|_{W_2^1} \) is denoted by \( \| u \|_{W_2^1} = \sqrt{\langle u, u \rangle_{W_2^1}} \), where \( u, v \in W_2^1 [a, b] \).

In [10], it has been proved that \( W_2^1 [a, b] \) is also a complete reproducing kernel space and its reproducing kernel is

\[
K_x(y) = \frac{1}{2 \sinh (b-a)} \cosh (x + y - b - a) + \cosh (|x - y| - b + a).
\]
3 The analytical and approximate solution

In this section, we will give the representation of analytical solution of Eq. (2) and the implementation method in the reproducing kernel space $W^2_2 [a,b]$.

In Eq. (2), it is clear that $L : W^2_2 [a,b] \rightarrow W^1_2 [a,b]$ is a bounded linear operator. Let $\varphi_i (x) = K_{x_i} (x)$ and $\psi_i (x) = L^* \varphi_i (x)$, where $L^*$ is the conjugate operator of $L$ and $\{x_i\}_{i=1}^{\infty}$ is dense on $[a,b]$. In terms of the properties of $K_x (y)$, one obtains $\langle u (x) , \psi_i (x) \rangle_{W^2_2} = \langle u (x) , L^* \varphi_i (x) \rangle_{W^2_2} = \langle Lu (x) , \varphi_i (x) \rangle_{W^2_2} = Lu (x_i) , i = 1, 2, \ldots$.

The orthonormal system $\{ \tilde{\psi}_i (x) \}_{i=0}^{\infty}$ of the space $W^2_2 [a,b]$ can be derived from Gram-Schmidt orthogonalization process of $\{ \psi_i (x) \}_{i=0}^{\infty}$ as follows:

$$\tilde{\psi}_i (x) = \sum_{k=1}^{i} \beta_{ik} \psi_k (x) , \quad (5)$$

where $\beta_{ik}$ are orthogonalization coefficients such that $\beta_{ii} > 0 , i = 1, 2, \ldots$.

Through the next theorem the subscript $y$ by the operator $L$ indicates that the operator $L$ applies to the function of $y$.

**Theorem 3.1** If $\{x_i\}_{i=0}^{\infty}$ is dense on $[a,b]$, then $\{ \psi_i (x) \}_{i=1}^{\infty}$ is a complete function system of $W^2_2 [a,b]$ and $\psi_i (x) = L_y R_x (y)|_{y=x_i}$.

**Proof.** Notice that

$$\psi_i (x) = L^* \varphi_i (x) = \langle L^*_i \varphi (y) , R_x (y) \rangle = \langle \varphi_i (y) , L_y R_x (y) \rangle = L_y K_x (y)|_{y=x_i} .$$

Clearly, $\psi_i (x) \in W^2_2 [a,b]$.

Now, for each fixed $u (x) \in W^2_2 [a,b]$, let $\langle u (x) , \psi_i (x) \rangle = 0 , (i = 1, 2, \ldots)$, that is $\langle u (x) , L^* \varphi_i (x) \rangle = \langle Lu (\cdot) , \varphi_i (\cdot) \rangle = Lu (x_i) = 0$. Note that $\{x_i\}_{i=0}^{\infty}$ is dense on $[a,b]$, therefore $Lu (x) = 0$. It follows that $u (x) = 0$ from the existence of $L^{-1}$. So, the proof of the Theorem is complete.

**Theorem 3.2** If $\{x_i\}_{i=0}^{\infty}$ is dense on $[a,b]$ and $u (x) , v (x) \in W^2_2 [a,b]$ are the solutions of Eq. (2), then $u (x) , v (x)$ satisfy the following form, respectively

$$u (x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} F_1 (x_k , u(x_k) , v(x_k) , Tu(x_k)) \tilde{\psi}_i (x) , \quad (6)$$

$$v (x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} F_2 (x_k , u(x_k) , v(x_k) , Sv(x_k)) \tilde{\psi}_i (x) . \quad (7)$$
Proof. \( u(x) \) can be expanded to Fourier series in terms of orthonormal basis \( \{ \hat{\psi}_i(x) \}_{i=0}^{\infty} \) in \( W_2^2[a,b] \) as \( u(x) = \sum_{i=1}^{\infty} \langle u(x), \hat{\psi}_i(x) \rangle \hat{\psi}_i(x) \).

Since the space \( W_2^2[a,b] \) is Hilbert space so the series \( \sum_{i=1}^{\infty} \langle u(x), \hat{\psi}_i(x) \rangle \hat{\psi}_i(x) \) is convergent in the norm of \( \| \cdot \|_{W_2^2} \). Also, note that \( \langle w(x), \phi_i(x) \rangle = w(x_i) \) for each \( w(x) \in W_2^1[a,b] \). Hence, we have

\[
\begin{align*}
    u(x) &= \sum_{i=1}^{\infty} \langle u(x), \hat{\psi}_i(x) \rangle_{W_2^2} \hat{\psi}_i(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle u(x), \psi_k(x) \rangle_{W_2^2} \hat{\psi}_i(x) \\
    &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle u(x), L^* \varphi_k(x) \rangle_{W_2^2} \hat{\psi}_i(x) \\
    &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle Lu(x), \varphi_k(x) \rangle_{W_2^2} \hat{\psi}_i(x) \\
    &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle F_1(x, u(x), v(x), Tu(x)), \varphi_k(x) \rangle_{W_2^2} \hat{\psi}_i(x) \\
    &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} F_1(x_k, u(x_k), v(x_k), Tu(x_k)) \hat{\psi}_i(x).
\end{align*}
\]

In the same way, we can get

\[
\begin{align*}
v(x) &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} F_2(x_k, u(x_k), v(x_k), Sv(x_k)) \hat{\psi}_i(x).
\end{align*}
\]

Now, the approximate solution \( u_n(x), v_n(x) \) can be obtained by taking finitely many terms in the series representation of \( u(x), v(x) \), respectively, and

\[
\begin{align*}
u_n(x) &= \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} F_1(x_k, u(x_k), v(x_k), Tu(x_k)) \hat{\psi}_i(x), \quad (8) \\
v_n(x) &= \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} F_2(x_k, u(x_k), v(x_k), Sv(x_k)) \hat{\psi}_i(x). \quad (9)
\end{align*}
\]

If Eq. (1) is nonlinear, then the approximation solution of Eq. (1) can be obtained using the following iteration formula.

We construct the iterative sequences \( u_n(x), v_n(x) \), putting

\[
\begin{align*}
\begin{cases}
\forall \text{ fixed } u_0(x), v_0(x) \in W_2^2[a,b], \\
u_n(x) = \sum_{i=1}^{n} A_i \tilde{\psi}_i(x), \\
v_n(x) = \sum_{i=1}^{n} B_i \tilde{\psi}_i(x),
\end{cases}
\end{align*}
\]

\[
(10)
\]
where the coefficients $A_i$, $B_i$ of $\bar{\psi}_i (x)$, $i = 1, 2, ..., n$ are given as

$$
\begin{align*}
A_1 &= \beta_{11} F_1 (x_1, u_0 (x_1), v_0 (x_1), Tu_0(x_1)), \\
A_2 &= \sum_{k=1}^{2} \beta_{2k} F_1 (x_k, u_{k-1} (x_k), v_{k-1} (x_k), Tu_{k-1}(x_k)), \\
& \vdots \\
A_n &= \sum_{k=1}^{n} \beta_{nk} F_1 (x_k, u_{k-1} (x_k), v_{k-1} (x_k), Tu_{k-1}(x_k)),
\end{align*}
$$

(11)

$$
\begin{align*}
B_1 &= \beta_{11} F_2 (x_1, u_0 (x_1), v_0 (x_1), Sv_0(x_1)), \\
B_2 &= \sum_{k=1}^{2} \beta_{2k} F_2 (x_k, u_{k-1} (x_k), v_{k-1} (x_k), Sv_{k-1}(x_k)), \\
& \vdots \\
B_n &= \sum_{k=1}^{n} \beta_{nk} F_2 (x_k, u_{k-1} (x_k), v_{k-1} (x_k), Sv_{k-1}(x_k)).
\end{align*}
$$

(12)

**Theorem 3.3** Assume that $u (x), v(x) \in W_2^2 [a, b]$ are the solutions of Eq. (2) and $r_n (x), r_n^* (x)$ are the differences between the approximate solution $u_n (x), v_n (x)$ and the exact solution $u (x), v(x)$ respectively. Then, $r_n (x), r_n^* (x)$ are monotone decreasing in the sense of the norm of $W_2^2 [a, b]$. i.e. $r_n \to 0$ and $r_n^* \to 0$ as $n \to \infty$.

**Proof.** From Eq. (6), (8), it is obvious that

$$
||r_n (x)||_{W_2^2}^2 = ||u (x) - u_n (x)||_{W_2^2}^2
$$

$$=
\left\| \sum_{i=n+1}^{\infty} \beta_{i} F_i (x_i, u_{i-1} (x_i), v_{i-1} (x_i), Tu_{i-1}(x_i)) \bar{\psi}_i (x) \right\|_{W_2^2}^2
$$

$$=
\left\| \sum_{i=n+1}^{\infty} A_i \bar{\psi}_i (x) \right\|_{W_2^2}^2 = \sum_{i=n+1}^{\infty} (A_i)^2,
$$

and $||r_{n-1} (x)||_{W_2^2}^2 = \sum_{i=n}^{\infty} (A_i)^2$. Thus, $||r_n (x)||_{W_2^2} \leq ||r_{n-1} (x)||_{W_2^2}$. Similarly, from Eq. (7), (9), $||r_n^* (x)||_{W_2^2}^2 = ||v (x) - v_n (x)||_{W_2^2}^2 = \left\| \sum_{i=n+1}^{\infty} B_i \bar{\psi}_i (x) \right\|_{W_2^2}^2

= \sum_{i=n+1}^{\infty} (B_i)^2$ and $||r_{n-1}^* (x)||_{W_2^2}^2 = \sum_{i=n}^{\infty} (B_i)^2$. Thus, $||r_n^* (x)||_{W_2^2} \leq ||r_{n-1}^* (x)||_{W_2^2}$. Consequently, the differences $r_n (x), r_n^* (x)$ are monotone decreasing in the sense of $||\cdot||_{W_2^2}$. So, the proof of the theorem is complete.
**Remark** Since $W^2_2[a,b]$ is a Hilbert space, it is clear that $\sum_{i=1}^{\infty} (A_i)^2 < \infty$ and $\sum_{i=1}^{\infty} (B_i)^2 < \infty$. Therefore, the sequence $u_n, v_n$ are convergent.

### 4 Example

In this section, some numerical examples are studied to demonstrate the accuracy of the present method. The examples are computed using Mathematica 7.0. Results obtained by the RKHS method are found to be in good agreement with the exact solution.

#### Example 4.1 [19]

Consider the following system of Fredholm integro-differential equations

\[
\begin{cases}
    u'(x) = 15x + \frac{4}{5} + \frac{1}{0} 3(2x + t^2)(2v(t) - u(t))dt, \\
    v'(x) = 3x^2 + \frac{3}{10}x + 2 + \frac{1}{0} 3xt(2v(t) - u(t))dt, \\
    u(0) = 1, v(0) = -1.
\end{cases}
\]

The exact solution is $u(x) = 3x^2 + 1, v(x) = x^3 + 2x - 1$. Using RKHS method, taking $x_i = \frac{i-1}{N-1}$, $i = 1, 2, ..., N$ with the reproducing kernel function $R_x(y)$ on $[0, 1]$. The numerical results at some selected nods for $N = 100$ are displayed in Table 1 and Table 2.

#### Table 1. Results for Example 4.1.

<table>
<thead>
<tr>
<th>Node</th>
<th>Exact $u(x)$</th>
<th>Approximate $u_{100}(x)$</th>
<th>Absolute error</th>
<th>Relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.16</td>
<td>1.0768</td>
<td>1.076792182</td>
<td>$7.8180 \times 10^{-6}$</td>
<td>$7.26040 \times 10^{-6}$</td>
</tr>
<tr>
<td>0.32</td>
<td>1.3072</td>
<td>1.307203485</td>
<td>$3.4850 \times 10^{-6}$</td>
<td>$2.66600 \times 10^{-6}$</td>
</tr>
<tr>
<td>0.48</td>
<td>1.6912</td>
<td>1.691212741</td>
<td>$1.2741 \times 10^{-5}$</td>
<td>$7.53370 \times 10^{-6}$</td>
</tr>
<tr>
<td>0.64</td>
<td>2.2288</td>
<td>2.228812810</td>
<td>$1.2810 \times 10^{-5}$</td>
<td>$5.74749 \times 10^{-6}$</td>
</tr>
<tr>
<td>0.80</td>
<td>2.92</td>
<td>2.920018339</td>
<td>$1.8339 \times 10^{-5}$</td>
<td>$6.28048 \times 10^{-6}$</td>
</tr>
<tr>
<td>0.96</td>
<td>3.7648</td>
<td>3.764806153</td>
<td>$6.1530 \times 10^{-6}$</td>
<td>$1.63435 \times 10^{-6}$</td>
</tr>
</tbody>
</table>

#### Example 4.2 [19]

Consider the following system of Fredholm integro-differential equations

\[
\begin{cases}
    u'(x) = u(x) + \int_{0}^{(x+2)t} v(t) - e^{(x-t)}u(t) dt + g_1(x), \\
    v'(x) = u(x) + v(x) - \int_{0}^{(cos(4\pi x) sin(2\pi t) u(t) + cos(4\pi x + 2\pi t) v(t)) dt + g_2(x), \\
    u(0) = 1, v(0) = 1.
\end{cases}
\]

where $g_1(x)$ and $g_2(x)$ are chosen such that the exact solution is $u(x) = e^x, v(x) = e^{-x}$. 

Using RKHS method, taking $x_i = \frac{i-1}{N-1}$, $i = 1, 2, ..., N$ with the reproducing kernel function $R_x(y)$ on $[0, 1]$. The numerical results at some selected nodes for $N = 100$ are displayed in Table 3 and Table 4.

Table 2. Results for Example 4.1.

<table>
<thead>
<tr>
<th>Node</th>
<th>Exact $v(x)$</th>
<th>Approximate $v_{100}(x)$</th>
<th>Absolute error</th>
<th>Relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.16</td>
<td>$-0.675904$</td>
<td>$-0.675899784$</td>
<td>$4.2160 \times 10^{-6}$</td>
<td>$6.23757 \times 10^{-6}$</td>
</tr>
<tr>
<td>0.32</td>
<td>$-0.327232$</td>
<td>$-0.327233098$</td>
<td>$1.0980 \times 10^{-6}$</td>
<td>$3.35542 \times 10^{-6}$</td>
</tr>
<tr>
<td>0.48</td>
<td>0.070592</td>
<td>0.0705902865</td>
<td>$1.7135 \times 10^{-6}$</td>
<td>$2.42733 \times 10^{-6}$</td>
</tr>
<tr>
<td>0.64</td>
<td>0.542144</td>
<td>0.5421426031</td>
<td>$1.3969 \times 10^{-6}$</td>
<td>$2.57662 \times 10^{-6}$</td>
</tr>
<tr>
<td>0.80</td>
<td>1.112000</td>
<td>1.1120010759</td>
<td>$1.0759 \times 10^{-6}$</td>
<td>$9.67536 \times 10^{-7}$</td>
</tr>
<tr>
<td>0.96</td>
<td>1.804736</td>
<td>1.8047349873</td>
<td>$1.0127 \times 10^{-6}$</td>
<td>$5.61135 \times 10^{-7}$</td>
</tr>
</tbody>
</table>

Table 3. Results for Example 4.2.

<table>
<thead>
<tr>
<th>Node</th>
<th>Exact $v(x)$</th>
<th>Approximate $v_{100}(x)$</th>
<th>Absolute error</th>
<th>Relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.16</td>
<td>1.17351</td>
<td>1.1735052013</td>
<td>$5.66969 \times 10^{-6}$</td>
<td>$4.83139 \times 10^{-6}$</td>
</tr>
<tr>
<td>0.32</td>
<td>1.37713</td>
<td>1.3771257064</td>
<td>$2.05794 \times 10^{-6}$</td>
<td>$1.49437 \times 10^{-6}$</td>
</tr>
<tr>
<td>0.48</td>
<td>1.61607</td>
<td>1.6160753703</td>
<td>$9.68107 \times 10^{-7}$</td>
<td>$5.99049 \times 10^{-7}$</td>
</tr>
<tr>
<td>0.64</td>
<td>1.89648</td>
<td>1.8964792516</td>
<td>$1.62770 \times 10^{-6}$</td>
<td>$8.58276 \times 10^{-7}$</td>
</tr>
<tr>
<td>0.80</td>
<td>2.22554</td>
<td>2.2255378017</td>
<td>$3.12679 \times 10^{-6}$</td>
<td>$1.40496 \times 10^{-6}$</td>
</tr>
<tr>
<td>0.96</td>
<td>2.6117</td>
<td>2.6116908109</td>
<td>$5.66252 \times 10^{-6}$</td>
<td>$2.16814 \times 10^{-6}$</td>
</tr>
</tbody>
</table>

Table 4. Results for Example 4.2.

<table>
<thead>
<tr>
<th>Node</th>
<th>Exact $v(x)$</th>
<th>Approximate $v_{100}(x)$</th>
<th>Absolute error</th>
<th>Relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.16</td>
<td>0.852144</td>
<td>0.8521463004</td>
<td>$2.51143 \times 10^{-6}$</td>
<td>$2.94719 \times 10^{-6}$</td>
</tr>
<tr>
<td>0.32</td>
<td>0.726149</td>
<td>0.7261470138</td>
<td>$2.02327 \times 10^{-6}$</td>
<td>$2.78631 \times 10^{-6}$</td>
</tr>
<tr>
<td>0.48</td>
<td>0.618783</td>
<td>0.6187843089</td>
<td>$9.17094 \times 10^{-7}$</td>
<td>$1.48209 \times 10^{-6}$</td>
</tr>
<tr>
<td>0.64</td>
<td>0.527292</td>
<td>0.5272891361</td>
<td>$3.28794 \times 10^{-6}$</td>
<td>$6.23552 \times 10^{-6}$</td>
</tr>
<tr>
<td>0.80</td>
<td>0.449329</td>
<td>0.4493235017</td>
<td>$5.46242 \times 10^{-6}$</td>
<td>$1.21568 \times 10^{-5}$</td>
</tr>
<tr>
<td>0.96</td>
<td>0.382893</td>
<td>0.3828854552</td>
<td>$7.43078 \times 10^{-6}$</td>
<td>$1.94069 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

5 Conclusion

In this paper, reproducing kernel Hilbert space method is employed to solve system of Fredholm IDEs. It is evident from the numerical examples that the method proposed in this paper gives the accurate results. The numerical results are displayed to demonstrate the validity of this method. Moreover, the error of the approximate solution is monotone decreasing in the sense of the norm of $W_2^2[a, b]$. 
References


**Received: March 18, 2013**